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Extrapolation of Bandlimited Signals Using Slepian Functions

Amal Devasia [1], *Member, IAENG* and Michael Cada [2]

***Abstract*— An efficient and reliable method to extrapolate bandlimited signals up to an arbitrarily high range of frequencies is proposed. The orthogonal properties of linear prolate spheroidal wave functions (PSWFs), also known as Slepian functions, are exploited to form an orthogonal basis set needed for synthesis. Higher order piecewise polynomial approximation is used for the calculation of overlap integral required for obtaining the expansion coefficients accurately with very high precision. A PSWFs set having a fixed Slepian frequency is utilized for performing extrapolation. Numerical results of extrapolation of some standard test signals using our algorithm are presented, compared, discussed, and some striking conclusions are made.**

***Index Terms*—Signal extrapolation; Slepian series; Bandlimited signals; Linear prolate spheroidal wave functions; Overlap integral**

I. Introduction

Bandlimited signal extrapolation is a well-known problem in signal analysis for which several iterative and non-iterative solutions have been proposed in the past. Recent advancements in this field are mainly based on using Slepian functions [1-4] (also known as prolate spheroidal wave functions; henceforth abbreviated as PSWFs) as an orthogonal series expansion of the bandlimited signal. Improved numerical techniques and superior computational power have aided in the numerical evaluation of these functions, which otherwise would have been an insurmountable task. Significant research has been going on especially within the past decade on Slepian functions [5-8]. In [9], Senay *et al.* proposed sampling and reconstruction of bandlimited as well as non-bandlimited signals using Slepian functions. They discussed the idea of modifying the Whittaker-Shannon sampling theory by replacing the *sinc* basis by Slepian functions for reconstruction of signals. Further to this, in [10,11], they showed signal reconstruction using non-uniform sampling and level-crossing sampling with Slepian functions.

Considering signal extrapolation to be the subject of this paper and not just reconstruction or interpolation, we will shift our focus to the recent advancements in this context. While we consider the signals to be bandlimited in the Fourier transform domain, much attention has recently been on extrapolation of signals bandlimited in linear canonical transform (LCT) domain, this being a four-parameter family of linear integral transform [12,13] that generalizes Fourier transform as one of its special cases. For extrapolation of LCT bandlimited signals, several iterative and non-iterative algorithms have been proposed [14-17]. Most of the iterative algorithms are centered on modifying the Gerchberg-Papoulis (GP) algorithm [18,19] that relies on successive reduction of error energy. Although theoretical convergence of the result has been shown, there is still some uncertainty associated with the swiftness with which this is achieved. With respect to the non-iterative algorithms proposed in [15], the authors themselves admit that the extrapolation could become unstable with an increase in the number of observations. A comparison of the extrapolation of an LCT bandlimited signal, using an iterative GP algorithm and another algorithm based on signal expansion into a series of generalized PSWFs [17] is presented in [16]. The comparison showed better results for the iterative method (proposed by [16]) over the one described in [17], in terms of the normalized mean square error (NMSE). Gosse, in [20], performed Fourier bandlimited signal extrapolation by handling lower and higher frequencies of the signal separately. He used PSWFs for extrapolating lower frequency components while the higher frequencies were dealt with compressive sampling [21,22] algorithms. The efficiency of the proposed method was highly dependent on the correlation between low and high frequencies in the signal (it should be weak for better results), the existence of a sparse representation of higher frequencies in the Fourier basis, and on a reasonable choice of extrapolation domain.

In this paper, we propose a non-iterative and simple method for bandlimited signal extrapolation valid up to an arbitrarily high range of frequencies using Slepian functions. Although we concentrate mainly on Fourier bandlimited signals, it however might also be applied for LCT bandlimited cases as is shown in one of our results below. Several comparisons are made with the results obtained in earlier related publications. They show that, within the prescribed bandwidth, the proposed method is far superior over several other methods referenced in this paper. PSWFs for analysis purposes need to be computed accurately and with rather high precision. Here, we rely on a proprietary algorithm developed theoretically and implemented numerically by Cada [23], for accurately generating the linear prolate functions (one-dimensional PSWFs, henceforth abbreviated as LPFs) set with desired high precision. Once the LPFs set is obtained with the

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[1] Amal Devasia is a MASc. candidate in the Department of Electrical and Computer Engineering, Dalhousie University, Halifax, NS, B3H 4R2, Canada (e-mail: amal.devasia@dal.ca).

