# Equivalence Relations Between Absolute Riezs And The Product of Two Absolute Riezs Summability Methods

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*Abstract*— **The paper deals with the problem of inclusion and equivalence of absolute Riesz method with that of the product of two absolute Riesz summability methods. Necessary and Sufficient conditions concerning (Inclusion and Equivalence) of these two methods have been established. Examples to show that each of these inclusions may hold in only one way without the other have been given. An example to show that the equivalence may hold in some non trivial case have been given , and an example to show that even if each Riesz method is regular, the inclusion is not hold (in either way) have been constructed.**

**Index Terms — absolutely regular, absolute Riesz method, equivalence, sequence – to - sequence transformation, summable.**

## I. Introduction

Each sequence $\{r_n\}$ for which

$R_n = r_0 + r_1 + ........ + r_n \neq 0$ for each n defines $(\overline{N}.r)$, where.

$$t_n^{(r)} = \frac{1}{R_n}\sum_{k=0}^{n} r_k S_k \qquad n = 0,1,2,......... \quad (1)$$

The product $(\overline{N}.p)$ $(\overline{N}.q)$ which was first considered by the author ([1] 1980) is given by

$$t_n^{(p,q)} = \frac{1}{P_n}\sum_{k=0}^{n} S_k q_k \sum_{v=k}^{n} \frac{p_v}{Q_v} \qquad (2)$$

Let (A) be a sequence – to - sequence transformation given by

$$t_n = \sum_{k=0}^{n} \propto_{n,k} S_k \qquad (3)$$

If $t_n \to s \, an \, n \to \infty, \{S_n\}$ is said to be summable A to sum s, and if in addition $\{t_n\}$ is of bounded variation, then $\{S_n\}$ is said to be absolutely summable (A) or summbale |A|. (A) is said to be regular, if it sums every convergent series to its ordinary sum. It follows from Toeplitz's Theorem (Hordy [8], Theorem 2) that $(\overline{N}.p)$ is regular if, and only if,

$$|P_n| \to \infty \, as \; n \to \infty, and \sum_{k=0}^{n} |p_k| = O(|P_n|) \quad (4)$$

If whenever $\{S_n\}$ has a bounded variation if follows that $\{t_n\}$ has a bounded variation, and if the limits are preserved, we shall say that (A) is absolutely regular.
We shall write throughout (A)$\subseteq$(B) to mean that any series summable by (A) to sum s is necessarily summable (B) to the same sum. (A) and (B) are equivalent if (A) $\supseteq$ (B) and (B) $\supseteq$ (A). In this case we write A~ B. We shall write throughout for any sequence, $\Delta U_n = U_n - U_{n+1}$; $(n \geq 0)$

## II. INCLUSION AND EQUIVALENCE RELATIONS

On inclusion and equivalence relations of different summability methods much work has been done already e.g. ([2], [3], [4], [5], [6], [7] and [8]).

## III. OBJECT OF THE PAPER

In [2] the author obtained necessary and sufficient conditions for which $(\overline{N}, r) \underline{C} (\overline{N}, p)(\overline{N}, q)$ and conversely, and consequently for which $(\overline{N}, r) \sim (\overline{N}, p)(\overline{N}, q)$. The object of this paper is to obtain results involving absolute methods $|(\overline{N}, r)|$ and $|(\overline{N}, p)(\overline{N}, q)|$ analogous to those by the author [2; Theorems (3.1), (3.2) and (3.3)] , and to show that the inclusion may hold in only one way without the other , and that the equivalence is valid in some non – trivial case ,

Manuscript received July 08, 2014; revised July 23, 2014. This work was supported in part by Al Zaytoonah University of Jordan.
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finally, to show that even if $(\overline{N}, r)$, $(\overline{N}, p)$ and $(\overline{N}, q)$ are regular, the inclusion need not hold in either way. These results will be concluded in sections (6) and (7).

## IV. REQUIRED RESULT

This section is devoted to result that is necessary for our purposes.

