# The Two-phase Stefan Problem for the Heat Equation

K.Kaliyeva

*Abstract*—**This paper is devoted to the two-phase Stefan problem with the irregular diving boundary of the region. We consider a two-dimensional heat equation with two known boundary conditions in one at the left-hand-side and the other at the right-hand-side. We construct the Green's function in a dihedral angle for the heat equation with the coupled conditions on the fixed known boundary of division two-phases. Then, using this defined Green's function and its properties, we obtain integral representatives of the temperature distributions and the low of motion of the diving boundary. Uniqueness and regularity of the constructed analytic solution with the diving boundary have been proved in the weighted Sobolev space.**

*Index Terms*—**Green's function with irregular boundary, two-phase Stefan problem, motion of the diving boundary**

## I. Introduction

The two-phase Stefan problem consists of determining a temperature field and the low of motion of the diving boundary separating the two phases. It describes a solidification process involving various physical phenomena, including conduction with phase change which is characterized by a moving interface separating two phases and the two-phase heat and mass transfer process. We will start with solving the heat equation, which governs the temperature distribution in the liquid and the solid phases. The unknowns are the two temperature distributions and the position of an interface between the phases (free boundary).

The description of the moving interface problem includes the heat transfer equations for each phase with corresponding initial and boundary conditions which should be specified in each phase as well as on the interface. We have considered the weak formulation of the two-phase Stefan problem and presented the analytical method of solution in the dihedral angle. The classical solutions of the two dimensional two-phase Stefan problem are not expected to exist for all domains [1]. An investigation of the coupled problems for the heat conduction equation with irregular boundary showed that the Stefan problem can't be solved in the functional space with a regular metric in the dihedral angle. This fact motivates the study of the weak solutions in the weighted Sobolev space. For the numerical solution of the Stefan problem traditionally are used Galerkin methods which combined with suitable explicit Runge-Kutta time stepping schemes or the implicit Euler method for the temporal discretization. Linearized formulation of the Stefan problem is possible, if we consider the problem in a small interval of time t ∈ [0; T], assuming that the unknown boundary changes during this time slightly.

The mathematical theory of the local solvability for the one-phase Stefan problem for the one-dimensional heat equation on a small time interval was considered by A.M.Meirmanov [1], E.I.Hanzava [2], B.V.Basalii [4], E.V. Radcevich [5]. The existence theorem for a parabolic equation in a small time period was proven by A.M.Meirmanov. The solution was obtained by using the auxiliary "regularized" tasks. The obtained estimates for the solutions of the auxiliary problems allowed one to get the compactness of the solution in the space $C^{(2,1)}$. The Green's function in the dihedral angle for the heat equation was built in the Holder space by V.A Solonnikov and E.V.Frolovova [3]. These results are used to prove the solvability of boundary value problems for the heat equation in a dihedral angle.

The objective of this paper is to develop a similar theory and construct an analytical solution for the two-phase Stefan problem in the dihedral angle. The construction of the analytical solution for the two-dimensional two-phase Stefan problem will be based on the Green's function for the temperature distribution in the two phases in a domain with a fixed boundary. The properties and features of the constructed Green's function are essentially provided existence and uniqueness of the temperature field and the law of motion of the dividing boundary for the two-dimensional two-phase Stefan problem. This result gives some general properties and features for the temperature distributions and the law of motion of the irregular dividing boundary.

## II. The Weighted Sobolev Spaces

Let $D_T = \Omega \times (0,T)$ be measurable function

$$\Omega = \{r > 0, \varphi_0(r,t) < \varphi < \varphi_1(r,t)\} \times (0,T)$$

with boundary $\sigma = \partial\Omega$ where $\varphi_0(r,t), \varphi_1(r,t)$ are known curves. In domain $D_T$ the solution of the heat equation

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K.Kaliyeva. University College London, Centre for Languages & International Education, London, UK . E-mail: k.kaliyeva@live.ucl.ac.uk or kklara_09@mail.ru

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can be unlimited in irregular the point $r=0$. The irregular boundary is not sufficient to deal with the classical solution. We can come across initial-boundary value problems for parabolic equations whose domain is disturbed in the sense that some singularity appears. It is necessary to introduce the notion of weak derivatives and to work in the Sobolev spaces. Then we can ensure the correct behavior of solutions and represent their features in the weighted Sobolev spaces. In this case, the weighted Sobolev space is the best context allowing us to develop our work.

Let $L_\mu(Q_T)$ be the Banach space of measurable function $Q_T$ with weighted norm

$$\|u\|_{L_\mu(Q_T)}=\left\{\int_0^T dt\int_\Omega r^{2\mu}|u(r,\varphi,t)|^2\,dx\right\}^{\frac{1}{2}}$$

For $l\geq 0$ integer number and $\mu>0$ real number we define the weighted Sobolev space $H_{\mu-l}^{(l+1)}(\Omega)$ with respect to the norm

$$\|u_0\|_{H_{\mu-l-1}^{(l+2,l/2+1/4}(\sigma_T)}=\left\{\sum_{2s+j_1=l}\int_{\sigma_T} r^{2(\mu-2(l+1)+2s+j_1)}\left|D_\varphi^{j_2}D_{r_1}^{j_1}u_0(r_1,\varphi)\right|^2 d\Omega\right\}^{1/2}$$

where

$$j=(j_1,j_2),\quad |j|=j_1+j_2,\quad D_\varphi^{j_2}D_{r_1}^{j_1}=\frac{\partial^{|j|}}{\partial r_1^{j_1}\partial\varphi^{j_2}},$$

