Homotopy Perturbation Method for Solving Some Initial Boundary Value Problems with Non Local Conditions

A. Cheniguel and M. Reghioua

Abstract— **In this paper, initial boundary value problems with non local boundary conditions are presented. The homotopy perturbation method (HPM) is used for solving linear and non linear initial boundary value problems with non classical conditions. The obtained results as compared with previous works are highly accurate. Also HPM provides continuous solution in contrast to finite difference method, which only provides discrete approximations. It is found that this method is a powerful mathematical tool and can be applied to a large class of linear and nonlinear problem in different fields of science and technology**

Index Terms— **Homotopy perturbation method (HPM), Partial differential equations, Initial boundary value problems,**

I. Introduction

Recently, much attention has been to partial differential equations with non local boundary conditions, this attention was driven by the needs from applications both in industry and sciences. Theory and numerical methods for solving initial boundary value problems with nonlocal conditions were investigated by many researchers see [1-10, 12-14,16-18,22-27] and the reference therein. In the last decade, there has been a growing interest in the analytical new techniques for linear and nonlinear initial boundary value problems with non classical boundary conditions. The widely applied techniques are perturbation methods. J.He [20] has proposed a new perturbation technique coupled with the homotopy technique, which is called the homotopy perturbation method (HPM). In contrast to the traditional perturbation methods. a homotopy is constructed with an embedding parameter π 2 [0; 1], which is considered as a small parameter. HPM has gained reputation as being a powerful tool for solving linear or nonlinear partial differential equations. This method has been the subject of intense investigation during recent years and many researchers have used it in their works involving differential equations see in [11,15]. He [19], applied HPM to solve initial boundary value problems which is governed by the nonlinear ordinary (Partial) differential equations, the results show that this method is efficient and simple. Thus, the main goal of this work is to apply the homotopy perturbation method (HPM) for solving linear and nonlinear initial boundary value problems with nonlocal boundary conditions. The general form of equation is given as:

$$\frac{\partial u}{\partial t} = G(x,t,u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2})\; a < x < b, 0 < t \le T \qquad (1)$$

Subject to the initial condition:

$$u(x,0) = f(x), 0 \le t \le T \qquad (2)$$

And the non local boundary conditions

$$u(a,t) = \int_a^b \varphi(x,t)u(x,t)dx + g_0(t),\; 0 < t \le T \qquad (3)$$
$$u(b,t) = \int_a^b \psi(x,t)u(x,t)dx + g_1(t),\; 0 < t \le T \qquad (4)$$

Where $f, g_0, g_2, \varphi, \psi$ are sufficiently smooth known functions and T is a given constant.

II. Analysis of Homotopy Perturbation Method

To illustrate the basic ideas, let Ξ; and Ψ be the topological spaces. If ϕ and γ are continuous maps of the spaces Ξ into Ψ , it is said that ϕ is homotopic to γ;if there is continuous map $F: X \times [0,1] \longrightarrow Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for each ξ 2 Ξ, then the map is called homotopy between ϕ and γ.
We consider the following nonlinear partial differential equation:

$$A(u) - f(r) = 0, r\;\; \Omega \qquad (5)$$

Subject to the boundary conditions

$$B\left(u,\frac{\partial u}{\partial \eta}\right) = 0, r\; \in\; \Gamma \qquad (6)$$

Where A is a general differential operator. ϕ is a known analytic function, Γ is the boundary of the domain Ω and $\frac{\partial}{\partial \eta}$ denotes directional derivative in outward normal direction to Ω. The operator A, generally divided into two parts, Λ and N, where Λ is linear, while N is nonlinear. Using A =Λ +N, eq. (5) can be rewritten as follows:

$$\Lambda(\upsilon)+N(\upsilon)-\phi(\rho)=0$$
(7)

By the homotopy technique, we construct a homotopy defined as

$$H(v,p): \Omega \times [0,1] \longrightarrow R \qquad (8)$$

Which satisfies:

$$H(v,p) = (1-p)\big(L(v) - L(u_0)\big) + p\big(A(v) - f(r)\big), p \in [0,1], r\; \in \Omega \qquad (9)$$

