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Algorithm for Interval Linear Programming Involving Interval Constraints

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***Abstract*—In real optimization, we always meet the criteria of useful outcomes increasing or expenses decreasing and demands of lower uncertainty. Therefore, we usually formulate an optimization problem under conditions of uncertainty.**

In this paper, a new method for solving linear programming problems with fuzzy parameters in the objective function and the constraints based on preference relations between intervals is investigated. To illustrate the efficiency of the proposed method, a numerical example is presented.

I. INTRODUCTION

IN many real-life situations we come across problems with imprecise input values. Imprecisions are dealt with by various ways. One of them is interval based approach in which we model imprecise quantities by intervals, and suppose that the quantities may vary independently and simultaneously within their intervals. In most optimization problems, they are formulated using imprecise parameters. Such parameters can be considered as fuzzy intervals, and the optimization tasks with interval cost function are obtained [13], [14].

When realistic problems are formulated, a set of intervals may appear as coefficients in the objective function or the constraints of a linear programming problem. Theoretically, intervals can not be ordered, they can only be partially ordered and as a consequences, can not be compared. Therefore, we build a criterion for quantitative assessment of degree in which one interval is greater than another one. This criterion must be applicable for all cases of intervals. The problem of intervals ordering is an important problem because of its direct relevance to real world optimization problems. Therefore, the comparison of intervals is necessary when we have to make a a choice in practical applications. Numerous definitions of the comparison relation on intervals exist [2], [3], [7], [15], [18], [21], [22], [27], [28].

In this field, we find the foremost work in [17], [18] where two transitive relations were defined on intervals; the first one is the extension of '$<$' on the real numbers, and the second is the extension of set inclusion '$\subseteq$'. These order relations can not compare between overlapping intervals. Ishibuchi and Tanaka [11] suggested two order relations '$\leq_{LR}$' where the endpoints of the intervals are used and '$\leq_{mw}$' where the midpoint and width are used. Nevertheless, there exist a set of pair of intervals can not be compared using these order relations. Moreover, these order relations do not discuss 'how much greater' when one interval is known to be greater than another. From this point of view, there exists a number of papers discussing this topic [2], [3], [5], [7], [8], [27]. The authors use some quantitative indices to present the degree to which one interval is greater than another interval. In some cases, even several indices are used simultaneously. In this paper, we will use the method that was introduced in [2] - where the author introduced the so called $\mu-$ordering - to solve interval linear programming problems.

This paper considers linear programming problems with interval coefficients. For these problems, we can not apply the technique of the classical linear programming directly. Therefore, many researchers investigated interval linear programming problems on the basis of order relations between two intervals [5], [11], [12], [25]. Interval linear programming problems have been studied by several authors, such as Bitran [4], Steuer [26], Ishibuchi and Tanaka [10], [11], Nakahara et al. [19], Chanas and Kuchta [5], and Gen and Cheng [9]. For example, Ishibuchi and Tanaka [10], [11] studied linear programming problems where the objective function has interval coefficients and they transformed this problem into a standard biobjective optimization problem. In this paper, we study the linear programming problems that has interval coefficients in the objective function and in the constraints.

In fuzzy programming problems [6], [15] the constraints and objective function are viewed as fuzzy sets. In stochastic programming problems [15] the coefficients are viewed as random variables. However, the method presented in this paper has the advantage that the solution is more intelligible to the decision maker.

In this paper, we focus on a satisfactory solution approach based on the inequality relations that was introduced by [2] and to solve the interval linear programming problem. This paper is organized as follows: In Section 2, we state the interval linear programming problem. In Section 3, we introduce some basic properties and arithmetics of intervals, and give an elaborate study on inequality relation with interval coefficients in search of realizing the relation as a constraint of an optimization problem defined in an inexact environment. Finally, in Section 4, we describe the solution principle of interval linear programming problems.