Let $\sigma$ be some boundary area which is be the border $\partial\Omega$ of area $\Omega$. We define space $H_{\mu-l-1}^{(l+1/2,l/2+1/4)}(\sigma_T)$ of traces $\Phi(r,t)$ with Sobolev norm

$$\|\Phi\|_{H_{\mu-l-1}^{(l+2,l/2+1/4}(\sigma_T)}=\left\{\sum_{2s+j_1=l}\int_{\sigma_T} r^{2(\mu-2(l+1)+2s+j_1)}\left|D_t^s D_{r_1}^{j_1}\Phi(r_1,\tau)\right|^2 d\sigma_T\right\}^{1/2}+$$

$$+\left\{\sum_{2s+j_1=l}\int_0^T d\tau\int_{\sigma_x} d\sigma_x\int_\sigma \frac{r^{2(\mu-2(l+1)+2s+j_1)}}{|r-r_1|^2}\left|D_t^s D_r^{j_1}\frac{\Phi(r,\tau)}{\sqrt{r}}-D_t^s D_{r_1}^{j_1}\frac{\Phi(r_1,\tau)}{\sqrt{r_1}}\right|^2 d\sigma\right\}^{1/2}+$$

$$+\left\{\sum_{2s+j_1=l}\int_0^T dt\int_0^T d\tau\int_\sigma \frac{r^{2(\mu-2(l+1)+2s+j_1)}}{|t-\tau|^{3/2}}\left|D_t^s D_r^{j_1}\Phi(r,t)-D_\tau^s D_r^{j_1}\Phi(r,\tau)\right|^2 d\sigma\right\}^{1/2}+$$

$$+\left\{\sum_{2s+j_1=l}\int_0^T dt\int_0^T d\tau\int_\sigma \frac{r^{2(\mu-2(l+1)+2s+j_1)}}{|t-\tau|^{5/2}}\left|D_t^s D_r^{j_1}\Phi(r,t)-D_\tau^s D_r^{j_1}\Phi(r,\tau)\right|^2 d\sigma\right\}^{1/2}$$

$H_{\mu-l-1}^{(l+2,l/2+1)}(Q_T)$ space is defined as functions $u(x,t)$ in $Q_T$ with the weighted Sobolev norm

$$\|u\|_{H_{\mu-l-1}^{(l+2,l/2+1/4}(\sigma_T)}=\left\{\sum_{2s+j_1=l}\int_{\sigma_T} r^{2(\mu-2(l+1)+2s+j_1)}\left|D_t^s D_{r_1}^{j_1}D_\varphi^{j_2}u(r_1,\varphi,t)\right|^2 d\Omega\right\}^{1/2}$$

We could remark that there are many possibilities to define an extension of the Sobolev spaces.

## III. THE STATEMENT OF THE PROBLEM

Let us recall the setting the two-phase two dimensional Stefan problems in domain $D_T=\{D_1\cup D_2\}\times(0,T)$, where

$$D_1=\{r>0,\ 0<\varphi<\xi(r,t)\},\ D_2=\{r>0,\ \xi(r,t)<\varphi<\varphi_1\}$$

are an open and smooth bounded domain with boundaries $\partial D_1=\gamma_1\cup\Gamma$, $\partial D_2=\Gamma\cup\gamma_2$, there surface $\Gamma_T=\{r>0,\varphi=\xi(r,t)\}$ is unknown boundary, $\gamma_1=\{r>0,\varphi=0\}$, $\gamma_2=\{r>0,\varphi=\varphi_1\}$ are known boundary for $0<\varphi_0<\varphi_1<2\pi$

Constant $\varphi_0$ is an angle between the x-axis and the tangential curve $\xi(r,t)$.

The two-phase Stefan problem is stated as follows: to find the unknown surface $\Gamma_T$ for $t>0$ and the temperature $u(r,\varphi,t)\in D_T$ which satisfies the heat equation

$$\frac{\partial u}{\partial t}=a^2(\varphi)\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2}\right)+f(r,\varphi,t),\ x\in D_T \quad (1)$$

$$a^2(\varphi)=\begin{cases}a_1^2, if\ 0<\varphi<\xi(r,t)\\ a_2^2, if\ \xi(r,t)<\varphi<\varphi_1\end{cases}\quad \text{for } x\in D_T$$

with the initial conditions

$$u\big|_{t=0}=\begin{cases}u_{01}(r,\varphi),\ if\ 0<\varphi<\xi(r,t)\\ u_{02}(r,\varphi),\ if\ \xi(r,t)<\varphi<\varphi_1\end{cases}\quad (2)$$

on the known boundary

$$u_1\big|_{\gamma_T^{(1)}}=p_1(r,t)\quad,\quad u_2\big|_{\gamma_T^{(2)}}=p_2(r,t)\quad (3)$$

on the boundary (the Stefan conditions)

$$u_2\big|_{\varphi=\xi(r,t)-0}=u_1\big|_{\varphi=\xi(r,t)+0}\quad (4)$$

$$\kappa_2\frac{1}{r}\frac{\partial u_2}{\partial\varphi}\Big|_{\varphi=\xi(r,t)-0}-\kappa_1\frac{1}{r}\frac{\partial u_1}{\partial\varphi}\Big|_{\varphi=\xi(r,t)+0}=\rho q^*\frac{\partial\xi(r,t)}{\partial t}\quad (5)$$

There $\xi(r,t)$ is unknown function of the diving boundary; $q^*$, $\rho$ - are respectively the hidden heat and density of crystallization or melting.