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Or
$H(v,p) = L(v) - L(u_0) + pL(u_0) + p\big(N(v) - f(r)\big) = 0, p \in [0,1], r \in \Omega$
(10)
Where π 2 [0; 1] is an embedding parameter , υ_0 is an initial approximation of equation (5), which satisfies the boundary conditions. It follows from the equation (10) that
$H(v,0) = L(v) - L(u_0) = 0$ (11)
$H(v,1) = A(v) - f(r) = 0$ (12)
The changing process of π from 0 to 1 monotonically is a trivial problem. $H(v,0) = L(v) - L(u_0) = 0$ is continuously transformed to the original problem $H(v,1) = A(v) - f(r) = 0$. (13)
In topology, this process is known as continuous deformation. $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic. We use the embedding parameter π as a small parameter , and assume that the solution of equation (10) can be written as a power series of π :
$v = p^0 v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \cdots + p^n v_N + \cdots$ (14)
Setting π =1 we obtain the approximate solution of equation (5) as:
$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots + v_n + \cdots$ (15)
The series of equation (15) is convergent for most of the cases, but the rate of the convergence depends on the nonlinear operator N(υ). He (1999) has suggested that:

- The second derivative of N(υ) with respect to υ should be small because the parameter may be relatively large i.e π ! 1and the norm of $L^{-1}(\frac{\partial N}{\partial u})$ must be smaller than one in order for the series to converge.

III. EXAMPLES

A. Example 1

We consider the problem

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (4t^3 + 12t^2 - 4x^3 - 12x^2) \quad (16)$$
$0 < x < 1, 0 < t < T$
With the initial condition:
$$u(x,0) = x^4, \frac{\partial u}{\partial t}(x,0) = 0,\ 0 < x < 1, 0 < t < T \quad (17)$$
And the boundary conditions:
$$u(0,t) = \int_0^1 \varphi(x,t)u(x,t)dt + g_0(t) = 1 + \tfrac{1}{5}t^4 \quad (18)$$
Where $\varphi(x,t) = \frac{1}{5}$ and $g_0(t) = \frac{24}{25}$
$$u(1,t) = \int_0^1 \varphi(x,t)u(x,t)dt + g_1(t) = 1 + \tfrac{1}{6}t^4 \quad (19)$$
Where $\varphi(x,t) = \frac{1}{6}$ and $g_1(t) = \frac{29}{30}$
For solving this problem, we construct HPM as follows:
$$H(v,p) = (1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial t^2} - \frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2} - (4t^3 + 12t^3 - 4x^3 - 12x^2)\right) = 0 \quad (20)$$
The component v_i of (15) are obtained as follows:
$$\frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, v_0 = u(x,0) = x^4 \quad (21)$$

$$\frac{\partial v_1}{\partial t} + \frac{\partial^2 v_0}{\partial t^2} - \frac{\partial v_0}{\partial x} - \frac{\partial^2 v_0}{\partial x^2} - (4t^3 + 12t^2 - 4x^3 - 12x^2) = 0,$$
$$v_1(x,0) = 0 \quad (22)$$

$$\frac{\partial v_0}{\partial x} = 4x^3, \frac{\partial^2 v_0}{\partial x^2} = 12x^2, \frac{\partial^2 v_0}{\partial t^2} = 0$$

$$\frac{\partial v_1}{\partial t} = 4t^3 + 12t^2$$
Hence
$$v_1 = t^4 + 4t^3 \quad (23)$$

$$\frac{\partial v_2}{\partial t} + \frac{\partial^2 v_1}{\partial t^2} - \frac{\partial v_1}{\partial x} - \frac{\partial^2 v_1}{\partial x^2} = 0, v_2(x,0) = 0 \quad (24)$$

$$\frac{\partial^2 v_1}{\partial t^2} = 12t^2 + 24t, \frac{\partial v_1}{\partial x} = \frac{\partial^2 v_0}{\partial x^2} = 0$$
$$\frac{\partial v_2}{\partial t} = -12t^2 - 24t$$

Then, we have
$$v_2 = -4t^3 - 12t^2 \quad (25)$$

For the next component:
$$\frac{\partial v_3}{\partial t} + \frac{\partial^2 v_2}{\partial t^2} - \frac{\partial v_2}{\partial x} - \frac{\partial^2 v_2}{\partial x^2} = 0, v_3(x,0) = 0$$

$$v_3 = 12t^2 + 24t, v_4 = -24t, \quad (26)$$
And so on, we obtain the approximate solution as follows:
$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots + v_n + \cdots$$
And this leads to the following solution
$$u(x,t) = x^4 + t^4 \quad (27)$$
We can, immediately observe that this solution is exact.