II. PROBLEM STATEMENT AND NOTATION

LINEAR programming is a mathematical tool that handles the optimization of a linear objective function subject to linear constraints. Linear programming is an important area in applied mathematics which has a large number of applications in many industries. A linear programming problem (LP) can be formulated as follows:

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$$\begin{aligned} \max z &= c^t x \\ \text{s.t.} \quad Ax &\leq b \\ x &\geq 0 \end{aligned}$$

where c and x are n dimensional vectors, b is an m dimensional vector, and A is $m \times n$ matrix.

Since we are living in and uncertain environment, the coefficients of objective function c, the technical coefficients of matrix A, and the resource variable b are intervals. Therefore, the problem can be solved by interval linear programming approach.

The interval linear programming problem is formulated as:

$$\begin{aligned} \max z &= \mathbf{c}^t\, x \\ \text{s.t.} \quad \mathbb{A}\, x &\preceq \mathbf{b} \\ x &\geq 0 \end{aligned} \tag{1}$$

where x is the vector of decision variables, $\mathbb{A}$ is in interval matrix where all of its entries are intervals, **b** and **c** are interval vectors, the inequality $\preceq$ is given by interval comparison relation, and the objective function z is to be maximized in the sense of a given interval linear programming criteria.

Since the calculation of the solution is very expensive and difficult to find as to real world problems and the practical problem is not solved yet even though the feasible region is known, in optimization problems with several objectives, one mostly waives the determination of the set of efficient solutions and searches a so-called compromise solution straightforwardly.

Definition II.1. If $F : \Omega \to \mathbb{R}$ is a function of problem (1), where

$$\Omega = \{x : \mathbb{A}\, x \preceq \mathbf{b}, x \geq 0\},$$

then a feasible solution $\widetilde{x} \in \Omega$ is called a compromise solution if

$$F(x) \leq F(\widetilde{x}) \qquad \forall x \in \Omega.$$

It is clear that the midpoint of an interval is the expected value of that special fuzzy variable. While an interval denotes the uncertain return, and uncertain cost, the pessimistic return is denoted by the intervals left and right limits respectively. The previous interval programming model can be viewed as a combination of the fuzzy expected value model and the pessimistic decision model.

III. Order relations between intervals

In this section, we review some properties of interval analysis [1], [17], [18]. Throughout this paper, real numbers will be denoted by lower case letters and the upper case letters denote closed intervals. We begin by defining interval arithmetic and then give a brief discussion of some implementation consideration. Finally, we discuss the order relations between intervals. Let

$$A = [\underline{a}, \bar{a}] = \{x \in \mathbb{R} : \underline{a} \leq x \leq \bar{a}\}$$

denote a closed interval on the real line defined by finite points $\underline{a}$, and $\bar{a}$ with $\underline{a} \leq \bar{a}$.

Denote the *radius* of A by $r_A = (\bar{a} - \underline{a})/2$, and denote the *midpoint* of A by $m_A = (\underline{a} + \bar{a})/2$. Interval A is alternatively represented as $A = \langle m_A, r_A \rangle$. Let $\bullet$ denote one of the arithmetic operations $+, -, \times$ or $\div$, and let $A = [\underline{a}, \bar{a}]$ and $B = [\underline{b}, \bar{b}]$, then the generalization of ordinary arithmetic to closed intervals is known as interval arithmetic, and is defined by:

$$A \bullet B = \{a \bullet b : a \in A, b \in B\},$$

where we assume $0 \notin B$ in case of division.

We can see that

$$\begin{aligned} &A + B = [\underline{a} + \underline{b}, \bar{a} + \bar{b}], \\ &A - B = [\underline{a} - \bar{b}, \bar{a} - \underline{b}], \\ &kA = k[\underline{a}, \bar{a}] = \begin{cases} [k\underline{a}, k\bar{a}] & k \geq 0 \\ [k\bar{a}, k\underline{a}] & k < 0 \end{cases} \\ &A \cdot B = [\min\{\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}\}], \\ &A \div B = [\underline{a}, \bar{a}][1/\bar{b}, 1/\underline{b}], \qquad 0 \notin B. \end{aligned}$$

In order to solve interval linear programming problems, we must build a criterion for quantitative assessment of degree in which one interval is "greater" than another one. Theoretically, intervals cannot be compared, they can only be partially ordered. However, when intervals are used in practical applications or when a choice has to be made among alternatives, the comparison of intervals becomes necessary. In the literature, there are numbers of definitions of the comparing two real intervals [1], [2], [3], [5], [7], [16], [18], [24].