[2] Michael Cada is with the Department of Electrical and Computer Engineering, Dalhousie University, Halifax, NS, B3H 4R2, Canada (e-mail: michael.cada@dal.ca).

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corresponding eigenvalues (discussed later in this paper), they are employed in our algorithm for extrapolation. Here, we do not consider the storage of LPFs set as an issue (as put forward by Shi *et al.* in [16]) to be addressed, as it is not the subject of this paper.

Cada's algorithm exploits robust properties of certain formulae derived that are efficient, accurate and suitable for fast numerical evaluations of linear prolate functions and their eigenvalues. Previous methods [Slepian, Flammer [24], etc.] required lengthy cumbersome calculations with slowly converging series and necessary approximations that led to insurmountable numerical problems and/or failing when higher orders were concerned. Prolate functions and the eigenvalues change their properties drastically at certain parameter values (see below), which has caused described problems. Even standard professional high-quality commercial packages such as Mathematica or Matlab fail to compute these functions and eigenvalues correctly or at all for such a combination of parameters that is critical for extrapolation applications. Our algorithm enables to break through this numerical barrier and makes it possible to calculate linear prolate functions and their eigenvalues correctly for basically any parameters.

Signal extrapolation is an extension of a signal, $f(t)$, beyond the interval in which it is known to the observer. The region in which the signal is known is called the observation interval (here, $[-t_0, t_0]$). Bandlimited signals are bound in the frequency domain; their Fourier transform, $F(\omega)$, vanishes beyond a particular finite frequency interval. Thus if,

$$F(\omega) = 0, \quad |\omega| > \Omega, \tag{1}$$

then $f(t)$ is said to be Ω-bandlimited.

The rest of the paper is organized as follows. In section II, relevant properties of LPFs are briefly introduced. The various steps involved in our proposed extrapolation algorithm are described throughout in section III. Section IV is devoted to presenting the actual extrapolated results of various test signals, their comparison and error analysis. Finally, in section V, we conclude with our inferences and possible future prospects.

II. Preliminaries

Bandlimited signal extrapolation using PSWFs was first discussed by Slepian and his colleagues in [1]. They explained the use of PSWFs, or more precisely, linear prolate functions (LPFs) as an orthogonal basis set for decomposition and reconstruction of the signal using analysis and synthesis equations. Linear prolate functions are one-dimensional PSWFs denoted by $\Psi_n(c,t)$, where n is the order of LPF (non-negative integer), t is the time parameter and c is the bandwidth parameter also known as Slepian frequency. LPFs can be evaluated using:

$$\Psi_n(c,t) = \sqrt{\frac{\lambda_n(c)/t_0}{\int_{-1}^{1}(S_{0n}(c,t))^2 dt}}\, S_{0n}\left(c, \frac{t}{t_0}\right), \tag{2}$$

where $\lambda_n(c)$ is the eigenvalue of *sinc* kernel with $\Psi_n(c,t)$ as eigenfunction (a measure of concentration of signal in the observation interval [-t_0, t_0]), t_0 is the observation boundary of the interval in which the function is known, and $S_{m,n}(c,\eta)$ are the angular solutions of the first kind to Helmholtz wave equation [25]. The eigenvalue $\lambda_n(c)$ is given by,

$$\lambda_n(c) = \frac{2c}{\pi}[R_{0,n}(c,1)^2], \tag{3}$$

where $R_{m,n}(c,\varepsilon)$ are the radial solutions of the first kind to Helmholtz wave equation.

Numerical evaluation of the LPFs set along with their corresponding eigenvalues practically seemed very difficult as it involved finding precise numerical values of the angular ($S_{m,n}(c,\eta)$) and radial ($R_{m,n}(c,\varepsilon)$) solutions. For obtaining this, a typical power series expansion was used which was predetermined by the association of Legendre and spherical Bessel functions to the angular and radial solutions respectively. Interested readers are referred to [7,8,26] for more details on LPFs derivation. These LPFs along with their corresponding eigenvalues have been used in our extrapolation algorithm. It should be noted that since extrapolation relies heavily on values of Ψ_n's and λ_n's when $n > 2c/\pi$, it is of paramount importance that one computes them accurately and with high precision. Ours is the first algorithm that offers such a capability.