B. Example 2

Consider the following nonlinear reaction-diffusion equation:
$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = u^2 - \left(\frac{\partial u}{\partial x}\right)^2 \quad 0 < x < 1, 0 < t < T \quad (28)$$

Subject to the initial condition
$$u(x,0) = e^x \quad 0 < x < 1, \quad (29)$$
And the boundary conditions:
$$u(0,t) = \int_0^1 \varphi(x,t)u(x,t)dt + g_0(t) = e^{1+t} \quad (30)$$
With $\varphi(x,t) = 1$ and $g_0(t) = e^t$
$$u(1,t) = \int_0^1 \psi(x,t)u(x,t)dt + g_1(t) = \tfrac{1}{2}e^{1+t} \quad (31)$$
With $\psi(x,t) = \frac{1}{2}$ and $g_1(t) = \frac{1}{2}e^t$
Solving the equation (28) with the initial condition (29), yields:
$$\frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, v_0 = u_0 = e^x$$

$$\frac{\partial v_1}{\partial t} - \left(\frac{\partial v_0}{\partial x}\right)^2 - \frac{\partial^2 v_0}{\partial x^2} - v_0^2 = 0, v_1 = te^x, v_1(x,0) = 0$$

$$\frac{\partial v_2}{\partial t} - \left(\frac{\partial v_1}{\partial x}\right)^2 - \frac{\partial^2 v_1}{\partial x^2} - v_1^2 = 0, v_2 = \frac{t^2}{2!}e^x,$$
And we can deduce the remaining components as:

$$v_3 = \frac{t^3}{3!}e^x, \ldots, v_n = \frac{t^n}{n!}e^x,, \quad (32)$$

Using equation we get :

$$u(x,t) = e^x\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!} + \cdots\right)$$

And finally the approximate solution is obtained as :
$$u(x,t) = e^{x+t} \quad (33)$$

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C. Example 3

Consider the problem

$\frac{\partial u}{\partial t}=\frac{1}{6}\left(x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}\right)$ $0<x,y,z<1, 0<t<T$
(34)

Subject to the initial condition:

$u(x,y,z,0)=x^2y^2z^2$ (35)

And the boundary conditions

$u(0,y,z,t)=\int_0^1\int_0^1\int_0^1 u(,x,y,z,t)dxdydz+g_1=\frac{1}{27}e^t, g_1=0$

$u(1,y,z,t)=\int_0^1\int_0^1\int_0^1 u(,x,y,z,t)dxdydz+g_2=\frac{1}{27}e^t+\frac{1}{2}t,\ \ g_2=\frac{1}{2}t$

$u(x,0,z,t)=\int_0^1\int_0^1\int_0^1 u(,x,y,z,t)dxdydz+g_3=\frac{1}{27}(e^t+1),$
$g_3=\frac{1}{27}$

$u(x,1,z,t)=\int_0^1\int_0^1\int_0^1 u(,x,y,z,t)dxdydz+g_4=\frac{1}{27}(e^t+3),$
$g_4=\frac{1}{9}$

$u(x,y,0,t)=\int_0^1\int_0^1\int_0^1 u(,x,y,z,t)dxdydz+g_5=\frac{1}{27}e^t+\frac{1}{6},$
$g_5=\frac{1}{6}$

$u(x,y,1,t)=\int_0^1\int_0^1\int_0^1 u(,x,y,z,t)dxdydz+g_6=\frac{1}{27}e^t+\frac{1}{5}t,$
$g_1=\frac{1}{5}t,$ (36)

As above, we get the components of (15):

$\frac{\partial v_0}{\partial t}-\frac{\partial u_0}{\partial t}=0, v_0=x^2y^2z^2$ (37)

$$\frac{\partial v_1}{\partial t}-\frac{1}{6}\left(x^2\frac{\partial^2 v_0}{\partial x^2}+y^2\frac{\partial^2 v_0}{\partial y^2}+z^2\frac{\partial^2 v_0}{\partial z^2}\right)=0, v_1(x,0)=0$$

$$\frac{\partial v_1}{\partial t}=\frac{1}{6}(2x^2y^2z^2+2x^2y^2z^2+2x^2y^2z^2)=x^2y^2z^2$$

$$v_1=x^2y^2z^2\frac{t}{1!}$$

$$\frac{\partial v_2}{\partial t}-\frac{1}{6}\left(x^2\frac{\partial^2 v_1}{\partial x^2}+y^2\frac{\partial^2 v_1}{\partial y^2}+z^2\frac{\partial^2 v_1}{\partial z^2}\right)=0, v_2(x,0)=0$$