We assume that the initial temperature satisfies the following conditions:

$$u_0(r,\varphi)=O(r^{1-\mu}),\quad \frac{\partial u_0(r,\varphi)}{\partial r}=O(r^{-\mu})\quad if\ r\to 0$$

$$u_0(r,\varphi)=o(r^{1-\mu}),\quad \frac{\partial u_0(r,\varphi)}{\partial r}=o(r^{-\mu})\quad if\ r\to\infty$$

The boundary function for $i$=1,2 satisfy:

$$p_i(r,t)=O(r^{1-\mu}),\quad \frac{\partial p_i(r,t))}{\partial r}=O(r^{-\mu})\quad if\ r\to 0$$

$$p_i(r,t)=o(r^{1-\mu}),\quad \frac{\partial p_i(r,t)}{\partial r}=o(r^{-\mu})\quad if\ r\to\infty$$

and also the heat conservation law

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$$(-1)^{i+1}\left[r^{\mu-2(m+1)+j_1+2s+1}D_r^{j_1}D_\varphi^{2m}D_t^s f_{in}(r,\varphi,t)\right]_{t=0}+$$

$$+a_i^{2m}r^{\mu-2(m+1)+j_1+2s+1}\Delta_{r,\varphi}^{(m)}u_{0i}(r,\varphi)=0 \quad (6)$$

## IV. GREEN FUNCTION FOR THE HEAT EQUATION WITH FIXED KNOWN BOUNDARY

In this part we consider the two-phase two dimensional initial boundary value problem for heat equation in domain $\Omega_T=\{\Omega_1\cup\Omega_2\}\times(0,T)$, where $\Omega_1=\{r>0, 0<\varphi<\varphi_0\}$, $\Omega_2=\{r>0,\varphi_0<\varphi<\varphi_1\}$ are an open with boundaries $\partial\Omega_1=\gamma_1\cup\Gamma$, $\partial\Omega_2=\Gamma\cup\gamma_2$.

There $\Gamma=\{r>0,\varphi=\varphi_0\}$ is known boundary.

There $\gamma_1=\{r>0,\varphi=0\}$, $\gamma_2=\{r>0,\varphi=\varphi_1\}$ for $0<\varphi_0<\varphi_1<2\pi$.

Let us formulate the initial boundary value problem for heat equation: to find the temperature $u(r,\varphi,t), x\in D_T$ satisfying the relations:

$$\frac{\partial u}{\partial}=a^2(\varphi)\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2}\right)+f(r,\varphi,t) \qquad (7)$$

$$a^2(\varphi)=\begin{cases}a_1^2, if\, 0<\varphi<\varphi_0\\ a_2^2, if\, \varphi_0<\varphi<\varphi_1\end{cases}, x\in D_T \qquad (8)$$

with the initial conditions

$$u\,\big|_{t=0}=\begin{cases}u_{01}(r,\varphi), if\, 0<\varphi<\varphi_0\\ u_{02}(r,\varphi), if\, \varphi_0<\varphi<\varphi_1\end{cases} \qquad (9)$$

on the known boundary (coupled conditions)

$$u_1\,\big|_{\varphi=\varphi_0+0}=u_2\,\big|_{\varphi=\varphi_0-0} \qquad (10)$$

$$\kappa_1\frac{1}{r_1}\frac{\partial u_1}{\partial\varphi}\,\big|_{\varphi=\varphi_0+0}=\kappa_2\frac{1}{r_1}\frac{\partial u_2}{\partial\varphi}\,\big|_{\varphi=\varphi_0-0} \qquad (11)$$

The two-phase initial boundary value problem (7)-(11) is concerned with the fundamental solution for the heat equation. Fundamental solution of the heat equation is considered and many of its properties are consequence of the constructed Green's function. Particular attention will be to integral representation with their initial values for the two-phase initial boundary value problem.

Using transformations [4] : Laplace for $t$, Henkel for $r$ and Fourier for $\varphi$ we constructed the Green's function for the first initial boundary value problem for the heat equation in a dihedral plane angle which are obtained as:

$$G_{11}=G_1^{(0)}(r,\varphi,r_1,\varphi^|,t)-\tfrac{\kappa_1}{\kappa_2}G_{11}^*(r,\varphi,r_1,\varphi^|,t) \quad ,if \begin{cases}(r,\varphi)\in D_1\\(r_1,\varphi^|)\in D_1\end{cases}$$

$$G_{12}=G_{12}^*(r,\varphi,r_1,\varphi^|,t) \quad , \quad if \begin{cases}(r,\varphi)\in D_1\\(r_1,\varphi^|)\in D_2\end{cases}$$

$$G_{21}=G_{21}^*(r,\varphi,r_1,\varphi^|,t) \quad , \quad if \begin{cases}(r,\varphi)\in D_2\\(r_1,\varphi^|)\in D_1\end{cases}$$

$$G_{22}=G_2^{(0)}(r,\varphi,r_1,\varphi^|,t)-\tfrac{\kappa_2}{\kappa_1}G_{22}^*(r,\varphi,r_1,\varphi^|,t) \quad ,if \begin{cases}(r,\varphi)\in D_2\\(r_1,\varphi^|)\in D_2\end{cases}$$

where

$$G_m^{(0)}=\sum_{n=1}^{\infty}\frac{e^{-\frac{r^2+r_1^2}{4a_m^2t}}}{a_m^2t}I_{\lambda_n}\left(\frac{rr_1}{2a_m^2t}\right)\frac{\Phi_{mn}(\varphi)\Phi_{mn}(\varphi^|)}{\|\Phi_n\|^2}$$

$$G_{ij}^*(r,\varphi,r_1,\varphi^|,t)=\sum_{n=1}^{\infty}\frac{\Phi_{in}(\varphi)\Phi_{jn}(\varphi^|)}{\|\Phi_n\|^2}R_{ij}^{(n)}(r,r_1,t)$$