Theorem (5.1) (Mears [9] the sequence – to - sequence transformation given in (3) is absolutely regular if and only if

$$\propto_{n,k} \to o \; as\, n \to \infty \quad for\, each \; k \qquad (5)$$

$$\sum_{k=0}^{\infty} \propto_{n,k} \to 1 \;\; as\, n \to \infty, \qquad (6)$$

and

$$\sum_{n=0}^{\infty} \left| \sum_{v=k}^{\infty} \propto_{n,v} - \sum_{v=k}^{\infty} \propto_{n+1,v} \right| = O(1), (k \to \infty) \qquad (7)$$

## V. MAIN RESULTS

In this section we prove our main results:

Theorem (6.1): Let $|P_n| \to \infty, |Q_n| \to \infty$, and let $r_n \neq 0$; $(n \geq 0)$, then $|\overline{N}, r| \subseteq |(\overline{N}, p)(\overline{N}, q)|$ if, and only if

$$\sum_{n=k-1}^{\infty} \left| \gamma_{n,k-1} - \gamma_{n+1,k-1} \right| = O(1), \qquad (8)$$

where

$$\gamma_{n,n} = \frac{R_n \, q_n \, p_n}{P_n \, Q_n r_n}, \qquad (9)$$

and

$$\gamma_{n,k-1} = \frac{1}{P_n} \left\{ P_{k-1} + (Q_k - R_k \frac{q_k}{r_k}) \sum_{u=k}^{n} \frac{p_u}{Q_u} \right\}; \;\; n \geq k \qquad (10)$$

Further, if $\gamma_{n,k-1}$ is decreasing in n, then (8) is equivalent to

$$\gamma_{k,k} - \gamma_{N+1,k-1} = O(1). \quad \forall N \geq k \geq o. \qquad (11)$$

Proof:Let $t_n^{(r)}$ and $t_n^{(p,q)}$ be respectively the $(\overline{N}, r)$ and $(\overline{N}, p)(\overline{N}, q)$ transforms of $\{S_n\}$ Using the inversion formula in (1) to obtain $S_n$ in terms of $t_n^{(r)}$ and substitute this in (2) to get $t_n^{(p,q)}$ in terms of $t_n^{(r)}$, we have

$$t_n^{(p,q)} = \sum_{v=0}^{n`} Y_{n,v} t_v^{(r)} \qquad (12)$$

where

$$Y_{n,n} = \gamma_{n,n} \qquad (13)$$

$$Y_{n,v} = \frac{R_v}{P_n} \left\{ \frac{q_v}{r_v} \sum_{u=v}^{n} \frac{p_m}{Q_u} - \frac{q_{v+1}}{r_{v+1}} \sum_{u=v+1}^{n} \frac{p_u}{Q_u} \right\};$$
$$(0 \leq v \leq n-1), \qquad (14)$$

and

$$Y_{n,v} = o \qquad \text{otherwise} \qquad (15)$$

The special case in which $S_n = 1$ $(n \geq o)$ gives $t_n^{(r)} = 1 = t_n^{(p,q)}$ $(n \geq o)$ and (12) implies that

$$\sum_{v=0}^{n`} Y_{n,v} = 1 \qquad (16)$$

which implies (6) Using the hypothesis, one may get that $Y_{n,v} \to 0 \; as \; n \to \infty$ for each $v$, and (5) is satisfied. Therefore Theorem (5.1) implies the result if, and only if

$$\sum_{n=0}^{\infty} \left| \sum_{v=k}^{\infty} Y_{n,v} - \sum_{v=k}^{\infty} Y_{n+1,v} \right| = O(1) \qquad (17)$$

Using (15), we see that (17) is equivalent to

$$\sum_{n=k-1}^{\infty} \left| \sum_{v=k}^{\infty} Y_{n,v} - \sum_{v=k}^{\infty} Y_{n+1,v} \right| = O(1), \qquad (18)$$

or to:

$$|Y_{k,k}| + \sum_{n=k}^{\infty} \left| \sum_{v=0}^{k-1} Y_{n,v} - \sum_{v=0}^{k-1} Y_{n+1,v} \right| = O(1), \qquad (19)$$