$$\frac{\partial v_2}{\partial t}=\frac{1}{6}(2x^2y^2z^2+2x^2y^2z^2+2x^2y^2z^2)t=x^2y^2z^2t$$

$$v_2=x^2y^2z^2\frac{t^2}{2!}$$

$$\frac{\partial v_3}{\partial t}-\frac{1}{6}\left(x^2\frac{\partial^2 v_2}{\partial x^2}+y^2\frac{\partial^2 v_2}{\partial y^2}+z^2\frac{\partial^2 v_2}{\partial z^2}\right)=0, v_3(x,0)=0$$

$$\frac{\partial v_3}{\partial t}=\frac{1}{6}(2x^2y^2z^2+2x^2y^2z^2+2x^2y^2z^2)\frac{t^2}{2!}$$
$$=x^2y^2z^2\frac{t^2}{2!}$$

$$v_3=x^2y^2z^2\frac{t^3}{3!}$$

And we deduce the general form of v_n as follows :

$$\frac{\partial v_n}{\partial t}-\frac{1}{6}\left(x^2\frac{\partial^2 v_{n-1}}{\partial x^2}+y^2\frac{\partial^2 v_{n-1}}{\partial y^2}+z^2\frac{\partial^2 v_{n-1}}{\partial z^2}\right)$$
$$=0, v_n(x,0)=0$$

$$\frac{\partial v_n}{\partial t}=\frac{1}{6}(2x^2y^2z^2+2x^2y^2z^2+2x^2y^2z^2)\frac{t^{n-1}}{(n-1)!}$$
$$=x^2y^2z^2\frac{t^{n-1}}{(n-1)!}$$

$v_n=x^2y^2z^2\frac{t^n}{n!}$ (38)

Hence, the approximate solution is given by:

$u(x,y,z,t)=\lim_{p\to 1} v=v_0+v_1+v_2+\cdots+v_n+\cdots$

Now, the solution of (34) when π ! 1 reduces to :

$$u(x,y,z,t)=x^2y^2z^2\left(1+\frac{t}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots+\frac{t^n}{n!}+\cdots\right)$$

And the solution in a closed form is given by:

$u(x,y,z,t)=x^2y^2z^2e^t$ (39)

D. Example 4

As a last example, consider the following problem:

$u_{tt}=(u^{-1}u_x)_x$ $0<x,y,z<1, 0<t<T$ (40)

With the initial condition

$u(x,0)=\frac{1}{(1+x)^2}, u_t(x,0)$=0, (41)

And the boundary conditions:

$u(0,t)=\int_0^1\varphi(x,t)u(x,t)dt+g_0(t)=1+0.5t,$

With $\varphi(x,t)=1$ and $g_0(t)=0.5$

$u(1,t)=\int_0^1\psi(x,t)u(x,t)dt+g_1(t)=1+0.125t$

With $\psi(x,t)=0.25$ and $g_1(t)=0.875$

According to the HPM, we have:

$H(v,p)=(1-p)\left(\frac{\partial^2 v}{\partial t^2}-\frac{\partial^2 u_0}{\partial t^2}\right)+p\left(\frac{\partial^2 v}{\partial t^2}-\frac{\partial}{\partial x}\left(\left(v^{-1}\frac{\partial v}{\partial x}\right)\right)\right)=$
0 (20)

By equating the terms with the identical powers of π , yields

$p^0:\ \frac{\partial^2 v_0}{\partial t^2}-\frac{\partial^2 u_0}{\partial t^2}=0, \frac{\partial^2 v_0}{\partial t^2}=0, v_0=\frac{1}{(1+x)^2}$ (43)

$$p^1:\ \frac{\partial^2 v_1}{\partial t^2}-\frac{\partial}{\partial x}\left(\left({v_0}^{-1}\frac{\partial v_0}{\partial x}\right)\right)=0, v_1(x,0)\ =0,$$

$$\frac{\partial^2 v_1}{\partial t^2}=\frac{2}{(1+x)^2}$$

$$v_1=\frac{2t^2}{2!\,(1+x)^2}$$

$$p^2:\ \frac{\partial^2 v_2}{\partial t^2}-\frac{\partial}{\partial x}\left(\left({v_1}^{-1}\frac{\partial v_1}{\partial x}\right)\right)=0, v_2(x,0)\ =0,$$

$$\frac{\partial^2 v_2}{\partial t^2}=\frac{2}{(1+x)^2}\qquad\Rightarrow v_2=\frac{2t^2}{2!\,(1+x)^2}$$

