# New Solutions for the Massive Dirac Equation with Electric Potential, Employing Biquaternionic and Pseudoanalytic Functions

M.P. Ramirez T. *IAENG, Member*, V. Martinez-Garza-Garcia.

***Abstract*—Employing elements of Quaternionic Analysis and Pseudoanalytic Function Theory, we propose a method to obtain new solutions, both analytic and numerical, for the massive Dirac equation with an arbitrary electric potential, depending upon one spatial variable. We also discuss how to extend the results to other kind of potentials.**

***Index Terms*—Dirac Equation, Pseudoanalytic Functions, Quaternionic Analysis.**

## I. Introduction

The study of the Dirac equation acquired more relevance in the field of Nuclear Medicine with, *e.g.*, the increased need of producing radiopharmaceuticals. Particularly, the solutions of this equation make possible a better understanding of the charge exchange reaction of $^{18}O(p,n)^{18}F$ to form the $^{18}F$, employed in Positron Emission Tomography (see [8]), an it is clear that a deeper understanding of new solutions will make available wider knowledge of the production procedures for another isotopes employed in the field.

Therefore, the engineering sciences might pay special attention to the foundations that make possible to study the behavior of the quantum particles involved in such processes, and in this particular case, the solutions of the Dirac equation for massive particles (as it is the electron), reached when applying elements of Quaternionic Analysis [5] and Modern Pseudoanalytic Function Theory [4].

In general, the discovering of the Dirac equation was fundamental for understanding several branches of Theoretical and Applied Physics. Yet, from the mathematical point of view, it became specially challenging to obtain analytic solutions (precisely those that make clearer the behavior of quantum particles) when considering more general cases, due its mathematical complexity. An interesting alternative to deal with this problem was to rewrite the Dirac equation in such a form that the degree of the algebra generated by the Pauli-Dirac matrices was equal to 4, the number of such matrices.

A positive answer in this direction was provided in [5], where there were posed a set of matrix transformations for rewriting the massive Dirac equation in quaternionic form. Moreover, employing the elements of Quaternionic Analysis, new classes of solutions were found thereafter. Among those, we can find some studies for the cases when particles with spin $\frac{1}{2}$ are influenced by electric or scalar potentials [7].

**M.P. Ramirez T.** is with Asaji Audio Internacional S.A. de C.V., Guadalupe I.R. 687, C.P. 16030, Mexico, marco.ramirez@asajiaudio.com
**V. Martinez-Garza-Garcia** is with the Engineering Faculty of Universidad La Salle, B. Franklin 47, C.P. 06140, Mexico, vanmtzg@hotmail.com

This work is fully dedicated to extend the class of solutions first posed employing the ideas of [7], by establishing a connection with the Vekua equation [10], which plays a central role in the Pseudoanalytic Function Theory [1]. This connection will allow us to obtain a set of new solutions, both analytic and numerical, for the case when the particles are affected by an arbitrary electric potential, depending upon a single spatial variable.

After a brief study of some elements belonging to Quaternionic Analysis [5], and to Pseudoanalytic Function Theory [1][4], we expose how to rewrite the classical massive Dirac equation into a quaternionic form. Then we analyze two possible ways for obtaining new sets of solutions, by virtue of separating the quaternionic system into a pair of decoupled partial differential equations, that includes a special class of Vekua equation.

Thus, employing the so-called formal powers [1], we propose a technique for obtaining new solutions for the classical Dirac equation.

## II. Preliminaries

### A. Elements of quaternionic analysis

Following [5], let us consider the set of biquaternionic functions $\mathbb{H}(\mathbb{C})$, whose elements $q$ have the form:

$$q = \sum_{k=0}^{3} q_k \mathbf{e}_k, \tag{1}$$

where $q_k$ are all complex-valued functions $q_k = \mathrm{Re}q_k + i\mathrm{Im}q_k$, being $i$ the standard imaginary unit $i^2 = -1$, the element $e_0 = 1$, and the set $\{\mathbf{e}_k\}_{k=1}^{3}$ are the classical quaternionic units, possessing the properties:

$$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -1, \tag{2}$$
$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -1.$$

Notice that, by definition, the imaginary unit $i$ commutes with the quaternionic units: $i\mathbf{e}_k = \mathbf{e}_k i$.

