Shannon-entropy Control of Quantum Systems

Yifan Xing, Jun Wu

Abstract—**This paper proposes a new quantum control method which controls the Shannon entropy of quantum systems. We provide controller design method for the discrete Shannon entropy, which can drive the entropy to track a desired trajectory. We also give the necessary and sufficient conditions under which in very short time the entropy can only increase or decrease. Simulation is done on five dimensional quantum system to show the effectiveness of our algorithm.**

Index Terms—**Quantum control, quantum entropy, quantum information**

I. Introduction

QUANTUM control has become an important topic in quantum information [1, 2], molecular chemistry [3] and atomic physics [4]. Many control methods, including optimal control [5], Lyapunov control [6], learning control [7], feedback control [8] and incoherent control [9, 10], have been used to controller design of quantum systems. Based on classical probability density function (PDF) control, there is a developing research area on Shannon entropy control, which has achieved good performance in classical systems, such as stochastic control [11, 12], networked control [13] and biological control [14]. The extension of Shannon entropy control into quantum area may also enhance quantum control performance.

Shannon entropy in atomic calculations has further been related to various properties such as atomic ionization potential [15], molecular geometric parameters [16], chemical similarity of different functional groups [17], characteristics of correlation methods for global delocalizations [18], molecular reaction paths [19], orbital-based kinetic theory [20], highly excited states of single-particle systems [21] and nature of chemical bonds [22]. The consistency of the Shannon entropy when applied to outcomes of quantum experiments has been analyzed [23], and it is shown that Shannon entropy is fully consistent and its properties are never violated in quantum settings.

In the recent research about quantum sliding-mode control (SMC) [24, 25], a sliding mode is defined based on the fidelity with a desired eigenstate, and the goal is to maintain the state in the mode or drive it back into the mode after measurement. In fact, the fidelity here is directly related to Shannon entropy. There is also research about coherent control based on tracking control for two-level systems [26]. Since coherence corresponds to large entropy, while fidelity corresponds to small entropy, we can directly control the entropy to achieve the goal. If the entropy can track a desired trajectory, the state will be able to slide among different modes, rather than in one mode in the existing quantum SMC. For n-level systems which can not be depicted by Bloch sphere, such method can also provide a systematic way to maintain fidelity or coherence.

For the biological and physiological data-sets, quantifying disorder of the system has become popular as an intense area of promising recent research. In the recent study of a complexity measure for nonstationary signals [14], Shannon entropy has been used to distinguish “healthy” from “unhealthy” biological signals. The study has quantified the information evolution of transitions associated with probabilities assigned to each state, with a goal of providing single value (an entropy) to describe the information content. Similar approach can be adopted to systems where the change in parameter would be indicative of a change in the “health” of the system. For example, in the recent research about information theoretic measures of the electron correlation for both continuous [27] and discrete [28] cases, it is shown that Shannon entropy can also provide a new way to calculate electron correlation energy more accurately. An accurate description of atomic and molecular properties requires an explicit account of electron correlation, while there is no operator in quantum mechanics whose measurement gives the correlation energy. Since strong correlation corresponds to large entropy, we can also use Shannon entropy as a new approach to control quantum correlation.

Quantum von Neumann entropy is a good measure of entanglement, and it will reduce to Shannon entropy for the pure state case. It can provide a real time noise observation and a systematic guideline to make reasonable choice of control strategy. The von Neumann entropy is just a measure of the purity of the given density matrix without explicit reference to information contained in individual measurements [29]. While quantum Shannon entropy can reveal a great deal of information from the perspective of geometrical changes to the density [19]. It shows interesting features about the bond forming and breaking process that are not apparent from the conventional reaction energy profile. Recent research has studied about how to image and manipulate the shape of electronic wavefunction [30], and how to directly measure the quantum wavefunction for photons [31]. If the probability density function can be well measured and controlled in the future, we can directly control the detailed spatial distribution for both pure and mixed states. Sometimes, the detailed distribution may not be important, while we only need to make the distribution more ordered or disordered. This also calls for the control of the uncertainty, which can be directly reflected by Shannon entropy.

For simplicity, this paper only considers the pure state case.

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Section 2 shows the definitions of both discrete and continuous quantum Shannon entropies, and presents our control goal. Section 3 provides controller design method for discrete entropy. Section 4 shows the numerical simulation example. Concluding remarks are given in Section 5.

