Singular Optimal Control Problem of Stochastic Switching Systems

Charkaz Aghayeva and Melis Alpaslan

***Abstract*—The work is concerned with singular stochastic optimal control problem of switching systems. Necessary condition of optimality is an important tool for solution of optimal control problems in general. Along singular controls the first-order necessary condition for optimal controls cannot provide enough information to find the desired optimal controls. Therefore, for singular controls further optimality tests are required. Second order necessary condition of optimality for switching systems with uncontrolled diffusions and transversality conditions are established.**

***Index Terms*—stochastic-optimal-control, singular-switching-systems, necessary-conditions-of-optimality, transversality-conditions.**

I. Introduction

Stochastic differential equations find much exhibits in description of the real systems, which in one or another degree are subjected to the influence of the random noises. Systems with stochastic uncertainties have provided a lot of interest for problems of nuclear fission, communication systems , self-oscillating systems and etc., where the influences of random disturbances can not be ignored [9], [22].

Change of the structure of the system means that at some moment it may go over from one law of movement to another. After changing the structure, the characteristics of the initial condition of the system depends on its previous state. This situation joins them into a single system with variable structure [7], [16].

A switching systems have the benefit for modeling dynamic phenomena with the continuous law of movement. Recently, optimization problems for switching systems have attracted a lot of theoretical and practical interest [7], [10], [17], [26], [28].

Stochastic control problems have a variety of practical applications in fields such as physics, biology, economics, management sciences, etc. [1], [20]. The modern stochastic optimal control theory has been developed along the lines of Pontryagin's maximum principle and Bellman's dynamic programming [19], [29]. The stochastic maximum principle has been first considered by Kushner [23]. Earliest results on the extension of Pontryagin's maximum principle to stochastic control problems are obtained in [8], [11], [13], [21]. A general theory of stochastic maximum principle based on random convex analysis was given by Bismut [14]. Modern presentations of stochastic maximum principle with backward stochastic differential equations are considered in [15], [24], [25]. First-order necessary conditions of optimality for stochastic switching systems with uncontrolled diffusion coefficients have been studied by the author in [2], [3]. The problems with controlled diffusion coefficients are considered in [4], [5], [6].

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The first-order necessary conditions provide a basic tools to study the properties of stochastic optimal controls. However, in some cases the first-order necessary condition for stochastic optimal controls may be trivial and therefore it cannot provide enough information to find the desired optimal controls.

In this paper, backward stochastic differential equations have been used to establish singular maximum principle for stochastic optimal control problems of switching systems. Such kind of problems have been investigated in [24], [27]. for stochastic singular optimal control problems. In this paper, the singular optimal control problem of stochastic switching systems with uncontrolled diffusion coefficients is considered. We obtain second-order necessary condition of optimality and transversality conditions for such systems.

II. Preliminaries and Formulation of Problem

Throughout this paper, we use the following notations. Let **N** be some positive constant, R^n denotes the n dimensional real vector space, $|.|$ denotes the Euclidean norm in R^n and E represents the mathematical expectation. Assume that $w_t^1, w_t^2, ..., w_t^r$ are independent Wiener processes, which generate filtration $F_t^l = \bar{\sigma}(w_t^l, t_{l-1}, t_l), l = \overline{1,r}$, $0 = t_0 < t_1 < ... < t_r = T$. Let (Ω, F, P), $l = \overline{1,r}$ be a probability space with filtration$\{\ F_t, t \in [0,T]\ \}$, where $F_t = \bigcup_{l=1}^{r} F_t^l$. $L_F^2(a,b;R^n)$ denotes the space of all predictable processes $x_t(\omega)$ such that: $E\int_a^b |x_t(\omega)|^2 dt < +\infty$. $R^{m\times n}$is the space of linear transformations from R^m to R^n. Let $O_l \subset R^{n_l}$, $Q_l \subset R^{m_l}$, $l = \overline{1,r}$, be open sets. Consider the following stochastic control system:

$$dx_t^l = g^l\left(x_t^l, u_t^l, t\right)dt + f^l\left(x_t^l, u_t^l, t\right)dw_t,$$

$$t \in (t_{l-1}, t_l] \quad l = \overline{1,r}; \tag{1}$$

$$x_{t_{l-1}}^l = \Phi^{l-1}\left(x_{t_{l-1}}^{l-1}, t_{l-1}\right)\ l = \overline{2,r}; x_{t_0}^1 = x_0, \tag{2}$$

$$u_t^l \in U_\partial^l \equiv \left(u^l(\cdot,\cdot) \in \mathrm{L}_{F^l}^2 | u^l(t,\cdot) \in U^l \subset \mathrm{R}^{m_l}, a.c.\right) \tag{3}$$

Elements of U_∂^l, are called admissible controls. The problem is to find optimal inputs $\left(x^1, x^2, ..., x^r, u^1, u^2, ..., u^r\right)$ and switching sequence $t_1, t_2, ..., t_r$, such that the cost functional :

$$J(u) = \sum_{l=1}^{r} E\left[\varphi^l\left(x_{t_l}^l\right) + \int_{t_{l-1}}^{t_l} p^l\left(x_t^l, u_t^l, t\right)dt\right] \tag{4}$$

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is minimized on the decisions of the system (1)-(3), which are generated by all admissible controls $U = U^1 \times U^2 \times ... \times U^r$. Assume that the following requirements are satisfied:

AI. Functions $g^l, f^l, p^l,\ l = \overline{1,r}$ and their derivatives are continuous in (x, u, t).