A. *Properties of LPFs*

LPFs have many interesting properties of which some of the relevant ones related to this study are shown below.

(i) *Bandlimited*

Bandlimiting property of LPFs is denoted by a free bandwidth parameter (Slepian frequency) c given by:

$$c = \Omega_0 t_0, \tag{4}$$

where Ω_0 is the finite bandwidth of $\Psi_n(c,t)$ for a given order n.

(ii) *Symmetry*

LPFs exhibit even and odd symmetries based on their integer order n. If n is even, $\Psi_n(c,t)$ is even symmetric. If n is odd, then it is odd symmetric.

(iii) *Orthogonality*

LPFs are linearly independent and orthogonal over finite (5) as well as infinite (6) intervals, unlike, for example, trigonometric functions that are orthogonal only over a finite domain.

$$\int_{-t_0}^{t_0} \Psi_n(c,t)\,\Psi_m(c,t)\,dt = \begin{cases}\lambda_n(c) & for\ n = m,\\ 0 & otherwise.\end{cases} \tag{5}$$

$$\int_{-\infty}^{\infty} \Psi_n(c,t)\,\Psi_m(c,t)\,dt = \begin{cases}1 & for\ n = m,\\ 0 & otherwise.\end{cases} \tag{6}$$

where n, m are non-negative integers.

(iv) *Invariance to Fourier transform*

Fourier transforms of LPFs over both finite (7) and infinite (8) intervals are simply scaled versions of themselves.

$$\int_{-t_0}^{t_0} \Psi_n(c,t)e^{j\omega t}\,dt = j^n\left(\frac{2\pi\lambda_n(c)t_0}{\Omega_0}\right)^{1/2}\Psi_n\left(c, \frac{\omega t_0}{\Omega_0}\right) \tag{7}$$

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$$\int_{-\infty}^{\infty} \Psi_n(c,t)e^{j\omega t}\,dt = j^n\left(\frac{2\pi t_0}{\Omega_0}\right)^{1/2} \Psi_n\left(c, \frac{\omega t_0}{\Omega_0}\right) \qquad (8)$$

Expressions (7) and (8) show LPFs' invariance to Fourier transforms and are a further proof of their bandlimiting property.

III. Extrapolation algorithm

This section explains the various steps with which we implemented our extrapolation algorithm.

A. Analysis and synthesis

In general, any bandlimited signal can be decomposed into a linear combination of weighted orthogonal basis functions using the relation:

$$f(t) = \sum_{n=-\infty}^{\infty} \gamma_n\, \Phi_n(t) \qquad \text{(synthesis)}, \qquad (9)$$

where $f(t)$ is the signal, γ_n is a set of scalar coefficients, and $\Phi_n(t)$ is the orthogonal basis set.
The set of scalar coefficients is found from:

$$\gamma_n = \frac{\int_{-\infty}^{\infty} f(t)\Phi_n(t)dt}{\int_{-\infty}^{\infty} \Phi_n(t)\Phi_n(t)dt} \qquad \text{(analysis)}. \qquad (10)$$

Employing LPFs as the orthogonal basis set for a fixed Slepian frequency c yields:

$$\gamma_n(c) = \lambda_n^{-1}(c) \int_{-t_0}^{t_0} f(t)\Psi_n(c,t)dt \quad \text{(analysis)}, \ (11)$$

where $\int_{-t_0}^{t_0} f(t)\,\Psi_n(c,t)\,dt$ is known as the overlap integral, and $\gamma_n(c)$ are the scalar expansion coefficient for a given order n of LPF.

The synthesis equation, used for signal extrapolation, is given by:

$$f(t) \cong \sum_{n=0}^{N} \gamma_n(c)\ \Psi_n(c,t) \qquad \text{(synthesis)}, \ (12)$$

where N is the truncation value for the order n.
The analysis and synthesis equations described in (11) and (12) respectively become the basis of our signal extrapolation algorithm.