$$\Phi_n(\varphi)=\begin{cases}\dfrac{\sin\lambda_n\varphi}{\sin\lambda_n\varphi_0}, & если \quad 0<\varphi<\varphi_0\\[2mm] \dfrac{\sin\lambda_n(\varphi_1-\varphi)}{\sin\lambda_n(\varphi_1-\varphi_0)}, & если \quad \varphi_0<\varphi<\varphi_1\end{cases},$$

$$\|\Phi_n\|^2=\frac{\kappa_1\varphi_0}{2\sin^2\lambda_n\varphi_0}+\frac{\kappa_2(\varphi_1-\varphi_0)}{2\sin^2\lambda_n(\varphi_1-\varphi_0)}$$

$$R_{11}^{(n)}(r,r_1,t)=\int_0^t d\tau\int_0^\infty\left(\frac{\partial}{\partial\tau}-a_2^2\Delta_n\right)\frac{e^{-\frac{r_1^2+r_2^2}{\alpha^2\tau}}}{\alpha^2\tau}I_{\lambda_n}\left(\frac{r_1r_2}{2\alpha^2\tau}\right)\frac{e^{-\frac{r^2+r_2^2}{4a_i^2(t-\tau)}}}{a_i^2(t-\tau)}I_{\lambda_n}\left(\frac{rr_2}{2a_i^2(t-\tau)}\right)dr_2$$

$$R_{12}^{(n)}(r,r_1,t)=\int_0^t d\tau\int_0^\infty\left(\frac{\partial}{\partial\tau}-a_1^2\Delta_n\right)\frac{e^{-\frac{r^2+r_2^2}{4a_i^2(t-\tau)}}}{a_i^2(t-\tau)}I_{\lambda_n}\left(\frac{rr_2}{2a_i^2(t-\tau)}\right)\frac{e^{-\frac{r_1^2+r_2^2}{\alpha^2\tau}}}{\alpha^2\tau}I_{\lambda_n}\left(\frac{r_1r_2}{2\alpha^2\tau}\right)dr_2$$

$$\alpha^2=\frac{\kappa_1a_2^2+\kappa_2a_1^2}{\kappa_1+\kappa_2}$$

Here

$$I_\alpha(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{ix}{2}\right)^{2m+\alpha}$$

is second order Bessel function , $\Gamma$ - Euler gamma function

For the functions

$$g_m(r,\varphi,r_1,\varphi^|,t)=\sum_{i=1}^{\infty}\frac{e^{-\frac{r^2+r_1^2}{4a_m^2t}}}{a_m^2t}I_{\lambda_n}\left(\frac{rr_1}{2a_m^2t}\right)\frac{\Phi_{mn}(\varphi)\Phi_{m1n}(\varphi^|)}{\|\Phi_n\|^2}$$

for $i$=1,2 are proved the validity of the representation [6 ]

$$g_m(r,\varphi,r_1,\varphi^|,t)=\frac{e^{-\frac{r^2+r_1^2}{4a_m^2t}+\frac{rr_1}{2a_m^2t}\cos((\gamma_m-\varphi)-(\gamma_m-\varphi^|))}}{4\pi a_m^2t}-$$

$$-\frac{e^{-\frac{r^2+r_1^2}{4a_m^2t}+\frac{rr_1}{2a_m^2t}\cos((\gamma_m-\varphi)+(\gamma_m-\varphi^|))}}{4\pi a_m^2t}+$$

$$+\frac{e^{-\frac{r^2+r^2}{4a_m^2t}}}{2\pi a_m^2t}\left(\frac{e^{-\frac{rr_1}{2a_m^2t}\cos\theta_m}}{2\pi a_m^2t}\psi(p,\varphi,\theta_m)\right)+$$

$$\left.+\int_{d_m}e^{\frac{rr_1}{2a_m^2t}\cos z}\left(\frac{rr_1\sin z}{2a_m^2t}+\frac{rr_1 shz}{2a_m^2t}\right)\psi(p,\varphi,z)dz\right)$$

where

$$\psi(p,\varphi,z)=Jm\left\{\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\widetilde{\Phi}_m(p,\varphi)\widetilde{\Phi}_m(p,\varphi^|)e^{zp}}{p[\kappa_1 cthp\varphi_0+\kappa_2 cthp(\varphi_1-\varphi_0)]}dp\right)$$

$$\theta_m=\begin{cases}\varphi_0 & , \ if \quad m=1\\ \varphi_0-\varphi_1 & , \ if \quad m=2\end{cases}\qquad \gamma_m=\begin{cases}0 & , \ if \quad m=1\\ \varphi_1 & , \ if \quad m=2\end{cases},$$

$$d_m=\begin{cases}[\varphi_0,\pi] & , \ if \quad m=1\\ [\varphi_1-\varphi_0,\pi] & , \ if \quad m=2\end{cases}$$

In the integral of $\psi(p,\varphi,z)$ the function $\widetilde{\Phi}\ (p,\varphi)$ are defined as contour functions for m = 1,2 :

$$\widetilde{\Phi}\ (p,\varphi)=\begin{cases}\widetilde{\Phi}_1(p,\varphi)=\dfrac{shp\varphi}{shp\varphi_0} & , \quad if \quad 0<\varphi<\varphi_0\\ \widetilde{\Phi}_2(p,\varphi)=\dfrac{shp(\varphi_1-\varphi)}{shp(\varphi_1-\varphi_0)} & , \quad if \quad \varphi_0<\varphi<\varphi_1\end{cases}$$