AII. The derivatives of $g^l, f^l, p^l,\ l = \overline{1,r}$ are bounded by $N(1 + |x|)$.

AIII. Function $\varphi^l(x)$ and functions $\Phi^l(x,t) l = \overline{1, r-1}$ are continuously differentiable and their derivatives are bounded by $N(1 + |x|)$.

Furthermore the following requirements are assumed.

BI. Functions $g^l, f^l, p^l, l = \overline{1,r}$ are twice continuously differentiable with respect to x .

BII. Functions $g^l, f^l, p^l,\ l = \overline{1,r}$ and all their derivatives are continuous in (x, u). $g^l_x, g^l_{xx}, f^l_x, f^l_{xx}, p^l_{xx}$ are bounded and hold the condition:

$$(1 + |x|)^{-1} \left|g^l(x,u,t)\right| + \left|g^l_x(x,u,t)\right| + \left|f^l(x,u,t)\right| +$$

$$+\left|f^l_x(x,u,t)\right| + \left|p^l(x,u,t)\right| + \left|p^l_x(x,u,t)\right|\Big) \le N.$$

BIII.Functions $\varphi^l(x) : R^{n_l}$ and functions $\Phi^l(x,t) : R^{n_l} \times \mathrm{T} \to R^1, l = \overline{1, r-1}$ are twice continuously bounded differentiable.

Consider the sets:

$A_i = \mathrm{T}^{i+1} \times \prod\limits_{j=1}^{i} O_j \times \prod\limits_{j=1}^{i} \Lambda_j \times \prod\limits_{j=1}^{i} Q_j,\ i = \overline{1,r},$

with the elements

$$\pi^i = (t_0, t_1, t_i, x^1_t, x^2_t, ..., x^i_t, u^1, u^2, ..., u^i).$$

Definition 1 The set of functions $\left\{x^l_t = x^l\left(t, \pi^l\right),\ t \in [t_{l-1}, t_l]\right., l = \overline{1,r}$ is said to be a solution of equations (1)-(2) with variable structure which corresponding to an element $\pi^r \in \mathrm{A}_r$, if function $x^l_t \in O_l$ on the interval $[t_{l-1}, t_l]$ satisfies the condition (2) on point t_l, while it is absolutely continuous on the interval $[t_{l-1}, t_l]$ with probability 1 and satisfies the equation (1) almost everywhere.

Definition 2 The element $\pi^r \in A_r$ is said to be admissible if the pairs $\left(x^l_t,\ u^l_t\right), t \in [t_{l-1}, t_l],\quad l = \overline{1,r}$ are the solutions of system (1)-(3).

Definition 3 Let A^0_r be the set of admissible elements. The element $\tilde{\pi}^r \in A^0_r$, is said to be an optimal solution of problem (1)-(4) if there exist admissible controls $\tilde{u}^l_t, t \in [t_{l-1}, t_l]$, $l = \overline{1,r}$ and solutions of system (1)-(2) such that pairs$\left(\tilde{x}^l_t, \tilde{u}^l_t\right)$, $l = \overline{1,r}$ minimize the functional (4).

III. Maximum Principle

The following is revised and improved from the proof of Theorem 2 in [2].

Theorem 1 Suppose that, conditions AI-AIII hold and

$$\pi^r = (t_0, t_1, t_r, x^1_t, x^2_t, ..., x^r_t, u^1, u^2, ..., u^r)$$

is an optimal solution of problem (1)–(4). Then,

a) there exist random processes $(\psi^l_t, \beta^l_t) \in L^2_F(t_{l-1}, t_l; R^{n_l}) \times L^2_F(t_{l-1}, t_l; R^{n_l \times n_l})$ which are the solutions of the following adjoint equations:

$$\left\{\begin{array}{l} d\psi^l_t = -H^l_x(\psi^l_t, x^l_t, u^l_t, t)dt + \beta^l_t dw_t,\ \ t_{l-1} \le t < t_l, \\ \psi^l_{t_l} = -\varphi^l_x(x^l_{t_l}) + \psi^l_{t_{l+1}} \Phi^l_x(x^l_{t_l}, t_l),\ l = \overline{1, r-1}, \\ \psi^r_{t_r} = -\varphi^r_x(x^r_{t_r}), \end{array}\right. \tag{5}$$