One of the order relations is defined as an extension of '$<$' on the real line as $A \leq B$ if and only if $\bar{a} \leq \underline{b}$, and another one as an extension of the concept of set inclusion; $A \leq B$ if and only if $\underline{a} \geq \underline{b}$ and $\bar{a} \leq \bar{b}$. These order relations cannot compare between overlapping intervals.

Another approach of the problem of ranking two intervals was defined in [11] as follows $A \leq_{LR} B$ if and only if $\underline{a} \leq \underline{b}$ and $\bar{a} \leq \bar{b}$. The authors suggested another order relation $\leq_{mr}$ if $\leq_{LR}$ can not be applied as follows: $A \leq_{mr} B$ if and only if $m_A \leq m_B$ and $r_A \leq r_B$.

The order relations $\leq_{LR}$ and $\leq_{mr}$ are antisymmetric, reflexive and transitive and hence, define partial ordering between intervals. But they did not compare the pairs of intervals for which both $\leq_{LR}$ and $\leq_{mr}$ fail.

Other definitions of order relations are stated in the following definition.

Definition III.1.
1) $A \leq B$ if and only if $\bar{a} \leq \bar{b}$ and $m_A \leq m_B$.
2) $A \leq B$ if and only if $\underline{a} - \epsilon \leq \bar{b}$.
3) $A \leq B$ if and only if $\bar{a} - \epsilon \leq \underline{b}$.
4) $A \leq B$ if and only if $\underline{a} + \bar{a} \leq \underline{b} + \bar{b}$.

A new useful method for ordering fuzzy numbers, has been proposed in [2]. If we let $\mathscr{I}$ be the set of all closed and bounded intervals on the real line $\mathbb{R}$, then the method is based on a measure function ($\mu-$function), that is defined from $\mathscr{I} \times \mathscr{I}$ to $\mathbb{R}$, and is defined as follows:

Definition III.2. If $A, B \in \mathscr{I}$, and $\mu : \mathscr{I} \times \mathscr{I} \longrightarrow \mathbb{R}$ is

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defined by:

$$\mu(A,B)=\begin{cases} m_B-m_A+2\,\mathrm{sgn}(m_B-m_A), \\ \text{if } r_B+r_A=0 \\ \frac{m_B-m_A}{r_B+r_A}+\mathrm{sgn}(m_B-m_A), \\ \text{if } m_A\neq m_B \text{ and } r_B+r_A\neq 0 \\ \frac{r_B-r_A}{\max\{r_B,r_A\}}, \\ \text{if } m_A=m_B \text{ and } r_B+r_A\neq 0 \end{cases}$$

then the order relation $\leq_\mu$ over intervals is defined by:

$$A \leq_\mu B \text{ if and only if } \mu(A,B)\geq 0.$$

This definition leads to the following theorem that has been proved in [2].

Proposition III.1. 1) *If A and B are real numbers, then $\leq_\mu$ is the ordinary inequality relation "$\leq$" on the set of real numbers.*
2) $\mu(A,B)=0$ *iff* $A=B$.
3) *If* $0<\mu(A,B)\leq 1$ *then* $A\subset B$, *(proper subset).*
4) *If* $1<\mu(A,B)\leq 2$ *then* $A\bigcap B\neq\phi$*;*
Moreover, if $1<\mu(A,B)\leq 2-\frac{2\min\{r_A,r_B\}}{r_B+r_A}$*, then .*

$$\begin{cases} A\subset B & \text{if } r_B\geq r_A \\ B\subset A, & \text{if } r_B<r_A \end{cases}$$

5) $\mu(A,B)>2$ *iff* $A\bigcap B=\phi$.