$v_2=v_1$

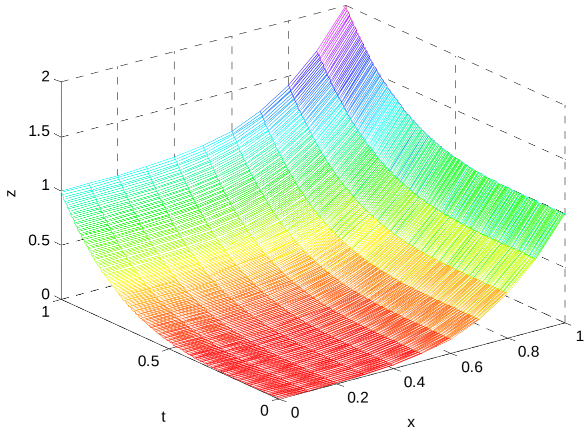
We then obtain the exact solution:

$u(x,t)=\frac{1+t^2}{(1+x)^2}$ (44)

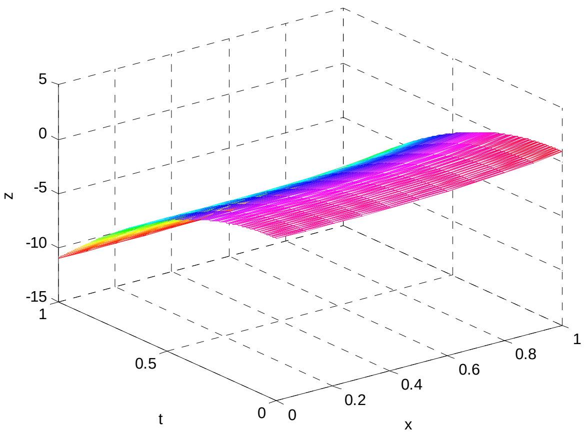
Table 1 Example 1

$h_x = \frac{1}{10}, h_t = \frac{1}{250}$,

x_i	u_{ex}	u_{hpm} *3-iterates*	$\lvert u_{ex} - u_{hpm} \rvert$
0.0	2.56×10^{-2}	-1.92000×10^{-3}	0.0224
0.1	0.0001	-9.2×10^{-5}	8.0×10^{-6}
0.2	0.0016	1.584×10^{-3}	1.6×10^{-5}
0.3	0.0081	7.908×10^{-3}	1.92×10^{-4}
0.4	0.0256	2.5408×10^{-2}	1.92×10^{-4}
0.5	0.0625	6.2308×10^{-2}	1.92×10^{-4}
0.6	0.1296	0.12941	0.00019
0.7	0.2401	0.23991	0.00019
0.8	0.4096	0.40941	0.00019
0.9	0.6561	0.65591	0.00019
1.0	1.0	0.99981	0.00019



Variation of $u_{ex} = x^4 + t^4$ for different values of x and t

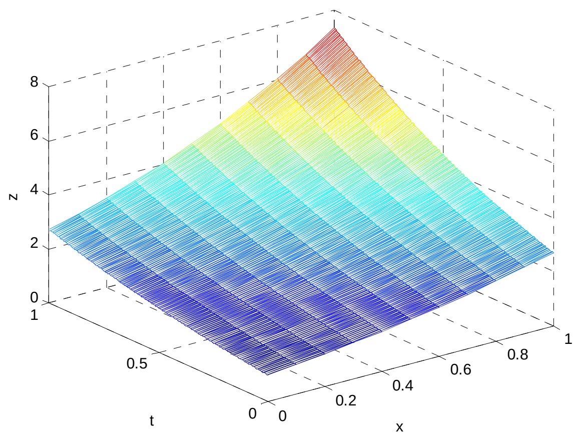


Variation of $u_{hpm}=x^4 + t^4 - 12t^2$for different values of x and t

Table 2 Example 2

$h_x = \frac{1}{10}, h_t = \frac{1}{250}$,

x_i	u_{ex}	u_{hpm} 5-iterates	$\lvert u_{ex} - u_{hpm} \rvert$
0.0	1.004	1.004	0
0.1	1.1096	1.1096	0
0.2	1.2263	1.2263	0
0.3	1.3553	1.3553	0
0.4	1.4978	1.4978	0
0.5	1.6553	1.6553	0
0.6	1.8294	1.8294	0
0.7	2.0218	2.0218	0
0.8	2.2345	2.2345	0
0.9	2.4695	2.4695	0
1.0	2.7292	2.7292	0