b) $\forall\ \tilde{u}^l \in U^l,\ l = \overline{1,r}$, a.c. fulfills the maximum principle:

$$H^l(\psi^l_\theta, x^l_\theta, \tilde{u}^l, \theta) - H^l(\psi^l_\theta, x^l_\theta, u^l_\theta, \theta) \le 0, \text{ a.e. } \theta \in [t_{l-1}, t_l]; \tag{6}$$

c) following transversality conditions hold:

$$\psi^{l+1}_{t_l} \Phi^l_t\left(x^l_{t_l}, t_l\right) = 0, \quad a.c., \, l = \overline{1, r-1}, \tag{7}$$

where $H^l\left(\psi_t, x_t, u_t, t\right) = \psi_t g^l\left(x_t, u_t, t\right) + \beta_t f^l\left(x_t, t\right) - p^l\left(x_t, u_t, t\right)$, $t \in [t_{l-1}, t_l]$, $l = \overline{1,r}$.

The first order necessary condition of optimality is a powerful tool the study of optimal control problem, but is not always effective. For example, when the solution of adjoint equation is identically zero or the maximum principle is trivial, to investigate the corresponding optimal control problem is required additional information. Above mentioned cases are called singular and corresponding controls are s singular ones. Now we will discuss stochastic optimal control problem of of singular switching systems. At first we wiil give some results on linear quadratic stochastic optimal control problem. The questions of existence and uniqueness for adjoint differential equations can be done by using the results of [13], [12]. To this end, we introduce the following matrix-valued equations.

$$\begin{array}{l} d\Phi_t = A_t \Phi_t dt + B_t \Phi_t dw_t, \\ \Phi_0 = I, \end{array}$$

which has a unique solution Φ_t with $E \sup \|\Phi_t\|^{2s} < \infty, s \ge 1$, if A_t, B_t be the predictable and bounded matrices. Then it is known that (see [9]) the matrix Φ_t has an inverse and $\Psi_t = \Phi_t^{-1}$ is a solution of the equation

$$\begin{array}{l} d\Psi_t = -\left(\Psi_t A_t - \Psi_t B_t B_t\right) - \Psi_t B_t dw_t, \\ \Psi_0 = I. \end{array}$$

The following adjoint stochastic differential equation was introduced by [13].

Theorem 2 Let w is an n-dimensional Wiener processes on $(\Omega, \Im, P)$ adapted with filtrations $\Im_t = \bar{\sigma}(w_t, 0 \le t \le 1)$, Let $\xi : \Omega \to R^n$ be $\Im_1$-measurable and square integrable variable and let $a_t \in L^2_\Im(0,1; R^n)$. Then the stochastic differential equation

$$\begin{array}{l} dp_t = -\left(A^*_t p_t + B^*_t q_t - a_t\right) + q_t dw_t, \\ p_1 = \xi. \end{array}$$

has a unique solution $(p_t, q_t) \in L^2_\Im(0,1; R^n) \times L^2_\Im(0,1; R^{n \times n})$. Moreover, p_t and q_t can be represented as

$$p_t = -\Psi^*_1 E\left\{\Phi^*_1 \xi + \int\limits_t^1 \Phi^*_s a_s d\,s |\, \Im_t\right\}, q_t = -B^*_t p_t - \Psi^*_t g_t.$$

Where g_t is obtained from the relation

$$E\left\{\Phi^*_1 \xi + \int\limits_t^1 \Phi^*_s a_s d\,s |\, \Im_t\right\} = E\left\{\Phi^*_1 \xi + \int\limits_t^1 \Phi^*_s a_s ds\right\} +$$

$$+ \int\limits_0^t g_s dw_s.$$

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IV. The Second Order Necessary Condition

To state the main result of this paper, we need to introduce the following definition of stochastic singular switching systems in sense of maximum principle.

Definition 4 An admissible controls $u_t^l, l=1,...,r$ are said to be singular on control regions V^l, if each $V^l \subset U^l$ is nonempty and for *a.e.* $t \in [t_{l-1}, t_l)$, we have
$H_x^l(\psi_t^l, x_t^l, u_t^l, t) = H_x^l(\psi_t^l, x_t^l, v^l, t), \forall v^l \in V^l$
hold.