Because the quaternionic units do not commute, as shown in (2), we shall denote the multiplication by the right-hand side of the biquaternion $q$ by the biquaternion $p$ as follows:

$$M^p q = qp.$$

*1) The Moisil-Theodoresco operator:* The partial differential operator $D$ is introduced as:

$$D = \mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2 + \mathbf{e}_3\partial_3,$$

where $\partial_k = \frac{\partial}{\partial x_k}$, $k = 1, 2, 3$; is defined in the space of at least once-derivable quaternions, with respect to the spacial variables $(x_1, x_2, x_3)$.

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### B. Elements of pseudoanalytic functions

According to [1], let the pair of complex-valued functions $F$ and $G$ satisfy the condition:

$$\mathrm{Im}\left(\overline{F}G\right) > 0, \tag{3}$$

where $\overline{F}$ represents the complex conjugation of $F$: $\overline{F} = \mathrm{Re}F - i\mathrm{Im}F$. Thus, any complex function $W$ can be written as the linear combination:

$$W = \phi F + \psi G,$$

where $\phi$ and $\psi$ are real functions. A pair of functions satisfying the condition (3), is called a *generating pair*. L. Bers introduced the concept of the derivative of a function $W$ with respect to the generating pair $(F,G)$ according to the expression:

$$\partial_{(F,G)}W = \partial_z(\phi)\cdot F + \partial_z(\psi)\cdot G, \tag{4}$$

where $\partial_z = \partial_\chi - i\partial_\nu$. But this derivative will exist if and only if the following condition holds:

$$\partial_{\overline{z}}(\phi)\cdot F + \partial_{\overline{z}}(\psi)\cdot G = 0, \tag{5}$$

where $\partial_z = \partial_\chi + i\partial_\nu$. Introducing the notations:

$$\begin{aligned} A_{(F,G)} &= \frac{\overline{F}\partial_z G - \overline{G}\partial_z F}{F\overline{G} - G\overline{F}}, \\ a_{(F,G)} &= \frac{\overline{G}\partial_{\overline{z}} F - \overline{F}\partial_{\overline{z}} G}{F\overline{G} - G\overline{F}}, \\ B_{(F,G)} &= \frac{F\partial_z G - G\partial_z F}{F\overline{G} - G\overline{F}}, \\ b_{(F,G)} &= \frac{F\partial_{\overline{z}} G - G\partial_{\overline{z}} F}{F\overline{G} - G\overline{F}}; \end{aligned} \tag{6}$$

the equation (4) can be rewritten into:

$$\partial_{(F,G)}W = \partial_z W - A_{(F,G)}W - B_{(F,G)}\overline{W} \tag{7}$$

whereas the condition (5) will turn into

$$\partial_{\overline{z}}W - a_{(F,G)}W - b_{(F,G)}\overline{W} = 0. \tag{8}$$

This is the so-called Vekua equation, and it will play a central role in further paragraphs. Indeed, L. Bers proved that the general solution for the Vekua equation (8) accepts the representation:

$$W = \sum_{n=0}^{\infty} Z^{(n)}\left(a_n, z_0; z\right) \tag{9}$$

where the functions $Z^{(n)}\left(a_n, z_0; z\right)$ are called *formal powers*. More precisely, the formal power $Z^{(0)}\left(a_0, z_0; z\right)$, with complex coefficient $a_0$, center at $z_0$, and depending upon $z = \chi + i\nu$, is defined according to the expression:

$$Z^{(0)}\left(a_0, z_0; z\right) = \lambda F(z) + \mu G(z),$$

where $\lambda$ and $\mu$ are both constants that fulfill the condition:

$$\lambda F(z_0) + \mu G(z_0) = a_0.$$

The formal powers with higher exponents $n > 0$ are defined according to the recursive formulae:

$$Z^{(n+1)}\left(a_n, z_0; z\right) = nG(z)\mathrm{Re}\int_\Lambda F^*(z)Z^{(n)}\left(a_n, z_0; z\right)dz +$$
$$+ nF(z)\mathrm{Re}\int_\Lambda G^*(z)Z^{(n)}\left(a_n, z_0; z\right)dz;$$

where:

$$\begin{aligned} F &= p(\nu), & G &= ip^{-1}(\nu), \\ F^* &= -iF(\nu) & G^* &= -iG(\nu). \end{aligned}$$

Beside, $\Lambda$ represents a rectifiable curve going from $z_0$ until $z$, and $p(\nu)$ is a non-vanishing function within some domain $\Omega$.

