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A Graph-based Decentralization Proposal for Convex Non Linear Separable Problems with Linear Constraints

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***Abstract*—This document proposes several decentralization approaches for the Newton step graph-based model for convex non linear separable problems with linear constraints. The Newton step is well suited for this kind of problems, but when the problem size grows the NLP model will grow in a non linear manner. When this happens, the sparse matrix representation is the path to follow. Furthermore, decentralization schemes are suitable to keep the problem from growing exponentially. In this work a graph based method to achieve this decentralization is proposed. To this end, we have chosen to weak the links, which are part of the graph, as an alternative. These links eventually will guide the solution process in this approach, which implies to weak the links which are coupling the problems in order to achieve such decentralization. A deeper analysis of these links is done which leads to its complete understanding. It will be seen that the main effect of the link weakening operation is to allow the computation of the exact gradient. However, the solution will be reinforced by taking into account the second order information provided by the linking structure. Finally, different decentralisation schemes are presented based on the previous analysis.**

***Index Terms*—Non Linear Separable Programming, Multi-Agent Systems, Decentralised Systems.**

NOMENCLATURE

N	Number of decision variables.
L	Number of equality constraints.
M	Number of inequality constraints.
z_i	Decision variable i.
$\lceil z_i \rceil$	Upper limit value of variable z_i.
$\lfloor z_i \rfloor$	Lower limit value of variable z_i.
$\overline{z_i}$	Slack variable for z_i uppper bound.
$\underline{z_i}$	Slack variable for z_i lower bound.
Δz_i	Variable x_i increment.
$f(z)$	Objective function
$g_l(z)$	Equality constraint l.
$h_m(z)$	Inequality constraint m.
a_{li}	ith coefficient in equality constraint l.
b_{mi}	ith coefficient in inequality constraint m.
r_l	RHS of equality constraint l.
s_m	RHS of inequality constraint m.
λ_l	Dual variable for equality constraint l.
μ_m	Dual variable for inequality constraint m.
$\overline{\rho}_i$	Dual variable for z_i uppper bound.
$\underline{\rho}_i$	Dual variable for z_i lower bound.
$\wp(S)$	Power set of set S.
Γ_i	Set of variables connected to z_i.

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I. INTRODUCTION

Non linear separable problems (NLSP) are NLPs whose objective function can be decomposed as a sum of functions with only one variable each. The solution of NLSPs are based on very well grounded mathematical theories [1]. This document proposes a Newton step graph-based model to decentralize convex non linear separable problems (NLSP) with linear constraints. Previous works [2], [3], [4], [5] have presented a decentralised approach for NLPs applied to Electrical Power Systems, which rely on the auxiliary principle problem [6]. In this work a graph based method to achieve this decentralization is proposed. To this end, we have chosen to weak the links, which are part of the graph, as an alternative. These links eventually will guide the solution process in this approach, which implies to weak the links which are coupling the problems in order to achieve such decentralization. A deeper analysis of these links is done which leads to its complete understanding. It will be seen that the main effect of the link weakening operation is to allow the computation of the exact gradient. However, the solution will be reinforced by taking into account the second order information provided by the linking structure. This document is structured as follows. First, a graph based model for convex NLSPs with linear constraints is presented. Then, an equivalent graph representation is presented in order to simplify the graph. After this, different decentralisation schemes are presented based on the previous analysis. Finally some conclusions are withdrawn from this proposal.

II. A GRAPH-BASED MODEL FOR CONVEX NLSP

In this section a graph topology for the Newton step method is replicated just as proposed in [7] with the purpose of completeness. To this end, let us base the discussion with the NLSP described by model 1 . This NLSP consists of N variables, L equality constraints, and M inequality constraints.

$$\begin{array}{llcll} & \min\limits_{z_i} \quad \sum_{i=1}^{N} f_i(z_i) & & & \\ \textbf{st.} & g_l(z) & = & 0, & l = 1,2,...,L \\ & h_m(z) & \leq & 0, & m = 1,2,...,M \end{array} \quad (1)$$

where $f_i(z_i)$ are a non linear functions. Furthermore, we assume they have second order derivatives. On the other hand, $g_l(z)$ are linear equalities while $h_m(z)$ linear inequalities denoted as follows:

$$\begin{aligned}\sum_{l=1}^{L} a_{li} z_i &= r_l \\ \sum_{m=1}^{M} b_{mi} z_i &\leq s_m\end{aligned}$$

In general, a NLP solver starts by building the Lagrangian given by Eq. 2.