Theorem 3 Suppose that, conditions AI-AIII and BI-BIII hold. Let $\pi^r = (t_0, t_1, ..., t_r, x_{t_1}, x_{t_2}, ..., x_{t_r}, u^1, u^2, ..., u^r)$ is an optimal solution of problem (1)-(4), and $\mathbf{u} = (u^1, u^2, ..., u^r)$ be singular on the control region $\mathbf{V} = (V^1, V^2, ..., V^r)$. Then,

a) there exist random processes $(\psi_t^l, \beta_t^l) \in L^2_{F^l}(t_{l-1}, t_l; R^{n_l}) \times L^2_{F^l}(t_{l-1}, t_l; R^{n_l x n_l})$ and $(\Psi_t^l, \mathrm{K}_t^l) \in L^2_{F^l}(t_{l-1}, t_l; R^{n_l}) \times L^2_{F^l}(t_{l-1}, t_l; R^{n_l x n_l})$ which are the solutions of the following conjugate equations:

$$\begin{cases} d\psi_t^l = -H_x^l(\psi_t^l, x_t^l, u_t^l, t)dt + \beta_t^l dw_t, \ \ t_{l-1} \le t < t_l, \\ \psi_{t_l}^l = -\varphi_x^l(x_{t_l}^l) + \psi_{t_{l+1}}^l \Phi_x^l(x_{t_l}^l, t_l), \ l = \overline{1, r-1}, \\ \psi_{t_r}^l = -\varphi_x^l(x_{t_r}^l), \ l = \overline{1, r}; \end{cases} \tag{8}$$

$$\begin{cases} d\Psi_t^l = -[H_x^l\left(\Psi_t^l, x_t^l, u_t^l, t\right) + H_{xx}^l(\psi_t^l, x_t^l, u_t^l, t) \\ + f_x^{l*}(x_t^l, t)\Psi_t^l f_x^l(x_t^l, t)] \ dt + \mathrm{K}_t^l dw_t^l, t \in [t_{l-1}, t_l) \\ \Psi_{t_l}^l = -\varphi_{xx}^l(x_{t_l}^l) + \psi_{t_{l+1}}^l \Phi_{xx}^l(x_{t_l}^l, t_l), \ l = \overline{1, r-1}, \\ \Psi_{t_r}^l = -\varphi_{xx}^l(x_{t_r}^l) \end{cases} \tag{9}$$

b) a.e. $\theta \in [t_{l-1}, t_l]$ and $\forall \ \tilde{u}^l \in U^l$, $l = \overline{1, r}$, a.c. the second order maximum conditions fulfill:

$$\begin{aligned} &\Delta_{u^l} H_x^l(\psi_\theta^l, x_\theta^l, u^l, \theta) - H^l(\psi_\theta^l, x_\theta^l, u_\theta^l, \theta) + \\ &\Delta_{u^l} f^{l*}(x_\theta^l, u_\theta^l, \theta)\Psi_\theta^l \Delta_{u^l} f^l(x_\theta^l, u_\theta^l, \theta) \le 0 \end{aligned} \tag{10}$$

c) following transversality conditions hold.:

$$\psi_{t_l}^{l+1}\Phi_t^l\left(x_{t_l}^l, t_l\right) = 0\,, \, l = \overline{1, r-1}\,, \, a.c. \tag{11}$$

Here $\ \ H^l\left(\psi_t, x_t, u_t, t\right) = \psi_t g^l\left(x_t, u_t, t\right) +$
$+ \beta_t f^l\left(x_t, t\right) - p^l\left(x_t, u_t, t\right);$
$H^l(\Psi_t, x_t, u_t, t) = \Psi_t g^l(x_t, u_t, t) + g^{l*}\left(x_t, u_t, t\right)\Psi_t +$
$+ \mathrm{K}_t f^l\left(x_t, t\right) + f^{l*}\left(x_t, t\right)\mathrm{K}_t$.

Proof. Let $\bar{u}_t^l = u_t^l + \Delta\bar{u}_t^l \ l = \overline{1, r}$ be some admissible controls and $\bar{x}_t^l = x_t^l + \Delta\bar{x}_t^l \ l = \overline{1, r}$ be the corresponding trajectories of system (1)-(3). Let $0 = t_0 < t_1 < ... < t_r = T$ be switching sequence. Then following identities are obtained for some sequence $0 = \bar{t}_0 < \bar{t}_1 < ... < \bar{t}_r = T$:

$$\begin{cases} d\Delta\bar{x}_t^l = \left[\Delta_{\bar{u}^l} g^l(x_t^l, u_t^l, t) + g_x^l(x_t^l, u_t^l, t)\Delta\bar{x}_t^l\right. \\ \left.+ 0.5\Delta x_t^* g_{xx}^l(x_t^l, u_t^l, t)\Delta x_t \ dt +\right. \\ \left[f_x^l(x_t^l, t)\Delta\bar{x}_t^l + 0.5\Delta x_t^* f_{xx}^l(x_t^l, t)\Delta x_t \ \right] dw_t^l \ + \eta_t^1 \ , \\ \Delta\bar{x}_{t_0}^1 = 0, \\ \Delta\bar{x}_{t_{l-1}}^l = \Phi^{l-1}\left(\bar{x}_{t_{l-1}}^{l-1}, \bar{t}_{l-1}\right) - \Phi^{l-1}\left(x_{t_{l-1}}^{l-1}, t_{l-1}\right) \end{cases} \tag{12}$$