Solution of the initial-boundary value problems for the heat equation (6) is obtained in the integral forms of heat potentials:

$$u_1(r,\varphi,t)=\int_0^\infty\int_0^{\varphi_0}u_{01}(r_1,\varphi^|)G_{11}(r,\varphi,r_1,\varphi^|,t)dr_1d\varphi^|+$$
$$+\int_0^\infty\int_{\varphi_0}^{\varphi_1}u_{01}(r_1,\varphi^|)G_{12}(r,\varphi,r_1,\varphi^|,t)dr_1d\varphi^|+$$
$$+\int_0^t d\tau\int_0^\infty\int_0^{\varphi_0}f_1(r_1,\varphi^|)G_{11}(r,\varphi,r_1,\varphi^|,t-\tau)dr_1d\varphi^|+$$
$$+\int_0^t d\tau\int_0^\infty\int_{\varphi_0}^{\varphi_1}f_2(r_1,\varphi^|)G_{12}(r,\varphi,r_1,\varphi^|,t-\tau)dr_1d\varphi^|$$

$$u_2(r,\varphi,t)=\int_0^\infty\int_0^{\varphi_0}u_{01}(r_1,\varphi^|)G_{21}(r,\varphi,r_1,\varphi^|,t)dr_1d\varphi^|+$$
$$+\int_0^\infty\int_{\varphi_0}^{\varphi_1}u_{01}(r_1,\varphi^|)G_{22}(r,\varphi,r_1,\varphi^|,t)dr_1d\varphi^|+$$
$$+\int_0^t d\tau\int_0^\infty\int_0^{\varphi_0}f_1(r_1,\varphi^|)G_{21}(r,\varphi,r_1,\varphi^|,t-\tau)dr_1d\varphi^|+$$
$$+\int_0^t d\tau\int_0^\infty\int_{\varphi_0}^{\varphi_1}f_2(r_1,\varphi^|)G_{22}(r,\varphi,r_1,\varphi^|,t-\tau)dr_1d\varphi^|$$

The solution of the initial-boundary value problems (7)-(11) correctly are solved in the functional space with a weighted integral metric and established a priori estimates in the weighted Sobolev space $H_{\mu-l-1}^{(l+2,l/2+1)}(D_T)$

for $0<\varphi_0<\varphi_1<2\pi$ with the conditions

$$1/2<1+l-\mu<\lambda_0,\ \lambda_0=\min\{\lambda_1,1\},\quad \lambda_1>\frac{1}{2}$$

where $\mu>0$ the real number of exponent's weight, $\lambda_1$ - the smallest positive number of the transcendent equation:

$$\kappa_1 ctg\lambda\varphi_0+\kappa_2 ctg\lambda(\varphi_1-\varphi_0)=0$$

for the first initial-boundary value problem.

**Theorem 1.** Let $l\geq 0$ be integer number , $\mu>0$ real number and $\lambda_0=\min\{1,\lambda_1\}$ with following conditions

$$1/2<1+l-\mu<\lambda_0 \quad \text{for} \quad \lambda_1>\frac{1}{2}.$$

For $i$=1,2 functions $u_{0i}(r,\varphi)\in H_{\mu-l-1}^{(l+1)}(\Omega)$ satisfy the heat conservation law (6) of $l$ orders for $t=0$ then there exist a unique and regularity solution $u_i(r,\varphi,t)\in H_{\mu-l-1}^{(l+2,l/2+1)}(Q_T^{(i)})$ for the initial boundary value problem (7) - (11). Moreover, the function $u(r,\varphi,t)$ holds the following estimate

$$\sum_{i=1}^{2}\|u\|_{H_{\mu-l-1}^{(l+2,l/2+1)}(D_T)}\leq C\sum_{i=1}^{2}\left\{\|u_{0i}\|_{H_{\mu-l-1}^{(l+1}(D)}+\|f_i\|_{H_{\mu-l-1}^{(l,l/2)}(D_T)}\right\}$$

where $C$ is independent constant from $u_{0i},f_i$ and t.

## V. CONSTRICTION OF THE TEMPERATURE FIELD AND BOUNDARY OF THE PHASE TRANSITION

Distribution of the temperature for the Stefan problem (1)-(5) we will search as sum the heat potentials with unknown densities $\omega_i(r,\varphi,t)$ for $i$ =1,2 and unknown $\xi(r,t)$ boundary:

$$u_1(r,\varphi,t)=\int_0^\infty\int_0^{\xi(r_1,\tau)}u_{01}(r_1,\varphi^|)G_{11}(r,\varphi,r_1,\varphi^|,t)dr_1d\varphi^|+$$
$$+\int_0^\infty\int_{\xi(r_1,\tau)}^{\varphi_1}u_{02}(r_1,\varphi^|)G_{12}(r,\varphi,r_1,\varphi^|,t)dr_1d\varphi^|+$$
$$+\int_0^t d\tau\int_0^\infty dr_1\int_0^{\xi(r_1,\tau)}f_1(r_1,\varphi^|,\tau)G_{11}(r,\varphi,r_1,\varphi^|,t-\tau)d\varphi^|+$$
$$+\int_0^t d\tau\int_0^\infty dr_1\int_{\xi(r_1,\tau)}^{\varphi_1}f_2(r_1,\varphi^|,\tau)G_{12}(r,\varphi,r_1,\varphi^|,t-\tau)d\varphi^|+$$
$$+2a_1^2\int_0^t d\tau\int_0^\infty P_1(r_1,\tau)\left[\frac{\partial G_{11}(r,\varphi,r_1,\varphi^|,t)}{\partial\varphi^|}\right]_{\varphi^|=0}dr_1-$$
$$+2a_1^2\int_0^t d\tau\int_0^\infty P_2(r_1,\tau)\left[\frac{\partial G_{12}(r,\varphi,r_1,\varphi^|,t)}{\partial\varphi^|}\right]_{\varphi^|=\varphi_1}dr_1+$$
$$+\int_0^t d\tau\int_0^\infty dr_1\int_0^{\xi(r_1,\tau)}\omega_1(r_1,\varphi^|,\tau)G_{11}(r,\varphi,r_1,\varphi^|,t-\tau)d\varphi^|+$$
$$+\int_0^t d\tau\int_0^\infty dr_1\int_{\xi(r_1,\tau)}^{\varphi_1}\omega_2(r_1,\varphi^|,\tau)G_{12}(r,\varphi,r_1,\varphi^|,t-\tau)d\varphi^|+$$
$$+\int_0^t d\tau\int_0^\infty K_{\lambda_1}[\xi(r_1,\tau)]\left[G_{11}(r,\varphi,r_1,\varphi^|,t)+G_{12}(r,\varphi,r_1,\varphi^|,t)\right]_{\varphi^|=\xi(r_1,\tau)}dr_1$$