*Remark 1:* As it was proven in [1], let us consider $a_n = a_n' + ia_n''$, where $a_n'$ and $a_n''$ are real constants. Thus, any formal power $Z^{(n)}\left(a_n, z_0; z\right)$ can be expressed by the linear combination:

$$Z^{(n)}\left(a_n, z_0; z\right) = a_n' Z^{(n)}\left(1, z_0; z\right) + a_n'' Z^{(n)}\left(i, z_0; z\right).$$

This implies that we shall focus our attention into the construction of the formal powers $Z^{(n)}\left(1, z_0; z\right)$ and $Z^{(n)}\left(i, z_0; z\right)$, because any other formal power will arise from these two sets of functions.

## III. The massive Dirac equation with electric potential

Let us consider the massive Dirac equation with electric potential:

$$\left[\gamma_0\partial_t - \sum_{k=1}^{3}\gamma_k\partial_k + im + \gamma_0 u(x_1)\right]\Phi(t,x) = 0, \tag{10}$$

where $m$ represents the mass of the particle, $u$ denotes the electric potential only depending upon $x_1$, $\partial_t = \frac{\partial}{\partial t}$, $t$ represents the time variable, and $\gamma_k$, $k = 0,1,2,3$; are the Pauli-Dirac matrices:

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{11}$$
$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$
$$\gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

If we consider the time-harmonic representation of the function $\Phi$, solution of (10): $\Phi(t,x) = e^{i\omega t}\varphi(x)$, where $\omega$ represents the energy of the particle, the Dirac equation (10) will turn into:

$$\left[i\omega\gamma_0 - \sum_{k=1}^{3}\gamma_k\partial_k + im + \gamma_0 u(x_1)\right]\varphi(x) = 0, \tag{12}$$

Let us consider the matrix operators $\mathbf{A}$ and $\mathbf{A}^{-1}$ [5]:

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & i & i & 0 \end{pmatrix},$$

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$$\mathbf{A}^{-1} = \begin{pmatrix} 0 & -i & -1 & 0 \\ -1 & 0 & 0 & -i \\ 1 & 0 & 0 & -i \\ 0 & i & -1 & 0 \end{pmatrix}. \tag{13}$$

In [7] it was shown that applying the matrices $\gamma_k$, $k = \overline{0,3}$, $\mathbf{A}$ and $\mathbf{A}^{-1}$, to the differential operator of the Dirac equation (12) as follows:

$$-\mathbf{A}\gamma_1\gamma_2\gamma_3 \left[\gamma_0\partial_t - \sum_{k=1}^{3}\gamma_k\partial_k + im + \gamma_0 u(x_1)\right]\mathbf{A}^{-1},$$

we will obtain a biquaternionic Dirac equation of the form:

$$\left(D - M^{g(x_1)\mathbf{e}_1 + m\mathbf{e}_2}\right) f(x) = 0, \tag{14}$$

where $f(x)$ is a full biquaternionic function of the form (1), $g(x_1)$ represents

$$g(x_1) = iu(x_1) + i\omega,$$

and

$$f(x) = \mathbf{A}\varphi(x).$$

We shall remark that the matrix transformations $\mathbf{A}$ and $\mathbf{A}^{-1}$ provoke the reflection of the $x_3$-axis when applied [5]. More precisely, we will have that:

$$\mathbf{A}^{-1}\varphi(x_1, x_2, x_3) \to f(x_1, x_2, -x_3),$$

as well

$$\mathbf{A}f(x_1, x_2, x_3) \to \varphi(x_1, x_2, -x_3).$$

*A. New class of solutions of the biquaternionic Dirac equation employing pseudoanalytic functions*