where
$$\eta_t^1 = \int_0^1 \left[g_x^{l*}(\bar{x}_t^l, \bar{u}_t^l, t) - g_x^{l*}(x_t^l, u_t^l, t)\right] \ \Delta\bar{x}_t^l d\mu \ dt +$$
$$0.5\int_0^1 \Delta\bar{x}_t^{l*} \left[g_{xx}^{l*}(\bar{x}_t^l, u_t^l, t) - g_{xx}^*(x_t^l, u_t^l, t)\right] \ \Delta\bar{x}_t^l d\mu \ dt$$
$$+ \int_0^1 \left[f_x^{l*}(x_t^l + \mu\Delta\bar{x}_t^l, t) - f_x^{l*}(x_t^l, t)\right]\Delta\bar{x}_t^l d\mu dw_t^l +$$
$$+0.5 \int_0^1 \Delta\bar{x}_t^{l*} \left[f_{xx}^{l*}(\bar{x}_t^l, t) - f_{xx}^*(x_t^l, t)\right] \ \Delta\bar{x}_t^l d\mu dw_t^l.$$

According to Ito's formula [20] implies that following identities hold:

$$\begin{aligned} &d(\psi_t^{l*}\Delta\bar{x}_t^l\Delta\bar{t}_l) = d\psi_t^{l*}\Delta\bar{x}_t^l\Delta\bar{t}_l + \psi_t^{l*}d\Delta\bar{x}_t^l\Delta\bar{t}_l + \\ &\{\beta_t^{l*}[\Delta_{\bar{u}^l} f^l(x_t^l, t) + f_x^l(x_t^l, t)\Delta\bar{x}_t^l + 0.5\Delta\bar{x}_t^{l*} f_{xx}^l(x_t^l, t)\Delta\bar{x}_t^l] \times \\ &\Delta\bar{t}_l + \beta_t^{l*}\int_0^1 \left[\ f_x^l(x_t^0 + \mu\Delta\bar{x}_t^l, t) - \ f_x^l(x_t^l, t)\right]\Delta\bar{x}_t^l\Delta\bar{t}_l d\mu + \\ &0.5\beta_t^{l*}\int_0^1 \Delta\bar{x}_t^{l*}\left[\ f_{xx}^l(x_t^0 + \mu\Delta\bar{x}_t^l, t) - f_{xx}^l(x_t^0, t)\right]\Delta\bar{x}_t^l\Delta\bar{t}_l\}dt \end{aligned} \tag{13}$$

and

$$\begin{aligned} &d(\Delta\bar{x}_t^{l*}\Psi_t^l\Delta\bar{x}_t^l\Delta\bar{t}_l) = \Delta\bar{x}_t^{l*}\Psi_t^l\Delta\bar{x}_t^l + \Delta\bar{x}_t^{l*}d\Psi_t^l\Delta x_t^l\Delta\bar{t}_l + \\ &\Delta x_t^{l*}\Psi_t^l d\Delta\bar{x}_t^l\Delta\bar{t}_l + d\Delta\bar{x}_t^{l*}\Psi_t^l \cdot \Delta\bar{x}_t^l\Delta\bar{t}_l + \{\mathrm{K}_t^{l*}[f_x^l(x_t^l, t)\Delta\bar{x}_t^l \\ &+0.5\Delta\bar{x}_t^{l*} f_{xx}^l(x_t^l, t)\Delta\bar{x}_t^l]\Delta\bar{t}_l + [\Delta_{\bar{u}} f_x^l(x_t^l, t) + \\ &+ f_x^l(x_t^l, t)\Delta\bar{x}_t^l + 0.5\Delta\bar{x}_t^{l*} f_{xx}^l(x_t^l, t)\Delta\bar{x}_t^l] \cdot \Psi_t^l\Delta_{\bar{u}} f^l(x_t^l, t) + \\ &+ f_x^l(x_t^l, t)\Delta\bar{x}_t^l + 0.5\Delta\bar{x}_t^{l*} f_{xx}^l(x_t^l, t)\Delta\bar{x}_t^l]\Delta\bar{t}_l\}dt \end{aligned} \tag{14}$$

In order to establish the existence and uniqueness of solution of adjoint stochastic differential equations, it is enough to follow the method described in the article [11], to make use of the independence of Wiener processes $w_t^1, ..., w_t^r$ and **Theorem 2** in each interval $[t_{l-1}, t_l], l = 1, ..., r$.