$$u_2(r,\varphi,t)=\int_0^\infty\int_0^{\xi(r_1,\tau)}u_{01}(r_1,\varphi^|)G_{21}(r,\varphi,r_1,\varphi^|,t)dr_1d\varphi^|+$$
$$+\int_0^\infty\int_{\xi(r_1,\tau)}^{\varphi_1}u_{02}(r_1,\varphi^|)G_{22}(r,\varphi,r_1,\varphi^|,t)dr_1d\varphi^|+$$
$$+\int_0^t d\tau\int_0^\infty dr_1\int_0^{\xi(r_1,\tau)}f_1(r_1,\varphi^|,\tau)G_{21}(r,\varphi,r_1,\varphi^|,t-\tau)d\varphi^|$$
$$+\int_0^t d\tau\int_0^\infty dr_1\int_{\xi(r_1,\tau)}^{\varphi_1}f_2(r_1,\varphi^|,\tau)G_{22}(r,\varphi,r_1,\varphi^|,t-\tau)d\varphi^|+$$
$$+2a_1^2\int_0^t d\tau\int_0^\infty P_1(r_1,\tau)\left[\frac{\partial G_{21}(r,\varphi,r_1,\varphi^|,t)}{\partial\varphi^|}\right]_{\varphi^|=0}dr_1-$$


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$$-2a_1^2\int_0^t d\tau\int_0^\infty P_2(r_1,\tau)\left[\frac{\partial G_{22}(r,\varphi,r_1,\varphi^|,t)}{\partial\varphi^|}\right]_{\varphi^|=\varphi_1} dr_1+$$

$$+\int_0^t d\tau\int_0^\infty dr_1\int_0^{\xi(r_1,\tau)}\omega_1(r_1,\varphi^|,\tau)G_{21}(r,\varphi,r_1,\varphi^|,t-\tau)d\varphi^|+$$

$$+\int_0^t d\tau\int_0^\infty dr_1\int_{\xi(r_1,\tau)}^{\varphi_1}\omega_2(r_1,\varphi^|,\tau)G_{22}(r,\varphi,r_1,\varphi^|,t-\tau)d\varphi^|+$$

$$+\int_0^t d\tau\int_0^\infty K_{\lambda_1}\left[\xi(r_1,\tau)\right]\left[G_{21}(r,\varphi,r_1,\varphi^|,t)+G_{22}(r,\varphi,r_1,\varphi^|,t)\right]\Big|_{\varphi^|=\xi(r_1,\tau)}dr_1$$

These integral representations for the problem (1) - (5) in the form of heat potentials are satisfied initial and boundary conditions (2)-(3) and the first Stefan condition. Using the second Stefan conditions we get equation for motion of the diving boundary for two-phase Stefan problem

$$\frac{\partial\xi(r,t)}{\partial t}-\beta^2\left(\frac{\partial^2\xi(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial\xi(r,t)}{\partial r}-\frac{\lambda_1^2}{r^2}\xi(r,t)\right)=0$$

with coefficient

$$\beta^2=\frac{\kappa_1 a_1^2+\kappa_2 a_2^2}{2\rho(\kappa_1+\kappa_2)q^*}$$

Having been satisfied in each area head operators

$$L_i\left[\cdot\right]=\frac{\partial}{\partial t}-a_i^2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)$$

the integral representations for the problem (1) - (5) can determine as the Volterra and Fredholm integral equations of the second kind.

$$\omega_1(r,\varphi,t)+\int_0^t d\tau\int_0^\infty dr_1\int_0^{\xi(r_1,\tau)}r_1\omega_1(r_1,\varphi^|,\tau)H_{11}(r,\varphi,\tau,r_1,\varphi^|,t-\tau)d\varphi^|+$$

$$+\int_0^t d\tau\int_0^\infty dr_1\int_{\xi(r,\tau)}^{\varphi_1}r_1\omega_2(r_1,\varphi^|,\tau)H_{12}(r,\varphi,\tau,r_1,\varphi^|,t-\tau)d\varphi^|=F_1(r,\varphi,t)$$

$$\omega_2(r,\varphi,t)+\int_0^t d\tau\int_0^\infty dr_1\int_0^{\xi(r_1,\tau)}r_1\omega_1(r_1,\varphi^|,\tau)H_{21}(r,\varphi,r_1,\varphi^|,t-\tau)d\varphi^|+$$

$$+\int_0^t d\tau\int_0^\infty dr_1\int_{\xi(r,\tau)}^{\varphi_1}r_1\omega_2(r_1,\varphi^|,\tau)H_{22}(r,\varphi,\tau,r_1,\varphi^|,t-\tau)d\varphi^|=F_2(r,\varphi,t)$$