II. Preliminary

In quantum control, the state of a closed quantum system is represented by a state vector (wavefunction) $\psi(x,t)$ in a Hilbert space. Here for the space variable we only consider one dimensional position variable x. The evolution of the state obeys the Schrödinger equation

$$\imath\hbar\dot{\psi}(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x,t)\right]\psi(x,t), \quad (1)$$

where $\imath = \sqrt{-1}$, and the external potential field $U(x,t) \in \mathbf{R}$ is taken as the control term. For an infinite dimensional quantum system, the wavefunction $\psi(x,t)$ is the superposition of free Hamiltonian's eigenstates $\psi_i(x)$:

$$\psi(x,t) = \sum_{i=1}^{\infty} c_i(t)\psi_i(x),$$

where both the wavefunction and the coefficients should be normalized:

$$\int_{-\infty}^{\infty} |\psi(x,t)|^2 \mathrm{d}x = \sum_{i=1}^{\infty} |c_i(t)|^2 = 1. \quad (2)$$

Defining the state of the system as

$$C(t) = [c_1(t), c_2(t), \cdots, c_n(t), \cdots]^{\mathrm{T}},$$

the Schrödinger equation can be written as

$$\dot{C}(t) = \left[A + \sum_{i=1}^{k} B_i U_i(t)\right] C(t),$$

where both A and B_i are skew-Hermitian matrices. If the case with only one controller $U(t)$ can be well solved, it will be easier for multiple-controller cases. So this paper only considers the following case with one controller:

$$\dot{C}(t) = AC(t) + BU(t)C(t). \quad (3)$$

Assuming a system that consists of n states, in which the probability for the i-th state to happen is p_i, the traditional discrete Shannon entropy in information theory is defined as

$$S = -\sum_{i=1}^{n} p_i \ln p_i,$$

which shows the degree of randomness of the system. For example, when $p_1 = p_2 = \cdots = p_n = 1/n$, every state happens in the equal probability, which is a random system. In this situation, the Shannon entropy takes its maximum value $\ln n$. If $p_1 = 1$, the system is completely predictable, i.e., the first state always happens and the entropy takes its minimum value 0. We can also regard the entropy as the superposition of the uncertainties $\ln(1/p_i)$ because larger probability leads to smaller uncertainty. Similarly, the discrete quantum Shannon entropy can be defined as

$$S(t) = -\sum_{i=1}^{\infty} |c_i(t)|^2 \ln |c_i(t)|^2, \quad (4)$$

where $|c_i(t)|^2$ is the probability that the superposition state collapses to the i-th eigenstate upon quantum measurement. From definition (4) we know, the discrete entropy satisfies $S(t) \geq 0$. For n-level quantum systems, $S(t)$ reaches its maximum value $\ln n$ when $|c_1(t)|^2 = |c_2(t)|^2 = \cdots = |c_n(t)|^2 = 1/n$, and reaches its minimum value 0 when

$$|c_i(t)|^2 = \begin{cases} 1, & \text{for } i = k, \\ 0, & \text{for } i \neq k, \end{cases}$$

where k is a given integer. Here $0\ln 0$ is defined as 0, which can be seen from

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0}\frac{\ln x}{1/x} = \lim_{x\to 0}\frac{\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)}{\frac{\mathrm{d}}{\mathrm{d}x}(1/x)} = \lim_{x\to 0}(-x) = 0.$$

The control of $S(t)$ can be realized by controlling the probability density $|c_i(t)|^2$. In Section 3 we provide the method which can directly drive the entropy to track a desired trajectory.

For the continuous case, Shannon proposed that the entropy for a system with a probability distribution $p(x)$ in one dimension could be characterized by

$$S_c = -\int p(x)\ln p(x)\mathrm{d}x, \quad \int p(x)\mathrm{d}x = 1,$$

which measures the delocalization or the lack of structure in the respective distribution. Thus the entropy is maximal for uniform distribution, and is minimal when the uncertainty about the structure of the distribution is minimal. Since the quantum probability density can be denoted by a continuous function $|\psi(x,t)|^2$, the continuous quantum Shannon entropy can be defined as

$$S_c(t) = -\int_{-\infty}^{\infty} |\psi(x,t)|^2 \ln |\psi(x,t)|^2 \mathrm{d}x, \quad (5)$$

where integral is used to deal with continuous probability distribution.

Our control goal is to drive the entropy to track a desired trajectory.

III. Controller Design Method

Here we provide the controller design method which can drive the discrete entropy (4) to track a desired trajectory. Such control task is called "temporal control", which means not only the destiny should satisfy the requirement, but also the entropy at any instant of the entire process should follow the pre-specified value.