The stochastic processes ψ_t^l , $l = \overline{1, r}$, at the points $t_1, t_2, ..., t_r$ are defined as

$$\psi_{t_l}^l = -\varphi_x^l\left(x_{t_l}^l\right) + \psi_{t_l}^{l+1}\Phi_x^l(x_{t^l, t_l}), \ \psi_{t_r}^r = -\varphi_x^r(x_{t_r}^r) \tag{15}$$

and

$$\Psi_{t_l}^l = -\varphi_{xx}^l\left(x_{t_l}^l\right) + \psi_{t_l}^{l+1}\Phi_{xx}^l\left(x_{t^l, t_l}\right), \ \ \Psi_{t_r}^r = -\varphi_{xx}^r\left(x_{t_r}^r\right) \tag{16}$$

By assumptions AIII and BIII,using the expression (15) from the identity (12) , we obtain that (11) is true. Taking into consideration (11)-(16) the expression of increment of a cost functional (4) gets the form as indicated below:

$$\begin{aligned} &\Delta J\left(u\right) = \ \sum_{l=1}^r E\{\varphi^l\left(\bar{x}_{t_l}^l\right) - \varphi^l\left(x_{t_l}^l\right) + \int_{t_{l-1}}^{t_l}[p^l\left(\bar{x}_t^l, \bar{u}_t^l, t\right) \\ &-p^l\left(x_t^l, u_t^l, t\right)]dt\} = -\sum_{l=1}^r E\int_{t_{-1}}^{t_l}\{\Delta_{\bar{u}^l} H^l\left(\psi_t^l, x_t^l, u_t^l, t\right) \\ &+H_x^l\left(\psi_t^l, x_t^l, u_t^l, t\right)\Delta\bar{x}_{t_l}^l - \Delta\bar{x}_{t_l}^{l*} g_x^l\left(x_t^l, u_t^l, t\right) \ \Psi_t^l\Delta\bar{x}_{t_l}^l \\ &-0.5\Delta\bar{x}_{t_l}^{l*} f^{l*}\left(x_t^l, t\right)\Psi_t^l f_x^l\left(x_t^l, t\right)\Delta\bar{x}_{t_l}^l + \\ &+\Delta\bar{x}_{t_l}^{l*}\Delta_{\bar{u}^l} g^l\left(x_t^l, u_t^l, t\right) \ \Psi_t^l\Delta\bar{x}_{t_l}^l - \Delta\bar{x}_{t_l}^{l*} f_x^l\left(x_t^l, t\right) \ K_t^l\Delta\bar{x}_{t_l}^l \\ &+\psi_t^{l*}\Delta_{\bar{u}^l} g_x^l\left(x_t^l, u_t^l, t\right)\Delta\bar{x}_{t_l}^l - \Delta_{\bar{u}^l} p_x^l\left(x_t^l, u_t^l, t\right)\Delta\bar{x}_{t_l}^l\Delta\bar{t}_l\}dt \\ &+\sum_{l=1}^{r-1}\psi_{t_l}^{l+1}\Phi_t\left(x_{t_l}^l, t_l\right) + \sum_{l=1}^r \eta_{t_{l-1}}^{t_l}, \end{aligned} \tag{17}$$

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where

$$\eta_{t_{l-1}}^{t_l} = -E\int_0^1 (1-\mu)\left[\varphi_x^{l*}\left(\bar{x}_{t_l}^l\right) - \varphi_x^*\left(x_{t_l}^l\right)\right]\Delta\bar{x}_{t_l}^l d\mu +$$
$$+E\int_{t_{l-1}}^{t_l}\int_0^1 (1-\mu)\,\Delta_{\bar{x}} H_x^l\left(\psi_t^l, x_t^l u_t^l, t\right)\Delta\bar{x}_t^l\Delta\bar{t}_l d\mu dt -$$
$$\mathrm{E}\int_0^1 (1-\mu)\,\psi_{t_l}^{l+1}\left[\Phi_x^l\left(\bar{x}_{t_l}^l\right) - \Phi_x^l\left(x_t^l, t_l\right)\right]\Delta x_t^l\Delta\bar{t}_l d\mu -$$
$$\mathrm{E}\int_0^1 (1-\mu)\,\Delta\bar{x}_{t_l}^{l*}\left[\varphi_{xx}^{l*}\left(\bar{x}_{t_l}^l\right) - \varphi_{xx}^*\left(x_{t_l}^l\right)\right]\Delta\bar{x}_{t_l}^l d\mu +$$
$$\mathrm{E}\int_{t_{l-1}}^{t_l}\int_0^1 (1-\mu)\,\Delta\bar{x}_t^l\Delta_{\bar{x}} H_{xx}^l\left(\psi_t^l, x_t^l u_t^l, t\right)\Delta\bar{x}_t^l\Delta\bar{t}_l d\mu dt$$
$$-\ \mathrm{E}\int_0^1 (1-\mu)\,\Delta\bar{x}_{t_l}^{l*}\psi_{t_l}^{l+1}\left[\Delta_{\bar{x}}\Phi_{xx}^l\left(x_t^l, t_l\right)\right]\Delta\bar{x}_{t_l}^l\Delta\bar{t}_l d\mu \quad (18)$$