B. LPFs set

A LPFs set with a fixed Slepian frequency c was used as a potential orthogonal basis set along with its corresponding eigenvalues for the proposed algorithm. These functions were discretized in time for numerical implementation. Each of these sampled data has very high numerical precision of about 200 digits. The specifics are as follows:

($\Psi_n(c,t)$ *Set)*

$$c = 20\pi, n{:}\, 0 \rightarrow 101,\ t \rightarrow [-1.900, 1.900].$$

The time parameter t is specified with 3 digits of precision after the decimal point as the sampling period in the time axis is $0.001s$. Generally, for any LPFs set with a fixed c, as the order n increases, the concentration of the LPFs within $[-t_0, t_0]$ decreases. For $n = 2c/\pi$, the signal's maximum concentration reaches the boundary of the observation interval (see Fig. 1).

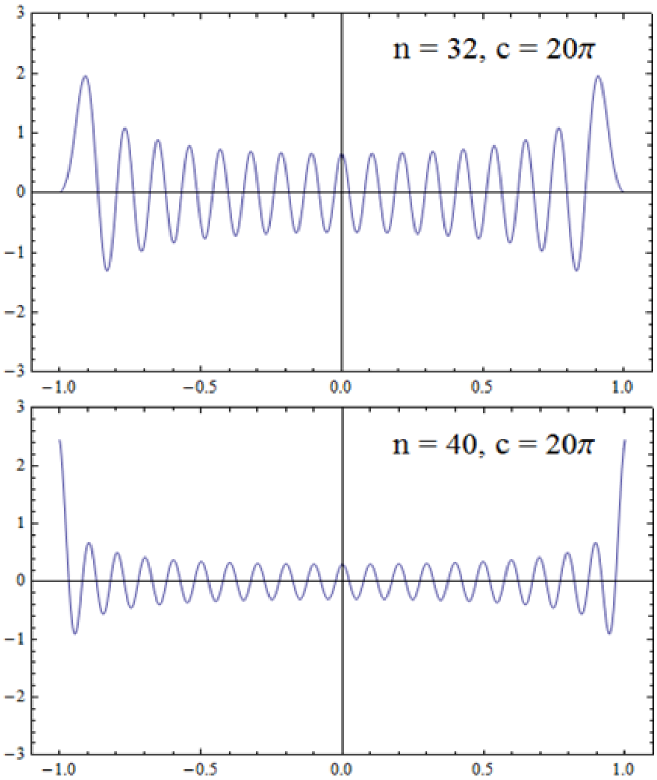

Fig. 1. Dependency of signal concentration on n for LPFs set with $c = 20\pi$.

C. Overlap integral

One can notice from the analysis (11) and synthesis (12) equations using LPFs that $f(t)$, the function to be extrapolated, is well defined in $[-t_0, t_0]$, i.e. $[-1,1]$ (t_0 chosen to be 1). Numerical values of $\lambda_n(c)$ and $\Psi_n(c,t)$ as a set for a given c are also known (see section II). The only unknown factor is an efficient method to calculate the overlap integral given by $\int_{-1}^{1} f(t)\,\Psi_n(c,t)\,dt$. Efficient estimation of overlap integral is of paramount importance in obtaining accurate results for extrapolation. As LPFs, $\Psi_n(c,t)$, and eigenvalues, $\lambda_n(c)$, are required to be of high precision for high orders of n, the eigenvalues tend to become extremely small, close to zero, which makes this essentially a problem of high precision numerical integration. If and when computed inaccurately, the coefficients of expansion, $\gamma_n(c)$, of the synthesis equation (12) assume extremely large values for such n's that are crucial and irreplaceable for extrapolation purposes. This, in turn, causes enormous numerical errors that thus render extrapolated signals completely incorrect and unusable.