First we can get the time derivative of (4) as

$$\dot{S}(t) = -\sum_{i=1}^{\infty}[\ln|c_i(t)|^2 + 1]\frac{\mathrm{d}|c_i(t)|^2}{\mathrm{d}t}$$
$$= -\sum_{i=1}^{\infty}\frac{\mathrm{d}|c_i(t)|^2}{\mathrm{d}t}\ln|c_i(t)|^2 - \sum_{i=1}^{\infty}\frac{\mathrm{d}|c_i(t)|^2}{\mathrm{d}t},$$

where

$$\sum_{i=1}^{\infty}\frac{\mathrm{d}|c_i(t)|^2}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{i=1}^{\infty}|c_i(t)|^2\right] \equiv 0,$$

because the sum of probabilities should always equal 1. So we have

$$\dot{S}(t) = -\sum_{i=1}^{\infty}\frac{\mathrm{d}|c_i(t)|^2}{\mathrm{d}t}\ln|c_i(t)|^2 = -\sum_{i=1}^{\infty}\frac{\mathrm{d}[c_i^*(t)c_i(t)]}{\mathrm{d}t}\ln|c_i(t)|^2$$

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$$
\begin{aligned}
&= -\sum_{i=1}^{\infty}[\dot{c}_i^*(t)c_i(t) + c_i^*(t)\dot{c}_i(t)]\ln|c_i(t)|^2 \\
&= -2\sum_{i=1}^{\infty}\Re[\dot{c}_i(t)c_i^*(t)]\ln|c_i(t)|^2 .
\end{aligned} \tag{6}
$$

We can define a row vector

$$D(t) \triangleq [-\ln|c_1(t)|^2, -\ln|c_2(t)|^2, \cdots, -\ln|c_n(t)|^2, \cdots],$$

which leads to

$$
\begin{aligned}
\dot{S}(t) &= 2D(t)\Re[\dot{c}_1(t)c_1^*(t), \dot{c}_2(t)c_2^*(t), \cdots, \dot{c}_n(t)c_n^*(t), \cdots]^{\mathrm{T}} \\
&= 2D(t)\Re[\dot{C}(t)\circ C^*(t)] \\
&= 2D(t)\Re\{[AC(t) + BU(t)C(t)]\circ C^*(t)\} \\
&= 2D(t)\Re[AC(t)\circ C^*(t)] + 2U(t)D(t)\Re[BC(t)\circ C^*(t)].
\end{aligned}
$$

Here "$\circ$" denotes the Hadamard product which means the corresponding elements are multiplied:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \circ \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 b_1 \\ a_2 b_2 \\ \vdots \\ a_n b_n \end{bmatrix}.$$

Next we define

$$
\begin{aligned}
\alpha(t) &\triangleq 2D(t)\Re[AC(t)\circ C^*(t)], \\
\beta(t) &\triangleq 2D(t)\Re[BC(t)\circ C^*(t)],
\end{aligned} \tag{7}
$$

which gives

$$\dot{S}(t) = \alpha(t) + U(t)\beta(t). \tag{8}$$

So we can get the controller

$$U(t) = \frac{\dot{S}(t) - \alpha(t)}{\beta(t)}. \tag{9}$$

If the desired trajectory of $S(t)$ is known, then at any time we can use (9) to calculate the feedback controller. We can combine (3) and (9) together to solve the controller out without measuring $C(t)$, so such method belongs to open-loop control. When $\beta(t) \neq 0$ holds, it is always easy to use (9) to make the entropy track the desired trajectory. When $\beta(t) = 0$, there is possibility that in very short time, the entropy can only increase or decrease. In order to analyze the controllability when $\beta(t) = 0$, we first show the following proposition about the derivatives of both $\alpha(t)$ and $\beta(t)$.