$$\mathcal{L}(z) = \sum_{i=1}^{N} f_i(z_i) + \sum_{l=1}^{L} \lambda_l g_l(z) + \sum_{m=1}^{M} \mu_m h_m(x) \qquad (2)$$

This is the base to implement the Newton step, whose formulation is given by Eq- 3 [1]:

$$H(\mathcal{L}(z))\Delta z = -\nabla(\mathcal{L}(z)) \qquad (3)$$

Table I, shows the involved elements to compute the Newton step for model 1. Do notice we have introduced slack variables as well dual variables which are used to control the bounds of the decision variables. Here, $\underline{\rho}_i$ and $\underline{z_i}$ are used for the lower bound while $\overline{\rho}_i$ and $\overline{z_i}$ are used for the upper bound of z_i

This model can be represented, as derived in [7], with the graph shown in Fig. 1.

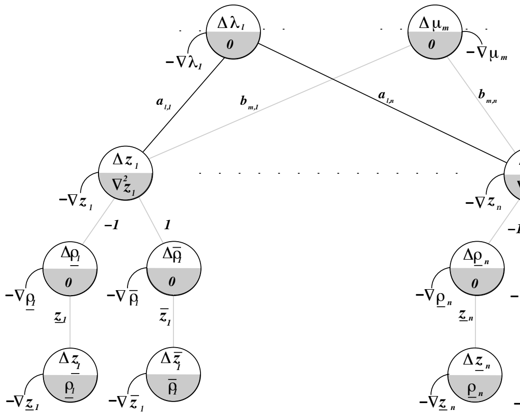

Fig. 1. Proposed topology for the Newton step method

Each constraint is represented by a dual variable and a set of links which represent the linear terms within the constraint. The terms in the constraints are represented by links which join the primal variables with the dual variables. The only difference between equality contraints and inequality constraints is the kind of links used to build the linking structure. In the case of equality constraints, the linking structure will be active along the whole solution process. On the other hand, the linking structure for an inequality constraint will be active only when such constraint is binding. This is represented by the gray color given to the links belonging to inequality constraints as opposed to the black links which belong to equality constraints. In table I, the same criterion has been imposed in the elements of $H(\mathcal{L}(z))$. As it can be noticed, it is full of empty spaces and gray color, the first are long term sparsity patterns and the second ones are temporary sparsity patterns awaiting to be exploited.

III. An Equivalent Graph Representation

In this section, once the characteristics of this graph have been analysed, a simpler graph model representation is derived in order to make its handling easier. This simplification is based on two main observations: first, the bounding structures are fixed, and second the different kind of variables can be represented in such a way that the content of that node will be inferred by its representation.

A. An Equivalent Graph Bounding Structure Representation

The subgraphs which represent the bound on the variables are well defined. Therefore a special graph notation will be derived in order to handle them, as shown in figure 2. Here all the links and nodes contained in such subgraph are embedded within the triangle. The link value is defined as follows:

$$link.value = \begin{cases} 1 & \text{if the constraint is lower binding,} \\ -1 & \text{if the constraint is upper binding,} \\ -- & \text{if is not binding.} \end{cases}$$

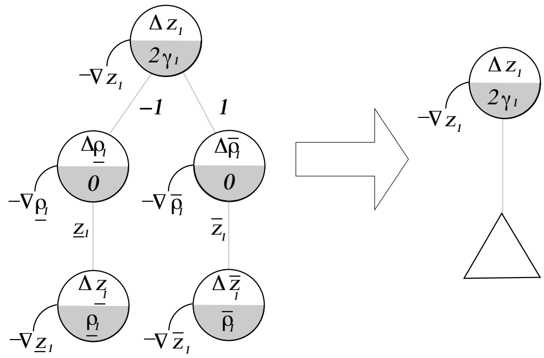

Fig. 2. Bound subgraph representation

This leads to the representation shown in figure 3.

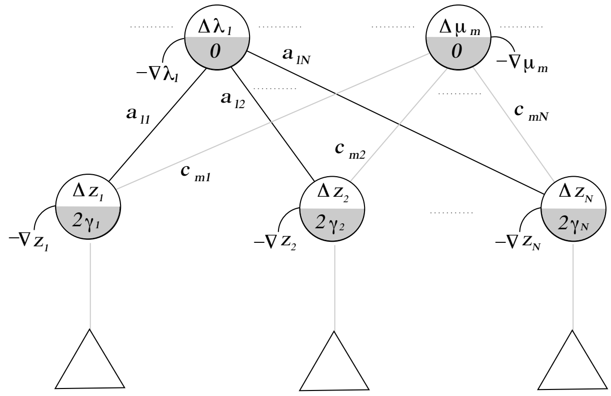

Fig. 3. Hessian topology - modified representation

B. An Equivalent Node Type Representation

In [7], it has been learnt that the gradient is embedded within the graph topology and the information attached to each node. Therefore, a simplification for the graph representation will be derived. The last section has presented a representation for the bounding structures which control the limits on the primal variables. Therefore, now the only