Following the procedure posed in [7], let us represent the biquaternion $f(x)$, solution of (14), in the form:

$$f = \alpha Q, \tag{15}$$

where $\alpha$ is a purely scalar function and $Q$ is a biquaternion. Substituting (15) into (14), and taking into account the Leibniz rule for deriving a quaternionic product [5], we obtain:

$$D\alpha \cdot Q + \alpha DQ - \alpha Q g\mathbf{e}_1 - \alpha Q m\mathbf{e}_2 = 0. \tag{16}$$

Let us suppose now, as appointed in [7], that the following equality holds:

$$DQ = Qm\mathbf{e}_2. \tag{17}$$

Thus, from (16) we immediately obtain:

$$D\alpha \cdot Q - \alpha Q g\mathbf{e}_1 = 0. \tag{18}$$

Once more, employing the procedure proposed in [7], let us assume that $Q$ possesses the particular form:

$$Q = Q_1 = q_0 + q_1\mathbf{e}_1, \tag{19}$$

then the quaternionic unit $\mathbf{e}_1$ will commute with the biquaternion $Q_1$: $Q_1\mathbf{e}_1 = \mathbf{e}_1 Q_1$. If we assume that $Q_1$ is not a *zero divisor* [5] (this is, that there exist $Q^{-1}$), the equation (18) can be simplified to:

$$D\alpha_1 - \alpha_1 g\mathbf{e}_1 = 0,$$

where the notation $\alpha_1$ enhances that, in further calculations, the form of this scalar function will be valid only when considering (19). Since the function $g = g(x_1)$, we have that:

$$\alpha_1 = K_1 e^{\int g(x_1)dx_1}, \tag{20}$$

where $K_1$ is an arbitrary complex constant. Let us consider now the equation (17) taking into account (19):

$$(\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2 + \mathbf{e}_3\partial_3)(q_0 + q_1\mathbf{e}_1) = (q_0 + q_1\mathbf{e}_1)m\mathbf{e}_2.$$

Performing standard calculations, it is possible to show that this equation is equivalent to the system:

$$\begin{aligned} \partial_2 q_0 + \partial_3 q_1 &= q_0 m, \\ \partial_3 q_0 - \partial_2 q_1 &= q_1 m; \end{aligned} \tag{21}$$

being $\partial_1 q_0 = \partial_1 q_1 = 0$. As a matter of fact, this system was also obtained in [7], where a certain class of solutions were posed. This work will analyze a wider class of new solutions, based upon pseudoanalytic functions.

For this purpose, let us multiply the second equation of (21) by the imaginary unit $i$, and let us add the result to the first equation of (21). The result can be written in the form:

$$(\partial_2 + i\partial_3)(q_0 - iq_1) - m(q_0 + iq_1) = 0.$$

Introducing the notations:

$$\begin{aligned} \partial_{\overline{z}_1} &= \partial_2 + i\partial_3, \\ W_1 &= q_0 - iq_1, \\ p_1 &= e^{-imx_3}; \end{aligned} \tag{22}$$

the last equality can be rewritten as a Vekua equation:

$$\partial_{\overline{z}_1} W_1 - \frac{\partial_{\overline{z}_1} p_1}{p_1}\overline{W}_1 = 0, \tag{23}$$

for which the following functions constitute a generating pair:

$$F_1 = p_1,\; G_1 = \frac{i}{p_1}.$$

Indeed, its characteristic coefficients (6) posses the form:

$$\begin{aligned} A_{(F_1,G_1)} &= a_{(F_1,G_1)} = 0; \\ B_{(F_1,G_1)} &= \frac{\partial_{z_1} p_1}{p_1}, \\ b_{(F_1,G_1)} &= \frac{\partial_{\overline{z}_1} p_1}{p_1}. \end{aligned}$$