Proposition 1. When $|c_i(t)|^2 \neq 0\, (\forall i)$ holds, we have

$$
\begin{cases}
\dot{\alpha}(t) = \alpha_1(t) + U(t)\alpha_2(t) \\
\dot{\beta}(t) = \beta_1(t) + U(t)\beta_2(t),
\end{cases}
$$

where

$$
\begin{cases}
\alpha_1(t) = 2D(t)\{\Re[A^2C(t)\circ C^*(t)] + AC(t)\circ A^*C^*(t)\} \\
\qquad + 4E(t)\{\Re[AC(t)\circ C^*(t)]\circ\Re[AC(t)\circ C^*(t)]\} \\
\alpha_2(t) = 2D(t)\Re[ABC(t)\circ C^*(t) + AC(t)\circ B^*C^*(t)] \\
\qquad + 4\{E(t)\circ\Re[BC(t)\circ C^*(t)]^{\mathrm{T}}\}\Re[AC(t)\circ C^*(t)] \\
\beta_1(t) = 2D(t)\Re[BAC(t)\circ C^*(t) + BC(t)\circ A^*C^*(t)] \\
\qquad + 4\{E(t)\circ\Re[AC(t)\circ C^*(t)]^{\mathrm{T}}\}\Re[BC(t)\circ C^*(t)] \\
\beta_2(t) = 2D(t)\{\Re[B^2C(t)\circ C^*(t)] + BC(t)\circ B^*C^*(t)\} \\
\qquad + 4E(t)\{\Re[BC(t)\circ C^*(t)]\circ\Re[BC(t)\circ C^*(t)]\},
\end{cases}
$$

and $E(t)$ is defined as

$$E(t) \triangleq \begin{bmatrix} \dfrac{-1}{|c_1(t)|^2} & \dfrac{-1}{|c_2(t)|^2} & \cdots & \dfrac{-1}{|c_n(t)|^2} & \cdots \end{bmatrix}.$$

Proof. From (7) we can get

$$
\begin{aligned}
\dot{\alpha}(t) &= 2\dot{D}(t)\Re[AC(t)\circ C^*(t)] + 2D(t)\frac{\mathrm{d}}{\mathrm{d}t}\Re[AC(t)\circ C^*(t)], \\
\dot{\beta}(t) &= 2\dot{D}(t)\Re[BC(t)\circ C^*(t)] + 2D(t)\frac{\mathrm{d}}{\mathrm{d}t}\Re[BC(t)\circ C^*(t)],
\end{aligned}
$$

where $\dot{D}(t)$, $\frac{\mathrm{d}}{\mathrm{d}t}\Re[AC(t)\circ C^*(t)]$ and $\frac{\mathrm{d}}{\mathrm{d}t}\Re[BC(t)\circ C^*(t)]$ can be calculated as follows:

$$
\begin{aligned}
\dot{D}(t) &= \begin{bmatrix} \dfrac{-1}{|c_1(t)|^2}\dfrac{\mathrm{d}|c_1(t)|^2}{\mathrm{d}t} & \dfrac{-1}{|c_2(t)|^2}\dfrac{\mathrm{d}|c_2(t)|^2}{\mathrm{d}t} & \cdots \\
\dfrac{-1}{|c_n(t)|^2}\dfrac{\mathrm{d}|c_n(t)|^2}{\mathrm{d}t} & \cdots \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{-1}{|c_1(t)|^2} & \dfrac{-1}{|c_2(t)|^2} & \cdots & \dfrac{-1}{|c_n(t)|^2} & \cdots \end{bmatrix} \circ \\
&\quad \begin{bmatrix} \dfrac{\mathrm{d}|c_1(t)|^2}{\mathrm{d}t} & \dfrac{\mathrm{d}|c_2(t)|^2}{\mathrm{d}t} & \cdots & \dfrac{\mathrm{d}|c_n(t)|^2}{\mathrm{d}t} & \cdots \end{bmatrix} \\
&= E(t)\circ\frac{\mathrm{d}}{\mathrm{d}t}[C^*(t)\circ C(t)]^{\mathrm{T}} \\
&= E(t)\circ[\dot{C}^*(t)\circ C(t) + C^*(t)\circ\dot{C}(t)]^{\mathrm{T}} = 2E(t)\circ\Re[\dot{C}(t)\circ C^*(t)]^{\mathrm{T}} \\
&= 2E(t)\circ\Re\{[AC(t) + BU(t)C(t)]\circ C^*(t)\}^{\mathrm{T}} \\
&= 2E(t)\circ\Re[AC(t)\circ C^*(t)]^{\mathrm{T}} + 2U(t)E(t)\circ\Re[BC(t)\circ C^*(t)]^{\mathrm{T}},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Re[AC(t)\circ C^*(t)] &= \Re\left\{\frac{\mathrm{d}}{\mathrm{d}t}[AC(t)\circ C^*(t)]\right\} \\
&= \Re[A\dot{C}(t)\circ C^*(t) + AC(t)\circ\dot{C}^*(t)] \\
&= \Re\{A[AC(t) + BU(t)C(t)]\circ C^*(t) \\
&\quad + AC(t)\circ[A^*C^*(t) + U(t)B^*C^*(t)]\} \\
&= \Re[A^2C(t)\circ C^*(t) + AC(t)\circ A^*C^*(t)] \\
&\quad + U(t)\Re[ABC(t)\circ C^*(t) + AC(t)\circ B^*C^*(t)],
\end{aligned}
$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\Re[BC(t)\circ C^*(t)]=\Re\left\{\frac{\mathrm{d}}{\mathrm{d}t}[BC(t)\circ C^*(t)]\right\}$$
$$=\Re[B\dot{C}(t)\circ C^*(t)+BC(t)\circ\dot{C}^*(t)]$$
$$=\Re\{B[AC(t)+BU(t)C(t)]\circ C^*(t)$$
$$+BC(t)\circ[A^*C^*(t)+U(t)B^*C^*(t)]\}$$
$$=\Re[BAC(t)\circ C^*(t)+BC(t)\circ A^*C^*(t)]$$
$$+U(t)\Re[B^2C(t)\circ C^*(t)+BC(t)\circ B^*C^*(t)].$$