Remark III.1. *If we have a maximization problem and $\mu(A,B)>0$, then interval B is preferred to A and for a minimization problem A is preferred to B, in terms of value.*

IV. Determination of a compromise solution of interval linear programming problem

In order to determine a compromise solution of the linear optimization problem (1), in literature preference functionals are proposed which transfer the infinitely many objective functions into a single objective function. The obvious way of doing this is to choose a single representative $\hat{c}$ of the interval interval $[\underline{c},\bar{c}]$.

Now we rewrite the interval LP problem (1) as follows

$$\begin{aligned} \max z &= \sum_{j=1}^{n}[\underline{c}_j,\bar{c}_j]x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n}[\underline{a}_{ij},\bar{a}_{ij}]x_j \preceq [\underline{b}_i,\bar{b}_i] \\ & \forall i=1,2,\ldots,m \\ & x_j\geq 0 \quad \forall j=1,\ldots,n \end{aligned} \tag{2}$$

each inequality constraint is first transformed into 2^{n+1} crisp inequalities and we get $\Omega_i=\{\Omega_i^k, k=1,2,\ldots,2^{n+1}\}$, which are the solutions to the i^{th} set of 2^{n+1} inequalities.

Now we define the maximum value range inequality by

$$\bar{\Omega}_i=\bigcup_{k=1}^{2^{n+1}}\Omega_i^k,$$

and the minimum value range inequality by

$$\underline{\Omega}_i=\bigcap_{k=1}^{2^{n+1}}\Omega_i^k.$$

The following example illustrates how to find the maximum and minimum value range inequality.

Example IV.1. Consider the inequality relation $[2,8]x\leq[4,12]$. Then we have

$$\Omega^1=\{x:2x\leq 4\}=\{x:x\leq 2\}$$
$$\Omega^2=\{x:2x\leq 12\}=\{x:x\leq 6\}$$
$$\Omega^3=\{x:8x\leq 4\}=\{x:x\leq 0.5\}$$
$$\Omega^4=\{x:8x\leq 12\}=\{x:x\leq 1.5\}$$

and we have the maximum value range inequality is

$$\bar{\Omega}_i=\bigcup_{k=1}^{2^2}\Omega^k=\{x:x\leq 6\},$$

and the minimum value range inequality is

$$\underline{\Omega}_i=\bigcap_{k=1}^{2^2}\Omega^k=\{x:x\leq 0.5\}.$$

Now we use the $\mu-$ordering to define $Ax\leq B$, where $A,B\in\mathscr{I}$ and x is a singleton variable. We say that $Ax\leq B$ if and only if $Ax\leq_\mu B$, i.e., $\mu(Ax,B)\geq 0$. However, in some cases we may look for an optimal constraint condition and to get higher satisfaction; therefore, we may like to increase the value of x to such an extent that $\mu(B,Ax)\geq\sigma$ for $\sigma\in[0,1]$, where σ may be interpreted as an assumed and fixed optimistic threshold. On the other hand, the right limit of Ax must not exceed the right limit of B, i.e., $\bar{a}x\leq\bar{b}$.

Now we propose a satisfactory crisp equivalent form of interval inequality relation as follows:

$$Ax\leq B \Rightarrow \begin{cases} \bar{a}x\leq\bar{b} \\ \mu(B,Ax)\leq\sigma, \quad \sigma\in[0,1] \end{cases}$$

Now we consider the following problem:

$$\begin{aligned} \max z &= \sum_{j=1}^{n}[\underline{c}_j,\bar{c}_j]x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n}[\underline{a}_{ij},\bar{a}_{ij}]x_j \preceq [\underline{b}_i,\bar{b}_i] \\ & \forall i=1,2,\ldots,m \\ & x_j\geq 0, \quad \forall j=1,\ldots,n \end{aligned}$$