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$\mathcal{L}(z) = \sum_{i=1}^{N} f_i(z_i) - \sum_{l=1}^{L} \lambda_l g_l(z) - \sum_{m=1}^{M} \mu_m h_m(z)$								
	$\nabla(\mathcal{L}(z))$	$H(\mathcal{L}(z))$						
		z_i	λ_l	μ_m	$\underline{\rho}_i$	$\overline{\rho}_i$	$\underline{z_i}$	$\overline{z_i}$
z_i	$\nabla_{z_i} f_i(z_i) - \lambda_l - \mu_m - \underline{\rho}_i + \overline{\rho}_i$	$\nabla(\mathcal{L}(z))^2_{z_i}$	a_{il}	b_{im}	-1	1		
λ_l	$g_l(z)$	a_{il}						
μ_m	$h_m(z)$	b_{im}						
$\underline{\rho}_i$	$\lfloor z_i \rfloor - z_i + \underline{z_i}^2/2$	-1					$\underline{z_i}$	
$\overline{\rho}_i$	$z_i - \lceil z_i \rceil + \overline{z_i}^2/2$	1						$\overline{z_i}$
$\underline{z_i}$	$\underline{\rho}_i \underline{z_i}$				$\underline{z_i}$		$\underline{\rho}_i$	
$\overline{z_i}$	$\overline{\rho}_i \overline{z_i}$					$\overline{z_i}$		$\overline{\rho}_i$

TABLE I
THE NEWTON STEP INGREDIENTS

nodes in the remaining graph are those representing the primal variables and the dual variables. As mentioned above, the primal variables set can be further divided into two sets. The first one represents those variables which are part of the objective function. The second one contains those primal variables which appear only within the constraints. An instance of these would be the variable representing the electrical angle δ in the electric power market example [10], [11]. These two subsets will be called objective and non-objective variables respectively. Therefore, there are three kinds of variables which have to be represented within the graph e.g. Primal, Dual, Non Objective. These representations are shown in figure 4. Based on the type of variable this node is representing, the information attached to it will be known. This information is as follows

- Objective variables: Attached to this node will be the information in order to be able to compute its gradient if there were no other external information,
- Dual variables: The information attached to this kind of node will be the right hand side of the constraints which will allow its gradient computation.
- Non objective variables: To this node there will be no additional information since its coefficients in the constraints are given by the values of the links which are attached to it, i.e. no second order information is available,

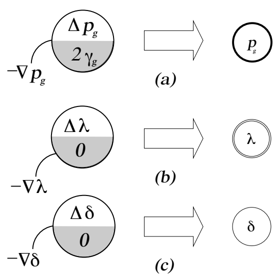

Fig. 4. Variable representation. (a) Primal variable, (b) Dual Variable, (c) Non Objective Variable

This leads to the representation shown in figure 5, where z_2 is assumed as a nonobjective variable. Based on this graph the appropriate classes for each type of variable can be defined. Once these definitions have been implemented, the operations to solve the graph can be implemented straightforward.

IV. The Graph and Its Decentralisation

In this section the graph and its decentralisation is addressed. To this end an operation over the links of the graph, called *link weakenning*, is defined. This operation is based on the expression derived in [7] as given in eq. 4

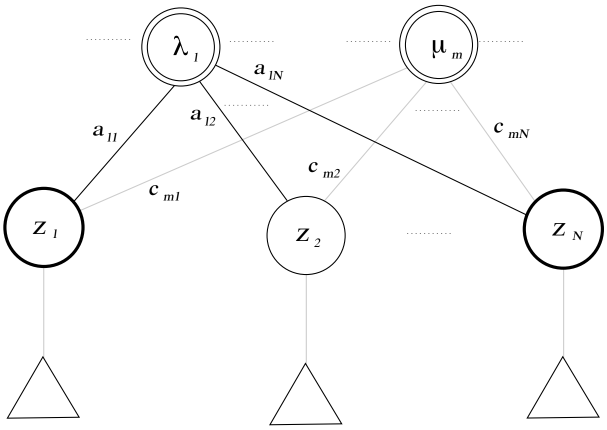

Fig. 5. Hessian topology - final equivalent representation

$$\Delta z_i = \frac{-\nabla_{z_i}\mathcal{L}(z)}{\frac{\partial^2\mathcal{L}(z)}{\partial^2 z_i}} - \sum_{\substack{\forall j \in \Gamma_i \\ (i,j)\in\mathcal{L}_k}} \frac{\frac{\partial^2\mathcal{L}(z)}{\partial z_i \partial z_j}}{\frac{\partial^2\mathcal{L}(z)}{\partial^2 z_i}} \Delta z_j \tag{4}$$

where $\mathcal{L}_k$ denotes the set of links which are taken into account for this process, do notice $|\mathcal{L}_k| = k$.