So we have

$$\dot{\alpha}(t)=2\{2E(t)\circ\Re[AC(t)\circ C^*(t)]^{\mathrm{T}}$$
$$+2U(t)E(t)\circ\Re[BC(t)\circ C^*(t)]^{\mathrm{T}}\}\Re[AC(t)\circ C^*(t)]$$
$$+2D(t)\{\Re[A^2C(t)\circ C^*(t)+AC(t)\circ A^*C^*(t)]$$
$$+U(t)\Re[ABC(t)\circ C^*(t)+AC(t)\circ B^*C^*(t)]\}$$
$$=2D(t)\{\Re[A^2C(t)\circ C^*(t)]+AC(t)\circ A^*C^*(t)\}$$
$$+4E(t)\{\Re[AC(t)\circ C^*(t)]\circ\Re[AC(t)\circ C^*(t)]\}$$
$$+U(t)\Big\{2D(t)\Re[ABC(t)\circ C^*(t)+AC(t)\circ B^*C^*(t)]$$
$$+4\{E(t)\circ\Re[BC(t)\circ C^*(t)]^{\mathrm{T}}\}\Re[AC(t)\circ C^*(t)]\Big\},$$

$$\dot{\beta}(t)=2\{2E(t)\circ\Re[AC(t)\circ C^*(t)]^{\mathrm{T}}$$
$$+2U(t)E(t)\circ\Re[BC(t)\circ C^*(t)]^{\mathrm{T}}\}\Re[BC(t)\circ C^*(t)]$$
$$+2D(t)\{\Re[BAC(t)\circ C^*(t)+BC(t)\circ A^*C^*(t)]$$
$$+U(t)\Re[B^2C(t)\circ C^*(t)+BC(t)\circ B^*C^*(t)]\}$$
$$=2D(t)\Re[BAC(t)\circ C^*(t)+BC(t)\circ A^*C^*(t)]$$
$$+4\{E(t)\circ\Re[AC(t)\circ C^*(t)]^{\mathrm{T}}\}\Re[BC(t)\circ C^*(t)]$$
$$+U(t)\Big\{2D(t)\{\Re[B^2C(t)\circ C^*(t)]+BC(t)\circ B^*C^*(t)\}$$
$$+4E(t)\{\Re[BC(t)\circ C^*(t)]\circ\Re[BC(t)\circ C^*(t)]\}\Big\}.$$

Hence Proposition 1 has been proved. □

Based on Proposition 1, we can get the conditions under which in very short time the entropy can only increase or decrease, which are shown in Theorem 1. It can be seen that Theorem 1 gives the necessary and sufficient conditions.

Theorem 1. In very short time, the entropy can only increase when

$$\begin{cases}\alpha(t)>0\\ \beta(t)=0\end{cases}\text{or}\begin{cases}\alpha(t)=\beta(t)=\beta_2(t)=0\\ \alpha_1(t)>0\\ \alpha_2(t)=-\beta_1(t)\end{cases}\quad\text{or}$$
$$\begin{cases}\alpha(t)=\beta(t)=0\\ \beta_2(t)>0\\ [\alpha_2(t)+\beta_1(t)]^2\le 4\alpha_1(t)\beta_2(t),\end{cases}$$

and can only decrease when

$$\begin{cases}\alpha(t)<0\\ \beta(t)=0\end{cases}\text{or}\begin{cases}\alpha(t)=\beta(t)=\beta_2(t)=0\\ \alpha_1(t)<0\\ \alpha_2(t)=-\beta_1(t)\end{cases}\quad\text{or}$$
$$\begin{cases}\alpha(t)=\beta(t)=0\\ \beta_2(t)<0\\ [\alpha_2(t)+\beta_1(t)]^2\le 4\alpha_1(t)\beta_2(t).\end{cases}$$