According to (8), (9), (15) and (16) , through the simple transformations expression (17) may be written as:

$$\Delta J(u) = -\sum_{l=1}^{s} E\int_{t_{l-1}}^{t_l}\left[\Delta_{\overline{u}^l} H^l\left(\psi_t^l, x_t^l, u_t^l, t\right) +\right.$$
$$\Delta_{\overline{u}^l} H_{x^l}^l\left(\psi_t^l, x_t^l, u_t^l, t\right)\Delta\bar{x}_t^l - \Delta\bar{x}_{t_l}^{l*} g_x^l\left(x_t^l, u_t^l, t\right)\ \Psi_t^l\Delta\bar{x}_{t_l}^l$$
$$-\ 0.5\Delta\bar{x}_{t_l}^{l*} f^{l*}\left(x_t^l, t\right)\Psi_t^l f_x^l\left(x_t^l, t\right)\Delta\bar{x}_{t_l}^l +$$
$$+\Delta\bar{x}_{t_l}^{l*}\Delta_{\bar{u}^l} g^l\left(x_t^l, u_t^l, t\right)\ \Psi_t^l\Delta\bar{x}_{t_l}^l -$$
$$\left.-\ \Delta\bar{x}_{t_l}^{l*} f_x^l\left(x_t^l, t\right)\ K_t^l\Delta\ \bar{x}_{t_l}^l\right]\Delta\bar{t}_l dt + \sum_{l=1}^{r}\eta_{t_{l-1}}^{t_l}$$

(19)

Consider the spike variations for $\tilde{u}^l \in L^2\left(\Omega, F^{\theta_l}, P; R^m\right)$:

$$\Delta u_{t,\varepsilon^l}^{\theta_l} = \begin{cases} 0,\ t\notin[\theta_l, \theta_l+\varepsilon_l), \varepsilon_l>0, \\ \tilde{u}^l - u_t^l,\ t\in[\theta_l, \theta_l+\varepsilon_l), \end{cases}$$

where ε_l are enough small numbers. For given special variation the expression (19) takes the form:

$$\Delta_\theta J(u) = \sum_{l=1}^{r} E\int_{\theta_l}^{\theta_l+\varepsilon_l}\left[\ \Delta_{\tilde{u}^l} H^l\left(\psi_t^l, x_t^l, u_t^l, t\right) +\right.$$
$$+\Delta_{\bar{u}^l} H_{x^l}^l\left(\psi_t^l, x_t^l, u_t^l, t\right)\Delta\bar{x}_t^l + \sum_{l=1}^{r}\eta_{\theta_l}^{\theta_l+\varepsilon_l} \quad (20)$$

Following lemma will be used in estimating for increment (20).

Lemma 1 Assume that conditions I-II are fulfilled. Then, the following is obtained:

$\lim\limits_{\varepsilon_l\to 0} E\left|x_{t,\varepsilon_l}^{\theta_l} - x_t^l\right|^2 \le N\varepsilon_l^2$, a.e. in $[t_{l-1}, t_l)$, $l=\overline{1,r}$.

Here $x_{t,\varepsilon_l}^{\theta_l}$ are the trajectories of system (1)-(2), corresponding to the controls $u_{t,\varepsilon_l}^{\theta_l} = u_t^l + \Delta u_{t,\varepsilon_l}^{\theta_l}$.

Proof

Let's denote the following: $\tilde{x}_{t,\varepsilon_l}^l = x_{t,\varepsilon_l}^{\theta_l} - x_t^l$.
It is clear that $\forall t\in[t_{l-1}, \theta_l)$ $\tilde{x}_{t,\varepsilon_l}^l = 0$, $\overline{l=1,r}$.
Then for $\forall t\in[\theta_l, \theta_l+\varepsilon_l)$

$$d\tilde{x}_{t,\varepsilon_l}^l = \left[g^l(\tilde{x}_{t,\varepsilon_l}^l, \tilde{u}^l, t) - g^l(x_t^l, u_t^l, t)\right]dt$$
$$+\left[f^l(\tilde{x}_{t,\varepsilon_l}^l, t) - f^l(x_t^l, t)\right]\ dw_t^l,$$
$$\tilde{x}_{\theta_l,\varepsilon_l}^l = -(g^l(x_{\theta_l}^l, \tilde{u}^l, \theta_l) - g(x_{\theta_l}^l, u_{\theta_l}^l, \theta_l))$$

or

$$\tilde{x}_{\theta_l+\varepsilon_l,\varepsilon_l}^l = \int_{\theta_l}^{\theta_l+\varepsilon_l}\ \left[g^l(\tilde{x}_{s,\varepsilon_l}^l, u^l, s) - g^l(x_s^l, u_s^l, s)\right]\ ds +$$
$$+\int_{\theta_l}^{\theta_l+\varepsilon_l}\ \left[g^l(x_\theta^l, u_{\theta_l}^l, \theta_l) - g^l(x_s^l, u_s^l, s)\right]\ ds +$$
$$+\int_{\theta_l}^{\theta_l+\varepsilon_l}\left[f^l(\tilde{x}_{s,\varepsilon_l}^l, s) - f^l(x_s^l, s)\right]\ dw_s^l +$$
$$+\int_{\theta_l}^{\theta v+\varepsilon_l}\ \left[g^l(x_s^l, \tilde{u}^l, s) -\ g^l(x_{\theta_l}^l, \tilde{u}^l, \theta_l)\right]\ ds$$