Furthermore, it is possible to analytically approach some of the first formal powers introduced in (9), since every one is a particular solution of (23), as posed in [1] and [4] (hereafter, for simplicity, we will consider the center of all formal powers $z_0 = 0$). More precisely:

$$\begin{aligned} Z_1^{(0)}(1, 0; z_1) &= e^{-imx_3}, \\ Z_1^{(0)}(i, 0; z_1) &= ie^{imx_3}, \\ Z_1^{(1)}(1, 0; z_1) &= x_2 e^{-imx_3} + \frac{i}{m}\sin(mx_3), \\ Z_1^{(1)}(i, 0; z_1) &= -\frac{1}{m}\sin(mx_3) + ix_2 e^{imx_3}, ... \end{aligned}$$

where $z_1 = x_2 + ix_3$. From here, and considering for instance the formal powers with coefficient 1, the quaternionic function $Q_1$ posed in (19), taking into account the notation introduced in (22), will have the form:

$$Q_1 = \mathrm{Re}Z_1^{(n)}(1, 0; z_1) - \mathbf{e}_1\mathrm{Im}Z_n^{(1)}(1, 0; z_1);$$

Thus, by virtue of (20), the function $f_1(x)$, solution of the quaternionic Dirac equation (14), will have the form:

$$f_1(x) = \tag{24}$$
$$= K_1\left(\mathrm{Re}Z_1^{(n)}\left(1,0;z_1\right) - \mathbf{e}_1\mathrm{Im}Z_n^{(1)}\left(1,0;z_1\right)\right)e^{\int g(x_1)dx_1}.$$

Still, there exists another possibility for approaching solutions of the equation (14) employing the idea posed in (15). Let us consider once more the equality (16), but let us suppose that the following equation holds:

$$DQ_2 = Q_2 g\mathbf{e}_1. \tag{25}$$

Then (16) will be simplified to the equation:

$$D\alpha_2 \cdot Q_2 = \alpha_2 m Q_2 \mathbf{e}_2. \tag{26}$$

In this case, it is convenient to consider $Q_2$ possessing the structure:

$$Q_2 = q_1'\mathbf{e}_1 + q_3'\mathbf{e}_3, \tag{27}$$

because in this way the quaternionic unit $\mathbf{e}_2$ will anticommute with $Q_2$: $Q_2\mathbf{e}_2 = -\mathbf{e}_2 Q_2$, therefore (26) can be written as:

$$D\alpha_2 \cdot Q_2 = -\alpha_2 m\mathbf{e}_2 Q_2.$$

If we assume that $Q_2$ is not a *zero divisor*, there will exist an inverse quaternion $Q_2^{-1}$, thus we can simply consider:

$$D\alpha_2 = -\alpha_2 m\mathbf{e}_2.$$

It is easy to check that the solution $\alpha_2$ has the form:

$$\alpha_2 = K_2 e^{-mx_2}. \tag{28}$$

We can now examine the equation (25), taking into account (27):

$$\left(\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2 + \mathbf{e}_3\partial_3\right)\left(q_1'\mathbf{e}_1 + q_3'\mathbf{e}_3\right) = \left(q_1'\mathbf{e}_1 + q_3'\mathbf{e}_3\right)g\mathbf{e}_1.$$

The system of equations, arising from the expression above, has the form:

$$\begin{aligned} \partial_1 q_1' + \partial_3 q_3' &= q_1' g, \\ \partial_3 q_1' - \partial_1 q_3' &= q_3' g, \end{aligned}$$

having $\partial_2 q_1' = \partial_2 q_3' = 0$. As performed before, when multiplying the first equation by $i$, and adding the result to the second equation, we will obtain the partial differential equation:

$$\left(\partial_3 + i\partial_1\right)\left(q_1' + iq_3'\right) - ig\left(q_1' - iq_3'\right) = 0.$$