Proof. Assuming the sampling period is T, $\dot{\alpha}(t)=\alpha_1(t)+U(t)\alpha_2(t)$ can be descretized as

$$\frac{\alpha(T)-\alpha(0)}{T}=\alpha_1(0)+U(0)\alpha_2(0),$$

which gives

$$\alpha(T)=\alpha(0)+T[\alpha_1(0)+U(0)\alpha_2(0)].$$

Similarly we can get

$$\beta(T)=\beta(0)+T[\beta_1(0)+U(0)\beta_2(0)].$$

If $U(t)$ remains constant as $U(0)$ in the first sampling period T, $\dot{S}(T)$ can be approximated as

$$\dot{S}(T)=\alpha(T)+U(0)\beta(T)$$
$$=\alpha(0)+T[\alpha_1(0)+U(0)\alpha_2(0)]$$
$$+U(0)\{\beta(0)+T[\beta_1(0)+U(0)\beta_2(0)]\}$$
$$=T\beta_2(0)U^2(0)+\{\beta(0)+T[\alpha_2(0)+\beta_1(0)]\}U(0)$$
$$+\alpha(0)+T\alpha_1(0).$$

Here $\dot{S}(T)$ is a once basic quadratic equation about $U(0)$, and the equation's discriminant Δ_1 can be calculated as

$$\Delta_1=\{\beta(0)+T[\alpha_2(0)+\beta_1(0)]\}^2-4T\beta_2(0)[\alpha(0)+T\alpha_1(0)].$$

If $\beta(0)\ne 0$, we have $\lim_{T\to 0}\Delta_1=\beta^2(0)>0$. So it is always easy to find $U(0)$ to make $\dot{S}(T)$ positive or negative, which means in very short time the entropy can both increase and decrease.

If $\beta(0)=0$, we can get $\dot{S}(T)=\alpha(0)+T\{\beta_2(0)U^2(0)+[\alpha_2(0)+\beta_1(0)]U(0)+\alpha_1(0)\}$, which yields $\lim_{T\to 0}\dot{S}(T)=\alpha(0)$. If $\alpha(0)>0$, we have $\lim_{T\to 0}\dot{S}(T)>0$, which means in very short time the entropy can only increase; similarly if $\alpha(0)<0$, the entropy can only decrease; if $\alpha(0)=0$, we have

$$\dot{S}(T)=T\{\beta_2(0)U^2(0)+[\alpha_2(0)+\beta_1(0)]U(0)+\alpha_1(0)\}.$$

Here $\beta_2(0)U^2(0)+[\alpha_2(0)+\beta_1(0)]U(0)+\alpha_1(0)$ is also a once basic quadratic equation about $U(0)$, and the equation's discriminant Δ_2 can be calculated as

$$\Delta_2=[\alpha_2(0)+\beta_1(0)]^2-4\alpha_1(0)\beta_2(0).$$

When $[\alpha_2(0)+\beta_1(0)]^2>4\alpha_1(0)\beta_2(0)$, we have $\Delta_2>0$, which means the entropy can both increase and decrease; when $[\alpha_2(0)+\beta_1(0)]^2\le 4\alpha_1(0)\beta_2(0)$, the discussion can be divided into 3 cases:

(a) If $\beta_2(0)>0$, the parabola opens upward, which means the entropy can only increase.
(b) If $\beta_2(0)<0$, the parabola opens downward, which means the entropy can only decrease.
(c) If $\beta_2(0)=0$, we have $[\alpha_2(0)+\beta_1(0)]^2\le 0$, which gives $\alpha_2(0)+\beta_1(0)=0$ and $\dot{S}(T)=T\alpha_1(0)$. So when $\alpha_1(0)>0$, the entropy can only increase; when $\alpha_1(0)<0$, the entropy can only decrease; when $\alpha_1(0)=0$, the entropy will remain constant in very short time.