Therefore from the conditions I and II, using the Gronwall's inequality we have

$$E\left|\tilde{x}_{\theta_l+\varepsilon_l,\varepsilon_l}^l\right|^2 \le N\left[\varepsilon_l^2\sup_{\theta_l\le t\le\theta_l+\varepsilon_l} E\left|x_{t,\varepsilon_l}^{\theta_l} - x_t^l\right|^2 +\right.$$
$$+\varepsilon_l^2\sup_{\theta_l\le t\le\theta_l+\varepsilon_l} E\left|x_t^l - x_{\theta_l}^l\right|^2 +$$
$$+\sup_{\theta_l\le t\le\theta_l+\varepsilon_l}\varepsilon_l^2 E\left|g^l(x_t^l, \tilde{u}^l, t) - g^l(x_{\theta_l}^l, \tilde{u}^l, \theta_l)\right|^2 +$$
$$+\varepsilon_l E\int_{\theta_l}^{\theta_l+\varepsilon_l}\left|f^l(x_t^l, t) - f^l(x_{\theta_l}^l, \theta_l)\right|^2 dt +$$
$$\left.+\varepsilon_l^2 E\int_{\theta_l}^{\theta_l+\varepsilon_l}\left|g^l(x_t^l, u_t^l, t) - g^l(x_{\theta_l}^l, u_{\theta_l}^l, \theta_l)\right|^2 dt\right]$$

Hence:

$$E\left|\tilde{x}_{t+\varepsilon_l,\varepsilon_l}^l\right|^2 \le N\varepsilon_l^2, \quad \varepsilon_l\to 0, \quad \forall t\in[\theta_l, \theta_l+\varepsilon_l)$$

Farther $\forall t\in[\theta_l+\varepsilon_l, t_l]$ we obtain:

$$d\tilde{x}_{t,\varepsilon_l}^l = \int_0^1 g_x^l(x_t^l + \mu\varepsilon_l\tilde{x}_{t,\varepsilon_l}^l, u_t^l, t)\tilde{x}_{t,\varepsilon_l}^l d\mu dt +$$
$$+\int_0^1 f_x^l(x_t^l + \mu\varepsilon_l\tilde{x}_{t,\varepsilon_l}^l, t)\tilde{x}_{t,\varepsilon_l}^l d\mu dt$$
$$\tilde{x}_{\theta_l+\varepsilon_l,\varepsilon_l}^l = -(g^l(x_{\theta_l+\varepsilon_l}^l, u_{\theta_l+\varepsilon_l}^l, \theta^l) - g(x_{\theta_l+\varepsilon_l}^l, \tilde{u}^l, \theta_l))$$

Hence: $E\left|\tilde{x}_{t,\varepsilon_l}^l\right|^2 \le \varepsilon_l^2 N$,for $\forall t\in[\theta_l+\varepsilon_l, t_l]$, if $\varepsilon_l\to 0$

Thus: $\sup\limits_{t_{l-1}\le t\le t_l} E\left|\tilde{x}_{t,\varepsilon_l}\right|^2 \le N\varepsilon_l^2$, $l=\overline{1,r}$.
Lemma 1 is proved.

By invoking the expression (18),using Lemma 1 following estimation is implied:

$$\eta_{\theta_l}^{\theta_l+\varepsilon_l} = o\left(\varepsilon_l{}^2\right), \ l=\overline{1,r}.$$

And according to singularity of controls $\bar{u}_t^l$, $l=\overline{1,r}$ from(20) for each l it follows that:

$$\Delta_{\theta^l} J(u) = -E\left[\psi_{\theta_l}^{l*}\Delta_{\tilde{u}^l}\ g^l(x_{\theta_l}^l, u_{\theta_l}^l, \theta_l) -\right.$$
$$-\Delta_{\tilde{u}^l} p^l(x_{\theta_l}^l, u_{\theta_l}^l, \theta_l) +$$
$$\left.0.5\Delta_{\tilde{u}^l} f^{l*}(x_{\theta_l}^l, u_{\theta_l}^l, \theta_l)\Psi_{\theta_l}^l\Delta_{\bar{u}^l} f^l(x_{\theta_l}^l, u_{\theta_l}^l, \theta_l)\ \right]\Delta\bar{t}_l +$$
$$+o(\varepsilon_l) \ge 0$$

Finally, due to the smallness and arbitrariness of ε_l (10) is fulfilled. Theorem 3 is proved.

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