Then, three different approaches to decentralise the graph are proposed. The first one is a complete decentralised, the second approach is based on the notion of the kind of variables within the graph (i.e. primal and dual variables); and the third approach will be based on agency definitions [9]. To this end let us refer to the system given in [8] as shown in figure 6(a). The graph representation corresponding to this example is shown in figure 6(b), where the minus sign represents a -1 value for the link.

A. Link Weakening

Before going into the decentralisation approaches, let us define the link weakening operation which allows the decentralisaton process. This operation labels the links with one of the following two labels.

- HARD - This labeling will be granted to those links which are not part of the decentralisation process. The graph reduction process will take into account these links,
- SOFT - These links provide the means to decentralise the graph. If the link posseses this property, then the reduction process will not pass through them. Nevertheless, by using its connectivity, they will provide a

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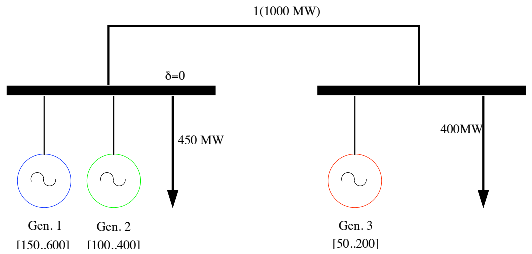
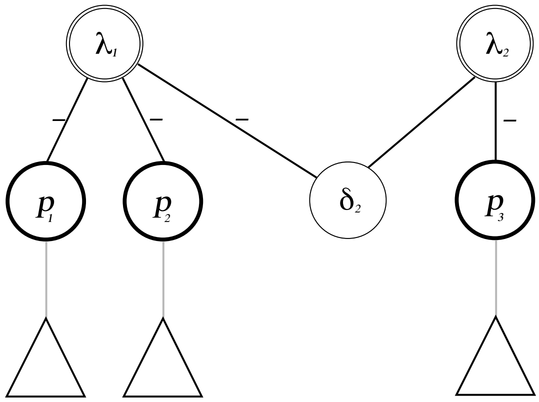


(a) Two nodes system.



(b) Its graph representation.

Fig. 6. Two nodes system and its graph representation

means to retrieve the actual value of the variable at the other end of the link which will allow the gradient to be computed, as described in [7].

B. A Gradient-oriented Approach

The first approach is to decentralise the graph in an extreme way by weakening all the links as shown in figure 7. This method leads to a model where the gradient method has to be applied at each node in the graph.

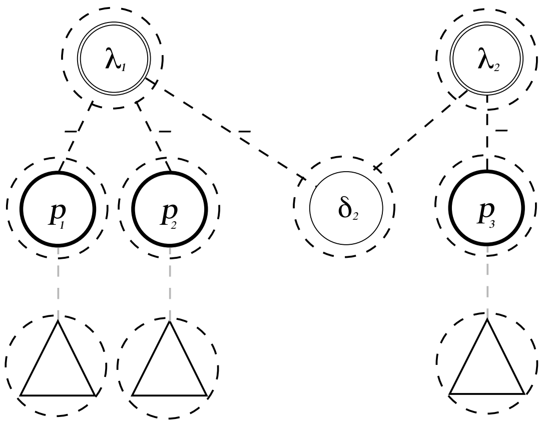


Fig. 7. An gradient-based decentralisation approach

From this figure and based on Eq. 4 we can assert it will become Eq. 5, where the second order information of all of its neighbours is disregarded. In this case $\mathcal{L}_k = \emptyset$. Therefore Eq. 4 becomes Eq. 5.

$$\Delta z_i = \frac{-\nabla_{z_i}\mathcal{L}(z)}{\frac{\partial^2\mathcal{L}(z)}{\partial^2 z_i}} \tag{5}$$

Nevertheless, those nodes which have proper second order information will be able to use it in order to speed up the convergence process. In particular, all the nodes related to primal variables have this information. Dual variables do not have second order information at all and therefore they will have to use Eq. 6

$$\Delta z_i = -\kappa \nabla_{z_i}\mathcal{L}(z) \tag{6}$$

The main drawback of gradient methods known also as steepest descent methods is the hardness to estimate κ. Do notice that for the primal nodes Eq. 7 holds.

$$\kappa = \frac{1}{\frac{\partial^2\mathcal{L}(z)}{\partial^2 z_i}} \tag{7}$$

C. A Dual-oriented Approach

The second natural approach to decentralise the graph is the dual-oriented approach. From the model proposed in section II it is known the dual variables are in only one layer so if a line across both layers is drawn dissecting the graph, the links which connected the dual variables with the primal variables will be weakened as shown in figure 8.