Write

$$A_{n,v} = \frac{1}{P_n} \frac{q_v}{r_v} \sum_{u-v}^{n} \frac{p_u}{Q_u} \qquad (20)$$

it follows from (14) that

$$Y_{n,v} = R_v A_{n,v} - R_v \, A_{n,v+1} \; ; o \leq v \leq n-1 \qquad (21)$$

this implies that

ISBN: 978-988-19253-7-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$\sum_{v=0}^{k-1} Y_{n,v} = R_o\, A_{n,o} - R_o\, A_{n,1} + R_1 A_{n,1} - R_1 A_{n,2} + \ldots + R_{k-1}\, A_{n,k-1} - R_{k-1}\, A_{n,k}$$

$$= r_o\, A_{n,0} + r_1\, A_{n,1} + \ldots + r_{k-1}\, A_{n,k-1} - R_{k-1}\, A_{n,k}$$

$$= \frac{1}{P_n}\left\{ q_o\left(\frac{p_o}{Q_o}\right) + (q_o+q_1)\frac{p_1}{Q_1} + \ldots + (q_o+q_1+\ldots+q_{k-1})\left(\frac{p_{k-1}}{Q_{k-1}}\right) + (q_o+q_1+\ldots+q_{k-1})\sum_{u=k}^{n}\frac{p_u}{Q_u}\right\} - \frac{R_{k-1}}{P_n}\frac{q_k}{r_k}\sum_{u=k}^{n}\frac{p_U}{Q_u}$$

$$= \frac{1}{P_n}\left\{ P_{k-1} + (Q_{k-1} - \frac{R_{k-1}\, q_k}{r_k} \sum_{u=k}^{n}\frac{p_u}{Q_u}\right\}$$

$$= \frac{1}{P_n}\left\{ P_{k-1} + (Q_k - \frac{R_k\, q_k}{r_k}) \sum_{u=k}^{n}\frac{p_u}{Q_u}\right\}$$

$$= \gamma_{n,k-1} \qquad ; n \ge k \qquad (22)$$

Using (9), (10) and (13) we see that $Y_{k,k} = \gamma_{k,k} = 1 - \gamma_{k,k-1}$, which implies that $\gamma_{k-1,k-1} - \gamma_{k,k-1} = O\ (1)$ if ,and only if $\gamma_{k-1,k-1} + \gamma_{k,k}$ is bounded, which is if, and only if $\gamma_{k,k} = \gamma_{k,k}$ is bounded. Using this, the result follows from (19) and (22).

Finally, if $\{\gamma_{n,k-1}\}$ is decreasing in $n$, then $\gamma_{n,k-1} - \gamma_{n+1,k-1} \ge o$ and the equivalence of (8) and (11) holds. This completes the proof.

Remark (6.1) We remark that $\gamma_{n,n} = O(1)$: is necessary (but not sufficient) condition for (8) to be satisfied.

Theorem (6.2) Let $|R_n| \to \infty$, $\{p_n\}$ *and* $\{q_n\}$ be nonzero sequences, then $|(\overline{N}, p)(\overline{N}, q)| \underline{C} |\overline{N}, r|$ if, and only if

$$\sum_{n=k-1}^{\infty} \left| B_{n,k-1} \right| = O(1) \qquad (23)$$

where

$$B_{n,n} = \frac{r_n\, P_n\, Q_n}{R_n\, p_n\, q_n} \qquad (24)$$

$$B_{n,n-1} = \frac{P_{n-1}}{R_n}\Delta\frac{r_{n-1}Q_{n-1}}{p_{n-1}q_{n-1}} - \Delta\frac{P_{n-1}r_{n-1}Q_{n-1}}{R_n\, p_n q_n} - \frac{P_{n-1}r_n Q_{n-1}}{R_n\, p_{n-1} q_n} \qquad (25)$$

$$B_{n,k} = (R_k - \frac{Q_k\, r_k}{q_k} - P_{k-1}\frac{Q_k}{p_k}\Delta\frac{r_k}{q_k})\Delta\frac{1}{R_n},\ o \le k \le n-2 \qquad (26)$$

and

$$B_{n,k} = 0 \qquad \text{otherwise} \qquad (27)$$

Further, if $r_n > 0$, $\{r_n\}$ decreasing, $\{P_n\}$ and $\{Q_n\}$ are increasing, then $|(\overline{N}, p)(\overline{N}, q)| \subseteq |\overline{N}, r|$ if, and only if $B_{n,n} = O\ (1)$.