In this problem, the satisfactory crisp equivalent system of constraints of the i^{th} interval constraint can be generated as

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follows:

$$\sum_{j=1}^{n} \bar{a}_{ij}x_j \leq \bar{b}_i \qquad \forall i$$
$$\sum_{j=1}^{n}(\underline{a}_{ij}+\bar{a}_{ij})x_j + \sigma\sum_{j=1}^{n}(\bar{a}_{ij}-\underline{a}_{ij})x_j \geq (\bar{b}_i+\underline{b}_i)-\sigma(\bar{b}_i-\underline{b}_i).$$

In order to solve the problem of interval linear programming (1), we noticed that the interval coefficient of the objective function, the constraints and the right hand of the constraints the uncertain return, the uncertain cost and the uncertain total resource respectively. Thus the interval linear programming problem is a problem where the objective function is to maximize the uncertain return under some uncertain constraints, where the uncertainty is described by intervals. The constraints , denote that the feasible solution to the problem is a solution such that the average costs and the costs in the worst case scenario are less than or equal to the average value and the maximal possible value of the uncertain resources. It is clear that the midpoint of an interval is the expected value of that special fuzzy variable. The interval programming problem can be viewed as a combination of the fuzzy expected value model and the pessimistic decision model. Therefore, for each interval in the objective function $\mathbf{c}_i$, a single representative $\hat{c}_i$ will be consider, and the best choice will be the midpoint of the interval, thus $\hat{c}_i = m_{\mathbf{c}_i}$, then the interval linear programming problem will be reduced into a linear programming problem as follows:

$$\begin{aligned}
\max z &= \sum_{j=1}^{n}\left(\frac{\underline{c}_j+\bar{c}_j}{2}\right)x_j \\
\text{s.t.} &\sum_{j=1}^{n} \bar{a}_{ij}x_j \leq \bar{b}_i \qquad \forall i \\
&\sum_{j=1}^{n}(\underline{a}_{ij}+\bar{a}_{ij})x_j + \sigma\sum_{j=1}^{n}(\bar{a}_{ij}-\underline{a}_{ij})x_j \\
&\geq (\bar{b}_i+\underline{b}_i)-\sigma(\bar{b}_i-\underline{b}_i). \\
&x_j \geq 0 \qquad \forall j
\end{aligned}$$

To estimate the efficiency of the proposed numerical method, we consider the following example.

Example IV.2. Consider the following interval linear programming problem:

$$\begin{aligned}
\max z &= [1,1.5]x_1 + [2.7,3]x_2 \\
\text{s.t.} \quad &[2,2.1]x_1 + [1.3,1.5]x_2 \leq [6,7] \\
&[3.2,3.3]x_1 + [4.1,4.2]x_2 \leq [8,11] \\
&x_1, x_2 \geq 0.
\end{aligned}$$

In order to solve the problem, we find the solution to the satisfactory crisp equivalent problem and chose $\sigma = 0.5$,

$$\begin{aligned}
\max z &= 1.25x_1 + 2.85x_2 \\
\text{s.t.} \quad &2.1x_1 + 1.5x_2 \leq 7 \\
&3.3x_1 + 4.2x_2 \leq 11 \\
&4.15x_1 + 2.9x_2 \geq 12.5 \\
&6.55x_1 + 8.35x_2 \geq 20.5 \\
&x_1, x_2 \geq 0.
\end{aligned}$$

The solution to the original problem is $x_1^* \simeq 2.62$, $x_2^* \simeq 0.56$ and $z^* \simeq [4.132, 5.61]$.

V. Conclusion

In this paper, we discuss a the solution of an interval linear programming problems. The solutions are based on order relation between intervals, and the solutions can be generated by solving a corresponding parametric linear programming problem. The new method introduced in this paper can find the solution of interval linear programming problems by choosing a good representative for each interval.

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