D. Calculation of overlap integral

A simple and efficient way to compute overlap integral is proposed and implemented to obtain satisfactory results. To make it simple and generic, polynomial approximation of the discrete samples of the scalar product $f(t)\,\Psi_n(c,t)$ was chosen. Our primary focus was to obtain the right polynomial approximation for the underlying common function, $\Psi_n(c,t)$ (for a given c), in any scalar product, irrespective of the bandlimited function $f(t)$. Another major task was to choose the right truncation value N for synthesis

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(12), which we determined by examining the behavior of the scalar coefficients $\gamma_n(c)$ obtained in analysis (11). After a series of thorough investigations using different kinds of polynomial interpolation and numerical integration techniques, it was found that piecewise polynomial approximation is best suited for the particular LPF set that was used.

Direct method of piecewise polynomial interpolation [27] is used. Given i discrete data points, (x_0, y_0) to (x_{i-1}, y_{i-1}), it can be approximated to a polynomial of order $i-1$ as:

$$y = \beta_0 + \beta_1 x + \cdots + \beta_{i-1} x^{i-1}, \tag{13}$$

where the coefficients (β) can be found by solving this linear system of equations:

$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{i-1} \end{pmatrix} = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{i-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{i-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{i-1} & x_{i-1}^2 & \cdots & x_{i-1}^{i-1} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{i-1} \end{pmatrix}. \tag{14}$$

Applying this technique to our problem, there are 1001 samples in the time interval $[0,1]$ and 2001 samples in $[-1,1]$. The whole interval of $[-1,1]$ is divided into 8 equal segments, thus there are 251 samples in each segment of the samples of the scalar product $f(t)\,\Psi_n(c,t)$. Applying piecewise polynomial approximation to each segment containing 251 samples, one obtains a polynomial of order 250 for each segment. The desired overlap integral is finally obtained by directly integrating each segment.

IV. Results and Discussion

For implementing our algorithm we used Mathematica, a software tool that is excellent for high precision computing. Extrapolation was carried out for some selected known test functions using the aforementioned LPFs set. The only restriction imposed on the selected signals was that its maximum frequency should be less than or equal to that of the corresponding LPFs set used for their extrapolation. In the results shown below, the same extrapolation formula was employed for both reconstructing the signal within as well as extrapolating it beyond the interval $[-1,1]$. To show the error estimates with respect to the original signal, common logarithm of the absolute error between the extrapolated and original data is also plotted. Extrapolation errors of the order of up to 10^{-2} or 10^{-3} are considered acceptable.

The range of extrapolation is limited mainly due to the series truncation, N, the value of which should be at least greater than $2c/\pi$ for performing extrapolation. We also verified the effective extrapolation range that can be achieved analytically by using simple sinusoidal signals, for which the analytical expressions are known [7]. We found that for the particular LPFs set used, the truncation value N was more or less equal to 100 (out of the total order of $n = 101$ being considered), which is much greater than $2c/\pi$ (with $c = 20\pi$, this equals 40 < 100), thus allowing signal extrapolation.

The signals for which the extrapolation is shown in Fig. 2 and 3, although not exactly bandlimited because of the Gaussian functions involved, also gave good results (as seen in Fig. 3), which are comparable to what was achieved using strictly bandlimited case (Fig. 4).

To find the effectiveness of our algorithm, the results obtained were also compared with those obtained in earlier publications. In [20], Gosse used the same test signal given here as signal 1 (see below) for extrapolation. He considered it as a composite signal composed of low (first term) and high (second term) frequencies, and extrapolated the lower frequency part using a truncated prolate series expansion, while the higher frequency part was handled by compressive sampling techniques. The effective extrapolation range in our case (for signal 1) is significantly improved when compared with their results (see [20], page 1277; and Fig. 2 of this paper). In the same context, the error analysis also shows better results as our method has absolute error magnitude around 10^{-38} (while it is 10^{-2} in [20]) within the reconstruction interval $[-1,1]$. We also obtained better ratios of error magnitudes (varying smoothly from the order of 10^{-36} to 10^{-3} using our algorithm, while oscillating between 10^{-2} and 10^{-1} in [20]) in the effective extrapolation region, i.e. outside $[-1,1]$ (see absolute error plots of Fig. 2).