Proof: Let $\{t_n^{(r)}\}$ and $\{t_n^{(p,q)}\}$ be respectively the $(\overline{N}, r)$ and $(\overline{N}, P)(\overline{N}, q)$ transforms of $\{S_n\}$. Write (2) in the form

$$t_n^{(p,q)} = \frac{1}{p_n}\sum_{v=0}^{n} p_v \cdot \frac{1}{Q_v}\sum_{R=o}^{v} q_k S_k = \frac{1}{P_n}\sum_{v=0}^{n} p_v\, A_v, say, \qquad (28)$$

we have

$$A_n = \frac{P_n\, t_n^{(p,q)} - P_{n-1}\, t_{n-1}^{(p,q)}}{p_n}, \qquad (29)$$

and

$$S_n = \frac{A_n\, Q_n - A_{n-1}\, Q_{n-1}}{q_n} \qquad (30)$$

Using (29) and (30) , it follows from (1) that

$$t_n^{(r)} = \sum_{v=0}^{n} F_{n,v}\, t_v^{(p,q)} \qquad (31)$$

where

$$F_{n,n} = B_{n,n} = \frac{r_n\, Q_n\, P_n}{R_n\, q_n\, \rho_n} \qquad (32)$$

$$F_{n,n-1} = \frac{P_{n-1}}{R_n}\left\{\frac{r_{n-1}\, Q_{n-1}}{q_{n-1}\, \rho_{n-1}} - \frac{Q_n\, r_n}{q_n\, p_n} - \frac{Q_{n-1}\, r_n}{p_{n-1}\, q_n}\right\} \qquad (33)$$

$$F_{n,v} = \frac{P_v}{R_n}\Delta\left(\frac{Q_v}{p_v}\Delta\frac{r_v}{q_v}\right),\ o \le v \le n-2 \qquad (34)$$

and

$$F_{n,v} = 0 \qquad \text{otherwise} \qquad (35)$$

(5) follows from the hypothesis and (34). The special case in which

$S_n = 1;\ (n \ge o)\ gives\ t_n^{(r)} = t_n^{(p,q)} = 1\ ;(n \ge o)$

and (31) implies that

$$\sum_{v=0}^{n} F_{n,v} = 1, \qquad (36)$$

$$|F_{k,k}| + |F_{k+1,k+1} + F_{k+1,k} - F_{k,k}| + |\sum_{v=0}^{k-1} P_v\, \Delta\left(\frac{Q_v}{p_v}\Delta\frac{r_v}{q_v}\right)| \sum_{n=k+1}^{\infty} |\Delta\frac{1}{R_n}| \qquad (37)$$

Write $G_v = \dfrac{Q_v}{P_v}\Delta\dfrac{r_v}{q_v}$ , we have

$$\sum_{v=0}^{k-1}(P_v\, G_v - P_v\, G_{v+1}) = p_o\, G_o + p_1\, G_1 + \ldots + p_{k-1}\, G_{k-1} - P_{k-1}\, G_k$$

$$= R_{k-1} - Q_{k-1}\frac{r_k}{q_k} - P_{k-1}\frac{Q_k}{p_k}\Delta\frac{r_k}{q_k}$$

$$= R_k - \frac{Q_k\, r_k}{q_k} - P_{k-1}\frac{Q_k}{p_k}\Delta\frac{r_k}{q_k} \qquad (38)$$

*Substitute this in (37) , we see that the left hand side of (7) reduces to*

$$\sum_{n=k-1}^{\infty} |\, B_{n,k-1}\, |$$

ISBN: 978-988-19253-7-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

where $B_{n,k}$ , as given by (24)-(27) .Hence , the result follows from Mears Theorem (5.1)

Next, assume that $|(\overline{N},p)(\overline{N},q)| \underline{C} |\overline{N},r|$ then (23) is satisfied which implies that $B_{n,n}$ = 0(1). Finally, let $B_{n,n}$ = 0(1), and write $B_{n,n-1}$ in the form

$$B_{n,n-1} = B_{n,n}\left(\frac{R_n}{R_{n+1}}\frac{p_n}{p_{n+1}}\frac{q_n}{q_{n+1}} - \frac{P_{n-1}}{P_n}\right) + \frac{R_{n-1}}{R_n}\left(1 - \frac{p_{n-1}\,q_{n-1}}{p_n\,q_n} - \frac{r_n\,q_{n-1}}{r_{n-1}\,q_n}\right)B_{n-1,n-1}$$