Varition of approximate solution for different values of x and t

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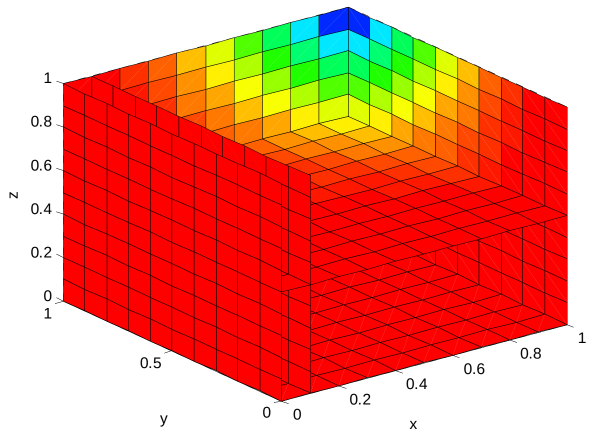
Table 3 Example 3

$h_x = h_y = h_z = \frac{1}{10}, h_t = \frac{1}{250}$

x_i	y_i	z_i	u_{ex}	u_{hpm} 5-Iterates	$\lvert u_{ex} - u_{hpm} \rvert$
0	0	0	0	0	0
0.1	0.1	0.1	1.004×10^{-6}	1.004×10^{-6}	0
0.2	0.2	0.2	6.4257×10^{-5}	6.4257×10^{-5}	0
0.3	0.3	0.3	7.3192×10^{-4}	7.3192×10^{-4}	0
0.4	0.4	0.4	4.1124×10^{-3}	7.1124×10^{-3}	0
0.5	0.5	0.5	1.5688×10^{-2}	1.5688×10^{-2}	0
0.6	0.6	0.6	4.6843×10^{-2}	4.6843×10^{-2}	0
0.7	0.7	0.7	0.11812	0.11812	0
0.8	0.8	0.8	0.26319	0.26319	0
0.9	0.9	0.9	0.53357	0.53357	0
1.0	1.0	1.0	1.004	1.004	0

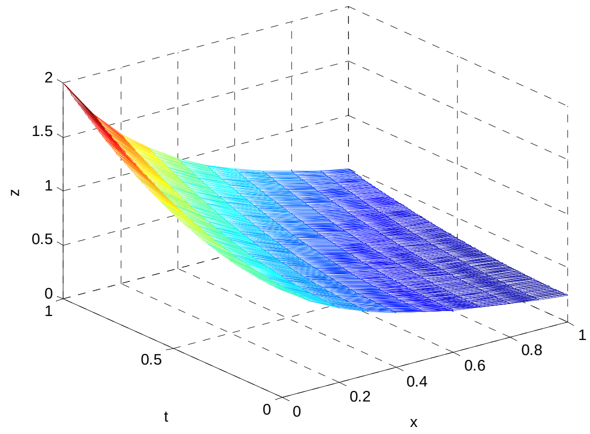


Variation of approximate solution for different values of x , y and z for t=0.004

Table 4 Example 4

$h_x = \frac{1}{10}, h_t = \frac{1}{250},$

x_i	u_{ex}	u_{hpm} 3-iterates	$\lvert u_{ex} - u_{hpm} \rvert$
0.0	1.0	1.0	0.0
0.1	0.82646	0.82645	0.00001
0.2	0.69446	0.69444	0.00002
0.3	0.59173	0.59172	0.00001
0.4	0.51021	0.51020	0.00001
0.5	0.44445	0.44444	0.00001
0.6	0.39063	0.39063	0.0
0.7	0.34603	0.34602	0.00001
0.8	0.30865	0.30864	0.00001
0.9	0.27701	0.27701	0.0
1.0	0.25	0.25	0.0



Variation of approximate solution for different values of x and t

IV. CONCLUSION

In this paper, we have made a detailed study of homotopy perturbation method. For this, we discussed in length its applications in solving various diversified initial boundary value problems with non local boundary conditions. This is employed without using linearization, discretization, transformation or restrictive assumptions. The results demonstrate the stability and convergence of the method, the obtained solutions are shown graphiclly. . Moreover, the method is easier to implement than the traditional techniques. It is worth mentioning that the technique and ideas presented in this paper can be extended for findng the analytic solution of the obstacle, unilateral and contact problems which arise in mathematical and engineering sciences.

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