Above all, we can get the conclusion in Theorem 1. □

In quantum mechanics, A is often chosen to be diagonal, thus all the elements in A are pure imaginary since A is skew-Hermitian. Assume $A = \mathrm{diag}\{a_{11}\iota, a_{22}\iota, \cdots, a_{nn}\iota, \cdots\}$, where $a_{ii} \in \mathbf{R}(\forall i)$ holds. We can get

$$\begin{aligned}
&\Re[AC(t)\circ C^*(t)] \\
&= \Re\left\{\begin{bmatrix} a_{11}\iota & & & & \\ & a_{22}\iota & & & \\ & & \ddots & & \\ & & & a_{nn}\iota & \\ & & & & \ddots \end{bmatrix}\begin{bmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \\ \vdots \end{bmatrix}\circ\begin{bmatrix} c_1^*(t) \\ c_2^*(t) \\ \vdots \\ c_n^*(t) \\ \vdots \end{bmatrix}\right\} \\
&= \Re\begin{bmatrix} \iota a_{11}|c_1(t)|^2 \\ \iota a_{22}|c_2(t)|^2 \\ \vdots \\ \iota a_{nn}|c_n(t)|^2 \\ \vdots \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix}.
\end{aligned}$$

From (7) we know $\alpha(t) \equiv 0$, which gives $\dot{S}(t) = U(t)\beta(t)$. So we can get the controller

$$U(t) = \frac{\dot{S}(t)}{\beta(t)}. \tag{10}$$

When $\beta(t) \neq 0$, if the desired trajectory of $S(t)$ is known, we can simply use (10) to get the controller. When $\beta(t) = 0$, the conditions under which in very short time the entropy can only increase or decrease are shown in Theorem 2.

Theorem 2. If A is diagonal, in very short time the entropy can only increase when

$$\begin{cases} \beta(t) = \beta_1(t) = 0 \\ \beta_2(t) > 0, \end{cases}$$

and can only decrease when

$$\begin{cases} \beta(t) = \beta_1(t) = 0 \\ \beta_2(t) < 0. \end{cases}$$

Proof. From $\dot{S}(t) = U(t)\beta(t)$ we know $\dot{S}(0) = U(0)\beta(0)$. So when $\beta(0) \neq 0$, it is always easy to choose $U(0)$ to make $\dot{S}(0)$ positive or negative. When $\beta(0) = 0$, we can get

$$\begin{aligned}
\dot{S}(T) &= U(0)\beta(T) = U(0)\{\beta(0) + T[\beta_1(0) + U(0)\beta_2(0)]\} \\
&= T[\beta_2(0)U^2(0) + \beta_1(0)U(0)].
\end{aligned}$$

The discussion can be divided into 3 cases:

(a) $\beta_2(0) = 0$:
We have $\dot{S}(T) = T\beta_1(0)U(0)$. If $\beta_1(0) \neq 0$, it is easy to choose $U(0)$ to make $\dot{S}(0)$ positive or negative; if $\beta_1(0) = 0$, the entropy will remain constant in very short time.

(b) $\beta_2(0) > 0$:
Here $\beta_2(0)U^2(0) + \beta_1(0)U(0)$ is also a once basic quadratic equation about $U(0)$, and the equation's discriminant Δ_3 can be calculated as $\Delta_3 = \beta_1^2(0)$. If $\beta_1(0) = 0$, we have $\Delta_3 = 0$, which means the entropy can only increase; if $\beta_1(0) \neq 0$, we have $\Delta_3 > 0$, which means the entropy can both increase and decrease in very short time.

(c) $\beta_2(0) < 0$:
Similarly we know if $\beta_1(0) = 0$, the entropy can only increase; if $\beta_1(0) \neq 0$, the entropy can both increase and decrease in very short time.

Above all, we can get the conclusion in Theorem 2. □

When the entropy has reached the target, it needs to be maintained constant. If A is diagonal, from (10) we know we only need $U(t) = 0$ to maintain the entropy constant. If A is non-diagonal, it is difficult to use (9) to keep the entropy constant especially when there exists disturbance. Then we need to develop the discrete control method to overcome the disturbance.