and observe that

$$\sum_{n=k+1}^{\infty}|B_{n,k-1}| \leq |R_k - \frac{Q_k\,r_k}{q_k} - P_{k-1}\frac{Q_k}{p_k}\Delta\frac{r_k}{q_k}|\frac{1}{R_{k+2}}$$

$$= |\frac{R_k}{R_{k+2}}| + |\frac{p_k R_k}{P_k\,R_{k+2}}||B_{k,k}| + |\frac{P_{k-1}\,R_k}{P_k\,R_{k+2}}||B_{k,k}| + |\frac{P_{k-1}\,r_{k+1}\,q_k R_k}{P_k\,r_k q_{n+1}\,R_{k+2}}||B_{k,k}|,$$

it follows from the assumptions that if $B_{n,n} = O(1)$ then $B_{n,n-1}$ and $\sum_{n=k+1}^{\infty}|B_{n,k-1}|$ are bounded. This implies that (23) is satisfied, and $|(\overline{N},p)(\overline{N},q)| \subseteq |\overline{N},r|$. This completes the proof.

The following Remake follows from Theorem (6.2)

Remark (6.2) We remark that the condition that $B_{n,n} = O(1)$ is necessary (but not sufficient) for (23) to be satisfied.

An immediate corollary of Theorems (6.1) and (6.2) is the following corollary:

Corollary(6.1):Let $|P_n| \to \infty, |Q_n| \to \infty, |R_n| \to \infty$ and let $p_n \neq 0, q_n \neq 0$ and $r_n \neq 0$ (all n ≥ 0) then $|\overline{N},r| \sim |(\overline{N},p)(\overline{N},q)|$ if, and only if (8) and (23) are satisfied.

## VI. EXAMPLES

In this section we will give four examples. In the first example, we will show that (8) is satisfied but (23) is not valid. In the second example we will show that (23) is satisfied but (8) is not hold. The third example will be given to show that both (8) and (23) are satisfied and thus the equivalence may hold. Finally, in the fourth example we will show that even if $(\overline{N},r),(\overline{N},p)$ and $(\overline{N},q)$ are regular, (8) and (23) need not be satisfied.

Example (7.1): Let

$$q_n = n!;(n \geq 0), r_o = 1, r_n = n!n(n \geq 1) and (\overline{N},p) be (C,1), \quad (39)$$

then $|(\overline{N},r)| \subseteq |(C,1)(\overline{N},q|$ but not conversely.

Proof: The result that $|(C,1)(\overline{N},q)| \not\subseteq |\overline{N},r|$ follows form (32), (39) together with Theorem (6.2) Next, we will show that (8) is satisfied, and Theorem (6.1) yields that $|(\overline{N},r)| \subseteq |(\overline{N},p)(\overline{N},q|$. Using (39) we have

$$P_n = 1, P_n = n+1, R_n = (n+1)!;(n \geq 0) \text{ and } Q_n \sim n! \quad (40)$$

Using (39) and (40), one can easily seen that $|\gamma_{k-1,k-1} - \gamma_{k,k-1}|$ is bounded. Using (39) and (40), it follows from (10) that

$$\gamma_{n,k-1} - \gamma_{n+1,k-1} = \frac{1}{(n+1)(n+2)}\left\{k + (Q_k - \frac{(k+1)!k!}{k!k})\sum_{u=k}^{n}\frac{1}{Q_u}\right\}$$

$$-(Q_k - \frac{(k+1)!k!}{k!k})\frac{1}{Q_{n+1}}.\frac{1}{n+2};\ n \geq k \quad (41)$$

Observe that $q_k + q_{k-1} = (k+1)k-1)!$ givens

$$(Q_k - \frac{(k+1)!k!}{k!k}) = Q_{k-2}, \quad (42)$$

this implies that the right hand side of (41) reduces to:

$$\frac{1}{(n+1)(n+2)}\left\{k + Q_{k-2}\sum_{u=k}^{n}\frac{1}{Q_u}\right\} - \frac{Q_{n-2}}{(n+2)Q_{n+1}} \quad (43)$$