Introducing the notations:

$$\begin{aligned} \partial_{\bar{z}_2} &= \partial_3 + i\partial_1, \\ W_2 &= q_1' + iq_3', \\ p_2 &= e^{i\int g dx_1}, \end{aligned} \tag{29}$$

the last differential equation will adopt the form of another Vekua equation:

$$\partial_{\bar{z}_2} W_2 - \frac{\partial_{\bar{z}_2} p_2}{p_2}\overline{W}_2 = 0, \tag{30}$$

for which a generating pair is composed by the functions:

$$F_2 = p_2,\ G_2 = \frac{i}{p_2},$$

and whose characteristic coefficients posses the form:

$$\begin{aligned} A_{(F_2,G_2)} &= a_{(F_2,G_2)} = 0; \\ B_{(F_2,G_2)} &= \frac{\partial_{z_2} p_2}{p_2}, \\ b_{(F_2,G_2)} &= \frac{\partial_{\bar{z}_2} p_2}{p_2}. \end{aligned}$$

The main difference of this case compared to the previous one, is that we can only analytically approach two formal powers, solutions of (30):

$$\begin{aligned} Z_2^{(0)}\left(1,0;z_2\right) &= e^{i\int g dx_1}, \\ Z_2^{(0)}\left(i,0;z_2\right) &= ie^{-i\int g dx_1}. \end{aligned}$$

Nevertheless, a variety of accurate numerical methods are available for approaching a considerable number of formal powers, taking into account the bounded domain $\Omega$ where the function $g(x_1)$ is defined, even for the cases when the boundary $\Gamma$ possesses non-smooth points. Some of the existing methods can be found, *e.g.*, in [2], [3] and [9].

Supposing we have compute a set of $2N+2$ formal powers within a bounded domain $\Omega$:

$$\left\{Z_2^{(n)}\left(1,0;z_2\right), Z_2^{(n)}\left(i,0;z_2\right)\right\}_{n=0}^{N},$$

where $z_2 = x_3 + ix_1$, and taking into account the notations introduced in (29), as well the functions $\alpha_2$ declared in (28) and $Q_2$ in (27), the solutions $f_2(x)$ of the quaternionic Dirac equation (14) will be constructed according to the expressions:

$$f_2(x) = \tag{31}$$
$$= K_2\left(\mathbf{e}_1\mathrm{Re}Z_2^{(n)}\left(1,0;z_2\right) + \mathbf{e}_3\mathrm{Im}Z_2^{(n)}\left(1,0;z_2\right)\right)e^{-mx_2}.$$

Thus, we have both analytic (24) and numerical (31) new classes of solutions for the quaternionic Dirac equation (14). For obtaining the solutions corresponding to the classical time-harmonic Dirac equation (12), it is only necessary to employ the linear operator $\mathbf{A}^{-1}$, introduced in (13), that transforms a full biquaternion $f$ solution of (14), into a four-dimensional vector function $\varphi$ solution of (12):

$$\varphi = \mathbf{A}^{-1}\left[f\right] = \begin{pmatrix} -i\tilde{f}_1 - \tilde{f}_2 \\ -\tilde{f}_0 - i\tilde{f}_3 \\ \tilde{f}_0 - i\tilde{f}_3 \\ i\tilde{f}_1 - \tilde{f}_2 \end{pmatrix}, \tag{32}$$

where $\tilde{f}_k$, $k = \overline{0,3}$; denotes the projection

$$f_k(x_1,x_2,x_3) \to \tilde{f}_k(x_1,x_2,-x_3).$$

More precisely, for the case of (24) we have:

$$\varphi_1 = \begin{pmatrix} iK_1\mathrm{Im}\tilde{Z}_n^{(1)}\left(1,0;z_1\right)e^{\int g(x_1)dx_1} \\ -K_1\mathrm{Re}\tilde{Z}_1^{(n)}\left(1,0;z_1\right)e^{\int g(x_1)dx_1} \\ K_1\mathrm{Re}\tilde{Z}_1^{(n)}\left(1,0;z_1\right)e^{\int g(x_1)dx_1} \\ -iK_1\mathrm{Im}\tilde{Z}_n^{(1)}\left(1,0;z_1\right)e^{\int g(x_1)dx_1} \end{pmatrix},$$

whereas for the case (31), the corresponding solution of (32) will be:

$$\varphi_2 = \begin{pmatrix} -iK_2\mathrm{Re}\tilde{Z}_2^{(n)}\left(1,0;z_2\right)e^{-mx_2} \\ -iK_2\mathrm{Im}\tilde{Z}_2^{(n)}\left(1,0;z_2\right)e^{-mx_2} \\ -K_2\mathrm{Im}\tilde{Z}_2^{(n)}\left(1,0;z_2\right)e^{-mx_2} \\ iK_2\mathrm{Re}\tilde{Z}_2^{(n)}\left(1,0;z_2\right)e^{-mx_2} \end{pmatrix}.$$