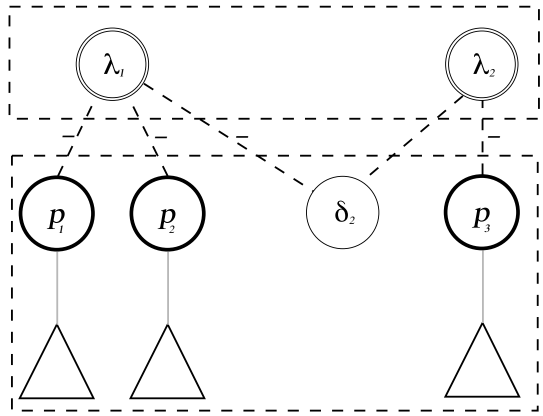


Fig. 8. An dual-oriented decentralisation approach

The only dual variables considered in this case are those related with constraints involving two or more primal variables (i.e. bound dual variables are not split from the primal variables set). Let us denote $\mathcal{D}$ as the set of those dual variables. Therefore Eq. 4 becomes Eq. 8, where all the second order information about the dual variables are disregarded by the primal variables. On the other hand, as the dual variables only have links with primal variables, they are now isolated just as in the gradient approach.

$$\Delta z_i = \frac{-\nabla_{z_i}\mathcal{L}(z)}{\frac{\partial^2\mathcal{L}(z)}{\partial^2 z_i}} - \sum_{\substack{\forall j \in \Gamma_i \\ z_j \notin \mathcal{D}}} \frac{\frac{\partial^2\mathcal{L}(z)}{\partial z_i \partial z_j}}{\frac{\partial^2\mathcal{L}(z)}{\partial^2 z_i}} \Delta z_j \tag{8}$$

D. An Agent-oriented Approach

In this approach the decentralisation process is made based on concepts drawn from the multiagent community. A classical definition for an agent is

"An agent is a computer system *situated* in an *environment*, and capable of *flexible autonomous action* in this *environment* in order to meet its *design objectives*" (adapted from [9]).

The interpretation in this work for an agent is an entity which posseses some states or variables and presents an independent and proactive behaviour, represented by their objective function. Furthermore, it is situated in an environment which he can sense and act accordingly. However, he is also constrained by its own limitations as well as the constrains presented by the environment. It is important to remark that the agents would be acting on behalf of each node. To cope with this paradigm, the graph is split into subsets of primal variables, links, and dual variables. Based on the membership of these components, the graph is decentralised as shown in figure 9. This approach was the one taken for [10], [11]

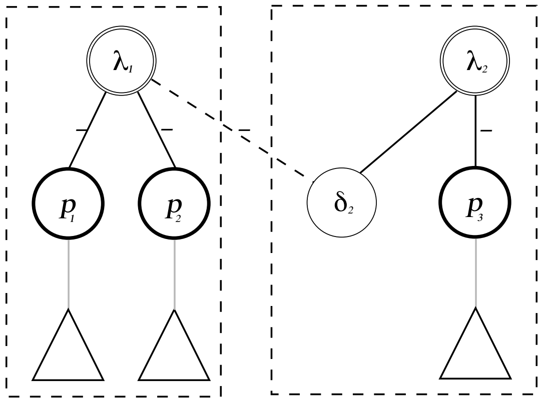


Fig. 9. An agend-based decentralisation approach

V. Concluding Remarks

This document has presented a graph-based approach to decentralize a convex NLSP with linear constraints. To this end the main concepts on how to decentralise the graph have been presented. The underlying decentralization principles have been presented based on the analysis of the equation represented by the node and its links. Finally, three decentralisation approaches have been described. The first one is a totally decentralised approach which will lead us to a gradient oriented model reinforced with its proper second order information. The second one is based on the type of variables (i.e. primal or dual), and leads us to a horizontal graph split. The last approach is based on concepts drawn from multi-agents community. Finally, even when it has been remarked that this approach is for convex problems, it can be used for non convex problems leading to local optimizers.

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