IV. Simulation Example

For the discrete entropy, we present simulation on a five-level quantum system

$$\begin{bmatrix} \dot{c}_1(t) \\ \dot{c}_2(t) \\ \dot{c}_3(t) \\ \dot{c}_4(t) \\ \dot{c}_5(t) \end{bmatrix} = \left\{\begin{bmatrix} -\iota & 0 & 0 & 0 & 0 \\ 0 & -1.2\iota & 0 & 0 & 0 \\ 0 & 0 & -1.3\iota & 0 & 0 \\ 0 & 0 & 0 & -2\iota & 0 \\ 0 & 0 & 0 & 0 & -2.15\iota \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -\iota & -\iota \\ 0 & 0 & 0 & -\iota & -\iota \\ 0 & 0 & 0 & -\iota & -\iota \\ -\iota & -\iota & -\iota & 0 & 0 \\ -\iota & -\iota & -\iota & 0 & 0 \end{bmatrix}U(t)\right\}\begin{bmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \\ c_4(t) \\ c_5(t) \end{bmatrix}. \tag{11}$$

For initial state $C(0) = [\iota/2, 1/2, 1/2, \sqrt{2}/4, \sqrt{2}/4]^{\mathrm{T}}$ with $S(0) = 1.56$, we expect that the entropy changes as follows in seven steps: (a) increases to 1.61; (b) keeps constant; (c) increases to 1.66; (d) keeps constant; (e) decreases to 1.61; (f) increases to 1.66; (g) keeps constant. If the step period $T = 0.01$, the controller can be calculated by (10) as

$$\begin{aligned}
U(t) = {} & [5 - 5\cdot 1(t-0.01) + 5\cdot 1(t-0.02) - 5\cdot 1(t-0.03) \\
& -5\cdot 1(t-0.04) + 10\cdot 1(t-0.05) - 5\cdot 1(t-0.06)] / \{2D(t) \\
& \Re[BC(t)\circ C^*(t)]\}.
\end{aligned} \tag{12}$$

Combining (12) with (11) gives the simulations for both $S(t)$ and $U(t)$, which are shown in Fig. 1 and Fig. 2 respectively.

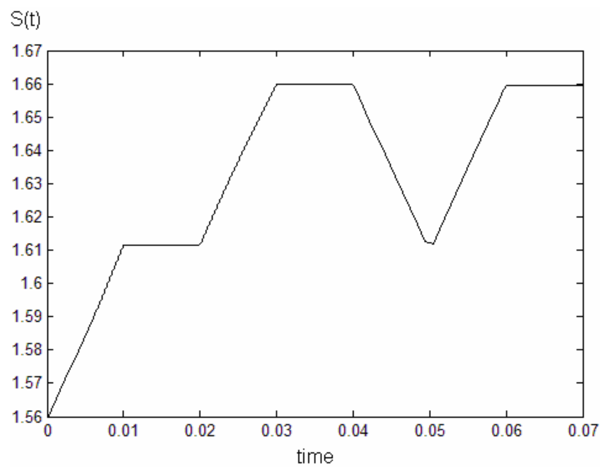

Fig. 1. Evolution of the entropy of system (11).

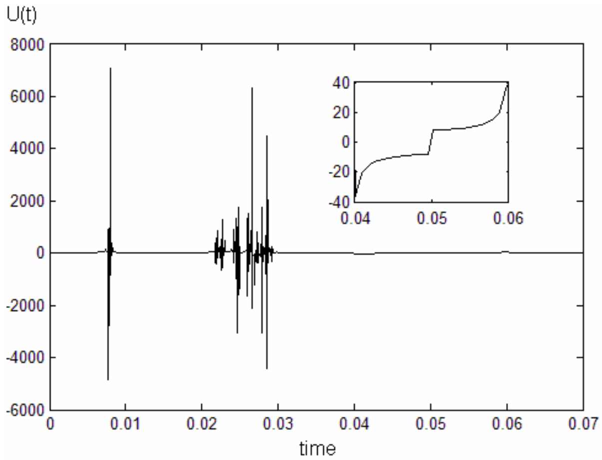

Fig. 2. Evolution of the controller (12).

We can see that for the five-level case, the continuous method can lead to very accurate tracking. Although at some instant $\beta(t)$ may be 0, which makes $U(t)$ very large, the entropy can still be driven to go along the desired trajectory.

V. Conclusion

This paper proposes a new quantum control method which controls the Shannon entropy of quantum systems. Simulation example evidenced the effectiveness of the method. A strength of our method is that it provides a direct control algorithm for discrete quantum entropy, rather than the indirect one via probability density function control. Our method provides a universal tool for entropy control, which can also contribute to classical information theory. Some immediate extensions of the method include quantum sliding-mode control and coherent control. In the future, we will study how to deal with the regions in which the entropy can only increase or decrease. The extension of the methods to the mixed state case deserves our future research. The applications in correlation energy and biological control are also of keen interests and currently being pursued.

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