Observe that $Q_{n+1} > (n+1)Q_{k-2};\ (k \leq n)$ , we see that

$$\frac{k}{(n+1)(n+2)} > \frac{Q_{k-2}}{(n+2)Q_{n+1}}$$

so that the quantity in (43) is greater than zero, and implies that $Y_{n,k-1} - Y_{n+1,k-1} > o;\ k \leq n$ This implies that the left hand side of (8) reduces to.

$$|\gamma_{k-1,k-1} - \gamma_{k,k-1}| + \gamma_{k,k-1} - \lim_{N\to\infty}\gamma_{N+1,k-1} \quad (44)$$

Using (9), (10), (39) and (40) one can easily seen that the first two terms of (44) are bounded, and

$$|\gamma_{N+1,k-1}| = |\frac{1}{P_{N+1}}(P_{k-1} + Q_{k-2}\sum_{u=k}^{N+1}\frac{1}{Q_u})|$$

$$\leq \frac{P_{k-1}}{N+2} + \frac{N+2-k}{N+2} = O(1).$$

Therefore, the Quantity in (44) is bounded, and (8) is satisfied. This completes the proof.

Example (7.2): let

$$q_n = 2^n, \rho_n = 3^n \text{ and } r_n = 2n+1;(n \geq o), \quad (45)$$

ISBN: 978-988-19253-7-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

then $|(\overline{N},P)(\overline{N},q)|\subseteq|(\overline{N},r)|$ but the converse is not true.

Proof Using (45), we have.

$$Q_n = 2^{n+1}-1, P_n = \frac{(3^{n+1}-1)}{2} \, and \, R_n = (n+1)^2 ;(n\geq 0) \qquad (46)$$

Using (45) and (46), we see that $\gamma_{n,n}$ given in (9) is not bounded. Using Remark (6.1) we see that (8) is not satisfied, and Theorem (6.1) implies that $|\overline{N},r|\not\subset(\overline{N},P)(\overline{N},q)|$. Next, we will show that (23) is satisfied, and Theorem (6.2) yields the result. Using (45) and (46), it is clear that $B_{n,n}$ and $B_{n,n-1}$ are bounded, and for o ≤ k ≤ n-2, the left hand side of (23) is equivalent to:

$$|B_{k-1,k-1}|+|B_{k,k-1}|+|(k+1)^2 - \frac{(2^{k+1}-1)(2k+1)}{2^k} -$$
$$\frac{(3^k-1)(2^{k+1}-1)}{2.\,3^k}\left[\frac{2k+1}{2^k}-\frac{2k+3}{2^{k+1}}\right]\left|\frac{1}{(k+2)^2}+o(1),\right.$$

which is clearly bounded, and (23) holds. This completes the proof.

Example (7.3) Let

$$r_n = n!n, p_n = 2^n, q_n = 3^n \qquad ;(n\geq 0), \qquad (47)$$

Then $\left|(\overline{N},r)\right| \sim \left|(\overline{N},p)(\overline{N},q)\right|$

Proof Using (47), we have

$$R_n = (n+1)!, P_n = 2^{n+1}-1 \, and \, Q_n = \frac{(3^{n+1}-1)}{2} \quad ;(n\geq o) \quad (48)$$

Using (47) and (48), it can be easily shown that

$$|B_{k-1,k-1}|\leq 3 \quad , |B_{k,k-1}|\leq \frac{5}{2}, \qquad (49)$$

and that

$$\sum_{n=k+1}^{\infty}|B_{n,k-1}|\succ\prec|(k+1)! = \frac{(3^{k+1}-1)}{2}\frac{k!k}{3^k} -$$
$$-\frac{(2^k-1)(3^{k+1}-1)}{2.2^k}\Delta\frac{k!k}{3^k}\frac{1}{(k+2)!}$$
$$\leq \frac{1}{k+2}+\frac{3}{2(k+2)}+\frac{3}{2(k+2)}+\frac{1}{2}\succ\prec\frac{1}{2}, \qquad (50)$$

thus (23) follows from (49) and (50), and Theorem (6.2) implies that $|(\overline{N},P)(\overline{N},q)|\underline{C}|\overline{N},r|$. Next, we will show that (8) is satisfied, and the result follows from theorem (6.1) together with corollary (6.1).