Here

$$H_{ij}(r,\varphi,r_1,\varphi^|,t)=\sum_{n=1}^\infty h_{ij}^{(n)}(r,r_1,t)\Phi_{in}(\varphi^|,r_1,t)\Phi_{jn}(\varphi,r,t)$$

$$H_{ij}(r,\varphi,r_1,\varphi^|,t)=\sum_{n=1}^\infty h_{ij}^{(n)}(r,r_1,t)\Phi_{in}(\varphi^|,r_1,t)\Phi_{jn}(\varphi,r,t)$$

$$h_{mi}^{(n)}(r,r_1,t)=d_m^{(n)}(r,r_1,t)G_{mi}^{(n)}+c_m^{(n)}(r,r_1,t)\frac{\partial G_{mi}^{(n)}}{\partial r}$$

$$d_m^{(n)}(r,r_1,t)=\frac{\partial\Phi_{mn}}{\partial\xi}L_0^{(m)}\left[\xi(r,t)\right]+a_m^2\frac{\partial^2\Phi_{mn}}{\partial\xi^2}\cdot\left(\frac{\partial\xi}{\partial r}\right)^2$$

$$c_m^{(n)}(r,r_1,t)=2a_m^2\frac{\partial\Phi_{mn}}{\partial\xi}\frac{\partial\xi}{\partial r}$$

$$L_0^{(m)}\left[\cdot\right]=\frac{\partial}{\partial t}-a_m^2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)$$

$$F_1(r,\varphi,t)=\int_0^\infty\int_0^{\xi(r_1,\tau)}u_{01}(r_1,\varphi^|)G_{11}(r,\varphi,r_1,\varphi^|,t)dr_1d\varphi^|+$$

$$+\int_0^\infty\int_{\xi(r_1,\tau)}^{\varphi_1}u_{02}(r_1,\varphi^|)G_{12}(r,\varphi,r_1,\varphi^|,t)dr_1d\varphi^|+$$

$$+\int_0^t d\tau\int_0^\infty dr_1\int_0^{\xi(r_1,\tau)}f_1(r_1,\varphi^|,\tau)G_{11}(r,\varphi,r_1,\varphi^|,t-\tau)d\varphi^|+$$

$$+\int_0^t d\tau\int_0^\infty dr_1\int_{\xi(r_1,\tau)}^{\varphi_1}f_2(r_1,\varphi^|,\tau)G_{12}(r,\varphi,r_1,\varphi^|,t-\tau)d\varphi^|+$$

$$+2a_1^2\int_0^t d\tau\int_0^\infty P_1(r_1,\tau)\left[\frac{\partial G_{11}(r,\varphi,r_1,\varphi^|,t)}{\partial\varphi^|}\right]_{\varphi^|=0}dr_1-$$

$$-2a_1^2\int_0^t d\tau\int_0^\infty P_2(r_1,\tau)\left[\frac{\partial G_{12}(r,\varphi,r_1,\varphi^|,t)}{\partial\varphi^|}\right]_{\varphi^|=\varphi_1}dr_1+$$

$$+\int_0^t d\tau\int_0^\infty K_{\lambda_1}\left[\xi(r_1,\tau)\right]\left[G_{11}(r,\varphi,r_1,\varphi^|,t)+G_{12}(r,\varphi,r_1,\varphi^|,t)\right]\Big|_{\varphi^|=\xi(r_1,\tau)}dr_1+$$

$$+\int_0^t d\tau\int_0^\infty K_{\lambda_1}\left[\xi(r_1,\tau)\right]\left[G_{11}(r,\varphi,r_1,\varphi^|,t)+G_{12}(r,\varphi,r_1,\varphi^|,t)\right]\Big|_{\varphi^|=\xi(r_1,\tau)}dr_1$$

$$F_2(r,\varphi,t)=\int_0^\infty\int_0^{\xi(r_1,\tau)}u_{01}(r_1,\varphi^|)G_{21}(r,\varphi,r_1,\varphi^|,t)dr_1d\varphi^|+$$

$$+\int_0^\infty\int_{\xi(r_1,\tau)}^{\varphi_1}u_{02}(r_1,\varphi^|)G_{22}(r,\varphi,r_1,\varphi^|,t)dr_1d\varphi^|+$$

$$+\int_0^t d\tau\int_0^\infty dr_1\int_0^{\xi(r_1,\tau)}f_1(r_1,\varphi^|,\tau)G_{21}(r,\varphi,r_1,\varphi^|,t-\tau)d\varphi^|+$$

$$+\int_0^t d\tau\int_0^\infty dr_1\int_{\xi(r_1,\tau)}^{\varphi_1}f_2(r_1,\varphi^|,\tau)G_{22}(r,\varphi,r_1,\varphi^|,t-\tau)d\varphi^|+$$

$$+2a_1^2\int_0^t d\tau\int_0^\infty P_1(r_1,\tau)\left[\frac{\partial G_{21}(r,\varphi,r_1,\varphi^|,t)}{\partial\varphi^|}\right]_{\varphi^|=0}^{(11)}dr_1-$$

$$-2a_1^2\int_0^t d\tau\int_0^\infty P_2(r_1,\tau)\left[\frac{\partial G_{22}(r,\varphi,r_1,\varphi^|,t)}{\partial\varphi^|}\right]_{\varphi^|=\varphi_1}dr_1+$$

$$+\int_0^t d\tau\int_0^\infty K_{\lambda_1}\left[\xi(r_1,\tau)\right]\left[G_{21}(r,\varphi,r_1,\varphi^|,t)+G_{22}(r,\varphi,r_1,\varphi^|,t)\right]\Big|_{\varphi^|=\xi(r_1,\tau)}dr_1$$

$$H_{ij}(r,\varphi,r_1,\varphi^|,t)=\sum_{n=1}^\infty h_{ij}^{(n)}(r,r_1,t)\Phi_{1n}(\varphi^|,r_1,t)\Phi_{1n}(\varphi,r,t)$$