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## IV. Conclusions

The variety of solutions for the massive Dirac equation with electric potential, employing Quaternionic Analysis and Pseudoanalytic Function Theory, is far to be fully characterized. This is, the assumptions made in the previous paragraphs, about the election of the $p$ functions in the formulae (22) and (29), are not unique for rewriting the quaternionic Dirac equation into a special kind of Vekua equation. For example, if we consider

$$p_1 = e^{m(x_2 - ix_3)},$$

the formal powers from which we extract the analytic solutions of the Dirac equation, will be different from those already shown. Indeed, the richness of this new class of solutions can be as wide as the classical and modern Pseudoanalytic Function Theory allow the construction of generating pairs.

This implies that, at least from the mathematical point of view, every single formal power could well indicate the behavior of a massive quantum particle with spin $\frac{1}{2}$. Thus, nevertheless the formal powers do not provide the general solution of the Dirac equation, they do provide an effective technique for simulating and comparing the dynamics of each particle, therefor it could be possible to appoint new behavior patterns in this area.

Finally, we shall enhance that the techniques posed in this work can be extended for analyzing the influence of other kinds of potentials, over quantum particles governed by the Dirac equation, such like scalar potentials, also studied in [7]. Moreover, there exist the possibility of generalizing the concept of the Vekua equation for vector spaces whose dimension is higher than two, as it was posed in [6]. Nevertheless there is no proposal for introducing the concept of formal powers for dimensions higher than two, the principles of the Pseudoalaytic Function Theory clearly make a positive contribution in the research of Theoretical Physics, thus we might consider them a powerful tool for expanding the frontiers of this branch of the Science.

## References

[1] L. Bers (1953), *Theory of Pseudoanalytic Functions*, IMM, New York University.

[2] H. M. Campos, R. Castillo-Perez, V. V. Kravchenko (2011), *Construction and application of Bergman-type reproducing kernels for boundary and eigenvalue problems in the plane*, Complex Variables and Elliptic Equations, 1-38.

[3] R. Castillo-Perez., V. Kravchenko, R. Resendiz V. (2011), *Solution of boundary value and eigenvalue problems for second order elliptic operators in the plane using pseudoanalytic formal powers*, Mathematical Methods in the Applied Sciences, Vol. 34, Issue 4.

[4] V. V. Kravchenko (2009), *Applied Pseudoanalytic Function Theory*, Series: Frontiers in Mathematics, ISBN: 978-3-0346-0003-3.

[5] V. Kravchenko (2003), *Applied Quaternionic Analysis*, Heldermann.

[6] V. Kravchenko, Marco P. Ramirez T. (2011), *On Bers generating functions for first order systems of mathematical physics*, Advances in Applied Clifford Algebras, Vol. 21, No. 3, pp. 547-559.

[7] V. Kravchenko, M. P. Ramirez T. (2000), *New Exact Solutions for the Massive Dirac Equation with Electric or Scalar Potential*, Mathematical Methods in the Applied Sciences, Vol. 23, 769-776.

[8] B. J. McParland (2010), *Nuclear Medicine Radiation Dosimetry: Advanced Theoretical Principles*, Springer verlag London Limited.

[9] M. P. Ramirez T., R. A. Hernandez-Becerril, M. C. Robles G., A. Bucio R. (2014), *Study of the Numerical Solutions for the Electrical Impedance Equation in the Plane: A Pseudoanalytic approach of the Forward Dirichlet Boundary Value Problem*, International Journal of Applied Mathematics, Hindawi (submited for publication). Available in electronic format at http://ArXiv.com

[10] I. N. Vekua (1962), *Generalized Analytic Functions,* International Series of Monographs on Pure and Applied Mathematics, Pergamon Press.

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