Using (47) and (48), we see that the first term of the left hand side of (8) is bounded. When n ≥ k, we have

$$\gamma_{n,k-1}-\gamma_{n+1,k-1} = \frac{p_{n+1}}{P_n P_{n+1}}\left[P_{k-1}+(Q_k-\frac{R_k q_k}{r_k})\sum_{u=k}^{n}\frac{\rho_u}{Q_u}\right]$$
$$-\frac{p_{n+1}}{Q_{n+1}P_{n+1}}(Q_k-\frac{Q_k q_k}{r_k}) \qquad (51)$$

Using (47) and (48), we have

$$Q_k - \frac{R_k q_k}{r_k} > o \qquad ;k\geq 3 \qquad (52)$$

and

$$\frac{p_{n+1}p_n}{P_n P_{n+1}Q_n} > \frac{p_{n+1}}{Q_{n+1}P_{n+1}} \qquad (53)$$

Using (52) and (53), it follows from (51) that $\{\gamma_{n,k-1}\}$ is decreasing in n, so by Theorem (6.1), (8) is satisfied if, and only if (11) is satisfied. Using (47) and (48), it can be easily seen that (11) is satisfied. This completes the proof.

Example (7.4) Let

$$p_n = \begin{cases} 2^n & n \text{ odd} \\ 1 & n \text{ even} \end{cases}, r_n = \begin{cases} 1 & n \ odd \\ 4^n & n \ even \end{cases} \qquad (54)$$

$and \; q_n = 3^n \; ;(n\geq o),$

Then it is clearly that each of $(\overline{N},p),(\overline{N},r)$ and $(\overline{N},q)$ is regular but neither (8) nor (23) is satisfied.

Proof Using (51), we have

$$P_n = \begin{cases} 2^{n+1}-1 & n \ odd \\ n+1 & n \ even \end{cases},$$
$$R_n = \begin{cases} n+1 & n \ odd \\ \dfrac{4^{n+1}-1}{3} & n \ even \end{cases} \qquad (55)$$
$$and \; Q_n = \frac{3^{n+1}-1}{2};(n\geq o),$$

Using (54) and (55), it follows from (9) that when n is odd then,

$$\gamma_{n,n} = \frac{(n+1).3^n.2^n}{(2^{n+1}-1)\dfrac{(3^{n+1}-1)}{2}.1} \neq O(1),$$

which by Remark (6.1) implies that (8) is not satisfied. Also, it follows from (24) that when n is even, then

$$B_{n,n} = \frac{4^n.(n+1)(\dfrac{3^{n+1}-1}{2})}{(\dfrac{4^{n+1}-1}{3}).1.3^n} \neq O(1)$$

which by Remark (6.2) implies that (23) is not satisfied. This completes the Proof.

## REFERENCES

[1] 1. AL - Madi, A.K. "On Translativity of the product of Riesz sumanobility Methods" Indian Jour. Pure Appl. Maths. 11(11) (1980) pp. 1444 – 1457.
[2] 2. AL- Madi, A.K. "Comparison theorems of the product of Riesz summability methods" Bull. Cal. Math. Soc. 78 pp 305 – 307 (1986).

ISBN: 978-988-19253-7-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

[3] 3. AL- Madi, A.K. "Comparison theorems of two absolute weighted mean summability methods" The Punjab University Jour. of Maths V (XXI) (1988), pp. 23-28.
[4] 4. AL- Madi, A.K. "Inclusion Relation between two absolute summability methods" Qatar Univ. Sci. Jour. No. 148 (2000).
[5] 5. Rhoades, B.E. "Some comparison theorems for absolute summability" Indian J. of Pure and Applied Maths 16 (1) (1984) 23-30.
[6] 6. Daz, G. "Product of NÖrlund Methods", Indian Jour. Maths. Vol (10) No. 1 (1968) PP 25 – 43.
[7] 7. Dikshit G.D. "On inclusion relation between Riesz and Norlund means" Indian Jour. Maths Vol 7 (1965) pp 73-81.
[8] 8. Hardy, G.H. "Divergent Series" Oxford (1949).
[9] 9. Mears, F.M. "Absolute regularity and the Norlund mean' Annals of Maths. 38 (1937) 595 – 601.


ISBN: 978-988-19253-7-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)