Solution of the Volterra and Fredholm integral equations of the second kind can obtained analytically, using integral [8]

$$\int_0^\infty x^{\gamma-1}e^{-\alpha x}I_{2\nu}(2c\sqrt{x}dx=\frac{\Gamma(\gamma+\nu+\frac{1}{2})}{\Gamma(2\nu+1)}c^{-1}e^{\frac{c^2}{2\alpha}}\alpha^{-\gamma}M_{-\gamma,\nu}(\frac{c^2}{\alpha})$$

where $\mathrm{Re}(\gamma+\nu+\frac{1}{2})>0$

$M_{\mu,\nu}(z)$ is Whittaker function:

$$M_{\mu,\nu}(z)=\frac{z^{\nu+\frac{1}{2}}}{2^{2\nu}B(\nu+\mu+\frac{1}{2},\nu-\mu+\frac{1}{2})}\int_{-1}^{1}(1+x)^{\nu-\mu-\frac{1}{2}}(1-x)^{\nu+\mu-\frac{1}{2}}e^{\frac{zx}{2}}dx$$

$B(\nu+\mu+\frac{1}{2},\nu-\mu+\frac{1}{2})$ is the Betta function.

Estimation for the kernel of the Volterra – Fredholm matrix integral equations of the second kind is obtained

$$\left|H_{ij}(r,\varphi,r_1,\varphi^{|},t)\right| \le M_0 \frac{e^{-\frac{(r-r_1)^2}{8a_0^2 t}}}{4a_0^2 t^{\frac{3}{2}}}$$

Resolvent of the integral equation has estimation

$$|R| \le M_0 e^{b_0\sqrt{t}} \qquad (12)$$

$M_0, b_0$ are the positive constants which depend on the certain coordinates of moving boundaries.

The existence and uniqueness of solution the Volterra–Fredholm integral equations can be proved by the estimation (12).The representation of the exact solution of the Volterra–Fredholm matrix integral equations of the second kind is defined in the reproducing kernel space $D_T$ .

$$\omega = R * F - F$$

## VI. Statement of the Main Result

**Theorem 2.** Let $l \ge 0$ be integer number, $\mu > 0$ is real number and $\lambda_0 = \min\{1, \lambda_1\}$ which satisfies following conditions $1/2 < 1 + l - \mu < \lambda_0$ for $\lambda_1 > \frac{1}{2}$. For $i$=1,2 functions $p_i(r,t) \in H_{\mu-l-1}^{(l+3/2, l/2+3/4)}(\gamma_T^{(i)})$, $u_{0i}(r,\varphi) \in H_{\mu-l-1}^{(l+1)}(D)$, $f_i(r,\varphi,t) \in H_{\mu-l-1}^{(l,l/2)}(D_T)$ satisfy the $l$ order heat conservation law (6) for $t=0$. Then there exist a unique and regularity solution $u(r,\varphi,t) \in H_{\mu-l-1}^{(l+2,l/2+1)}(D_T)$ for Stefan problem (1) - (5) and motion of the diving boundary $\xi(r,t)$ is defined as function:

$$\xi(r,t) = \int_0^\infty r_1 \xi_0(r_1) \frac{e^{-\frac{r^2+r_1^2}{4\beta^2 t}}}{\beta^2 t} I_{\lambda_1}\left(\frac{r r_1}{2\beta^2 t}\right) dr_1$$

here

$$\xi_0(r) = r\sin\varphi_0, 0 < \varphi_0 < \pi \ ,$$

$$\beta^2 = \frac{\kappa_1 a_2^2 + \kappa_2 a_1^2}{2 a_1^2 a_2^2 \rho q^*}$$

$q^*$ , $\rho$ - are respectively the hidden heat and density of crystallization or melting per unit mass of the solid. Moreover, $u(r,\varphi,t)$ the following estimate holds

$$\|u\|_{H_{\mu-l-1}^{(l+2,l/2+1)}(D_T)} \le C\Big\{\|u_0\|_{H_{\mu-l-1}^{(l+1)}(D)} + \|f\|_{H_{\mu-l-1}^{(l,l/2)}(D_T)} + $$
$$+ \|p\|_{H_{\mu-l-1}^{(l+3/2,l/2+3/4)}(\gamma_T)} + \left\|K_{\lambda_1}[\xi(r,t)]\right\|_{H_{\mu-l-1}^{(l+1/2,l/2+1/4)}(\Gamma_T)}\Big\}$$

There $C$ is constant which is independent from $u_0, f, p$ and $t$. $K_{\lambda_1}[\xi(r,t)]$ is the operator

$$K_{\lambda_1}[\cdot] = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\lambda_1^2}{r^2}$$

## VII. Conclusion

Experimental selection of cooling or heating parameters for the thermal regimes is costly and not always feasible process what is a starting point to consider the two-phase Stefan problem in the weak formulation, therefore important argument for research is to find an analytical solution. We have presented the analytic method of weak solution for the two-phase Stefan problem which expected to exist for all domain. This method analytic based on constructing Green's function and is required a good deal using the heat potential theory. For unknown two independent thermodynamically parameters (the temperature and the phase function) Green's function has played a key role in the obtained integral representation of the temperature field and the description of behavior of the phase transition temperature distribution. There are the main points that temperature field would satisfy the heat equation with their initial-boundary condition and the phase of diving boundary satisfies the law on the free boundary (the Stefan conditions) and they obtained equation of motion of the diving boundary deliver full information of the behavior for the phase change boundary. There the main mathematical difficulty associates the construction of the corresponding Green's function and motion of the diving boundary. Using a priori estimates we obtain an uniqueness and regularity of the weak solution of the two-phase Stefan problem. We take the first step in developing our analytical method for solving free boundary problem where each area of heat and mass transfer with moving boundaries are need in the constriction own Green's function of the model coupled task with the initial and fixed boundary conditions.

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