1) $f(t) = e^{-2t^2} e^{3t} \sin(\pi t)\cos(3\pi t) + 0.5[\sin(5\pi t) - \cos(7\pi t)]$

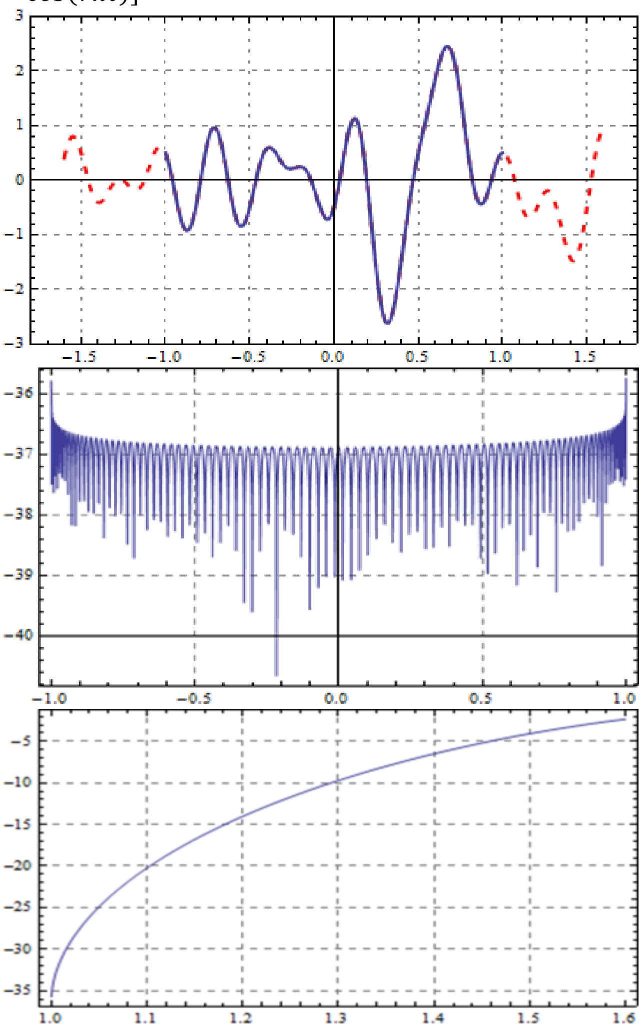

Fig. 2. $f(t)$ original [solid] and extrapolated [dashed] (top), logarithm of absolute error (bottom two) versus time; LPFs set with $c = 20\pi$ used.

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2) $f(t) = 3\,sin\left(17\pi t + \left(\frac{\pi}{5}\right)\right) - 5\,cos\left(13\pi t - \left(\frac{3\pi}{8}\right)\right) - 2e^{-\pi(t+7)^2} + 9e^{-\pi t^2}$

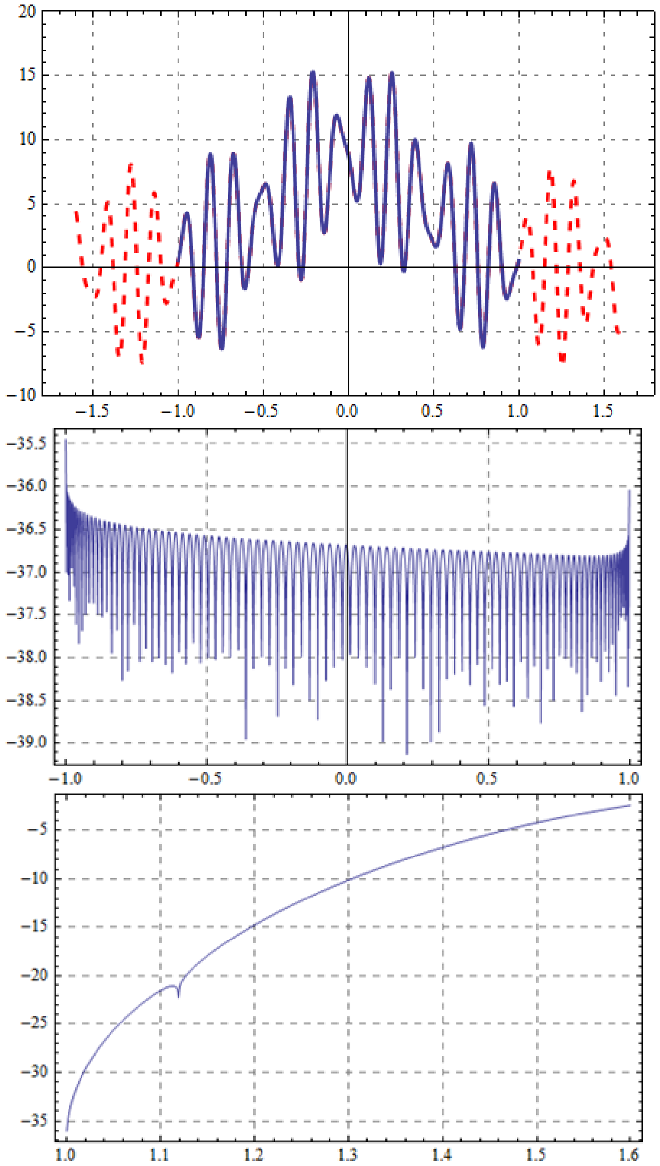
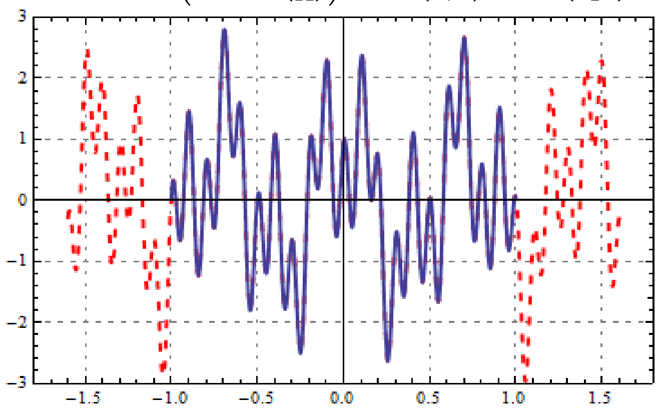


Fig. 3. $f(t)$ original [solid] and extrapolated [dashed] (top), logarithm of absolute error (bottom two) versus time; LPFs set with $c = 20\pi$ used.

3) $f(t) = cos\left(20\pi t - \left(\frac{\pi}{11}\right)\right) + cos\left(\frac{20\pi t}{7}\right) - cos\left(\frac{15\pi t}{2}\right)$



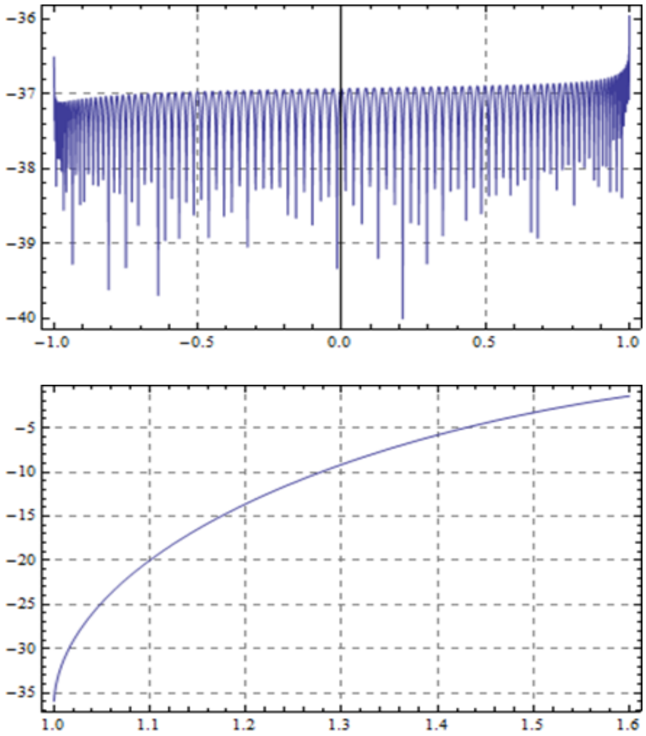


Fig. 4. $f(t)$ original [solid] and extrapolated [dashed] (top), logarithm of absolute error (bottom two) versus time; LPFs set with $c = 20\pi$ used.

4) $f(t) = \frac{20}{\pi} e^{j2t^2}\, sinc\left(\frac{20t}{\pi}\right)$

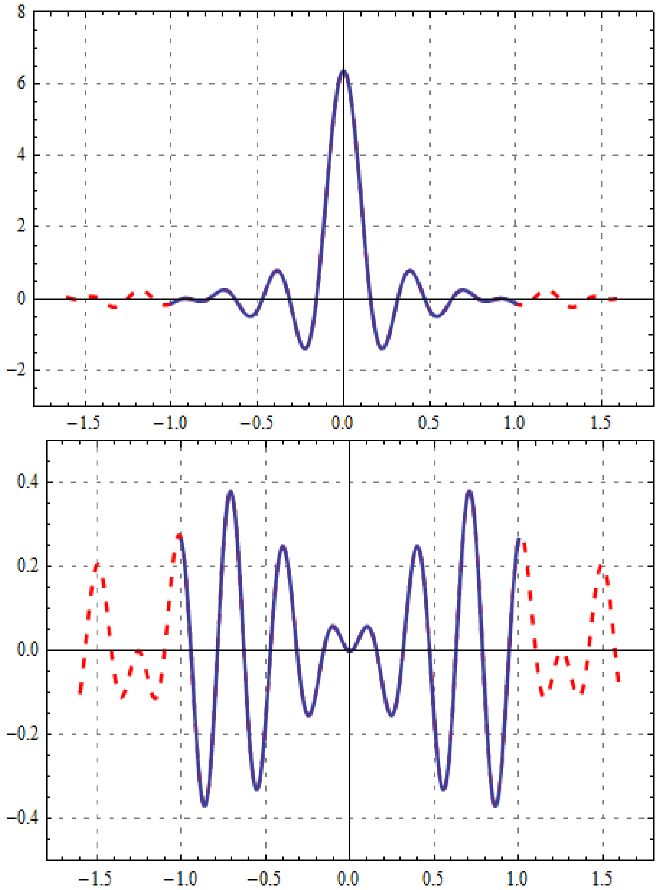


Fig. 5. Real part of $f(t)$ original [solid] and extrapolated [dashed] (top), imaginary part of $f(t)$ original [solid] and extrapolated [dashed] (bottom) versus time; LPFs set with $c = 20\pi$ used.

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As mentioned earlier, we also performed extrapolation on an LCT bandlimited signal to compare our method with existing ones. We chose the same signal used by Shi *et al.* in [16]. It is given as signal 4 (the *sinc* function used is a normalized *sinc* function) in the results. The extrapolation of the real and imaginary parts of the signal is shown in Fig. 5. The performance was measured using an error norm known as the normalized mean-square error (NMSE); following their [16] notation it is defined as:

$$NMSE = \frac{\|f_e - f\|^2}{\|f\|^2}, \tag{15}$$

where f_e is the extrapolated signal and f is the original signal. The NMSE of the extrapolated signal using our algorithm is 8.430 x 10^{-7}. The corresponding NMSE using the iterative algorithm proposed in [16] is 1.037 x 10^{-4}, and NMSE using the generalized PSWFs expansion method proposed in [17] is only 0.685. Thus our algorithm is shown performs superiorly even for LCT bandlimited signals, albeit it is not the fundamental goal of this paper.

V. Conclusion

We have presented an implemented robust and efficient algorithm for bandlimited signal extrapolation valid up to an arbitrarily high range of frequencies. We also looked into some background theory essential for establishing our results, as well as shared our perspectives while interpreting some observations made from the results. Even though the algorithm is complex in the sense that it involves time consuming calculations and tedious computations with big matrices of very high precision, the overall idea is simple and easy to execute, thanks to the current computational speeds and available system memory. We believe that the accuracy with which the LPFs were computed with very high precision allowed this method to work efficiently thus making it suitable for extrapolating signals within the prescribed bandwidth. Characterizing the Slepian functions (LPFs set) finely and precisely into an appropriate polynomial expression is the key with which, this method could be extended to other LPF sets. This is a promising development in the field of signal processing [14-17,28,29] and will be helpful in the characterization of both known and random bandlimited observations.

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