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A Numerical Mesh-less Collocation Based Orthogonal M. L. S. Shape Function

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***Abstract*— In the present paper, the collocation based orthogonal moving least squares shape function is developed to solve class of partial differential equations. The classical moving least squares function has a wide range in different mesh-less methods but there will a problem encountered, ill conditioned stiffness matrix. The collocation based orthogonal moving least squares function overcomes this problem. The later has so many advantages over the moving least squares shape function, such as its simplicity in mathematics and computations, practical and efficient, and fewer number of scattered nodes required.**

***Index Terms*— Mesh-free, collocation techniques, orthogonal moving least squares, weighted orthogonal polynomials.**

I. INTRODUCTION

PARTIAL differential equations are of practical importance in science and technology due to their wide applications in different branches of science. Due to the difficulties encountered when solving nonlinear partial differential equations and due to the evolution of computers and programming, most researchers prefer the numerical methods. The mesh-less methods have become interesting and promising in solving partial differential equations because only scatter nodes are required to describe the problem domain **[1]**. The power of mesh-free methods depends on the efficiency of the good approximation of the shape function and its derivative to represent the problem geometry **[2]**. In the collocation method, the discretized system of algebraic equations is obtained from the direct enforce of the approximate solution and its derivatives into the governing equations, such as, the smooth particle hydrodynamic **[3]**, finite difference method **[4-5]** and other mesh-free collocation methods **[6-10]**. Although using moving least squares shape function obtains best approximation for scatter nodes, there are some well-known disadvantages such as complex computations, lack the Kronecker delta function and the final system of algebraic equations is sometimes ill-conditioned. Most of the articles dealing with the improved moving least squares have been suggested that the use of orthogonal shape function instead of non-orthogonal one can mitigate the ill-conditioning problem when solving the algebraic system of equations **[11]**. Recently, Zhuang and Aaugard analytically proof that the use of orthogonal shape functions does not remove the ill-conditioning problem but it make the evaluation of interpolation coefficients easier **[12]**. More recently, Carley examined the moving least squares sense in the framework of orthogonal polynomials, as applied to the estimation of derivatives **[13]**. In the present paper, the collocation based orthogonal moving least squares shape function is developed to solve class of partial differential equations. The classical moving least squares function has a wide range in different mesh-less methods but there will a problem encountered, ill conditioned stiffness matrix. The collocation based orthogonal moving least squares function overcomes this problem. The later has so many advantages over the moving least squares shape function, such as its simplicity in mathematics and computations, practical and efficient, and fewer number of scattered nodes required. Two different examples were solved to check the proposed method and its results were compared with the analytical solutions, and a very good agreement were obtained.

II. APPROXIMATION USING MOVING LEAST SQUARES

Although, the use of orthogonal least squares shape function does not remove the ill-conditioned problem but it makes the evaluation of the interpolation coefficients more easier. The local approximation of the function $u(x)$ by the trial function $u^h(x)$ can be defined as follows:

$$u^h(x)=\sum_{j=1}^{m}P_j(x)a_j(x)=P^T(x)a(x) \tag{1}$$

The basis functions $P(x)$ are chosen from a complete polynomial, and the unknown coefficients are determined by solving the quadratic weighted least squares as a minimization of the weighted discrete L_2 − norm defined as:

$$J(x)=\sum_{i=1}^{n}w(x-x_i)\left(P^T(x_i)a(x_i)-u_i\right)^2 \tag{2}$$

In equation (2) n is the number of nodes that are included in the support domain of the point of interest x and $w(x-x_i)$ is a nonzero positive weight function. The variation of the stationary value of $J(x)$ with respect to $a(x)$ leads to the following set of linear equations:

$$\mathrm{A}(x)a(x)=\mathrm{B}(x)\mathrm{u} \tag{3}$$

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Now, to describe the improved moving least squares, let the discrete weighted inner product $\langle f, g\rangle$ of two functions $f(x)$ and $g(x)$ with respect to the weight function $w(x)$, $w(x) > 0$, is defined as follows:

$$\langle f, g\rangle = \sum_{i=1}^{n} w(x_i) f(x_i) g(x_i) \tag{4}$$

Then the polynomials $p_0(x), p_1(x),.., p_m(x)$ are generated by Schmidt orthogonal procedure form an orthogonal system of polynomials with respect to the above inner product**[17]** . This means that $(p_i, p_j) = 0, (i \neq j)$

Therefore, equation (3) can be written as follows:

$$\begin{bmatrix} \langle p_1, p_1\rangle & \langle p_1, p_2\rangle & \cdots & \langle (p_1, p_m\rangle \\ \langle p_2, p_1\rangle & \langle p_2, p_2\rangle & \cdots & \langle p_2, p_m\rangle \\ \ldots & \ldots & \ldots & \ldots \\ \langle p_m, p_1\rangle & \langle p_m, p_2\rangle & \cdots & \langle p_m, p_m\rangle \end{bmatrix} \begin{bmatrix} a_1(x) \\ a_2(x) \\ \ldots \\ a_m(x) \end{bmatrix} = \begin{bmatrix} \langle p_1, u_i\rangle \\ \langle p_2, u_i\rangle \\ \ldots \\ \langle p_m, u_i\rangle \end{bmatrix} \tag{5}$$

The moment matrix $\widetilde{A}$ can be reduced to a positive definite diagonal matrix, if we apply the following orthogonal condition

$$\langle p_k, p_j\rangle = \sum_{i=1}^{n} w(x_i) p_k(x_i) p_j(x_i) = \begin{cases} 0 & k \neq j \\ \widetilde{A}_k & k = j \end{cases} \tag{6}$$

$k,j = 1,2,3,\ldots\ldots\ldots m$

Due to the orthogonal condition (6), the system (5) can be written as:

$$\begin{bmatrix} \langle p_1, p_1\rangle & 0 & \ldots & 0 \\ 0 & \langle p_2, p_2\rangle & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \langle p_m, p_m\rangle \end{bmatrix} \begin{bmatrix} a_1(x) \\ a_2(x) \\ \ldots \\ a_m(x) \end{bmatrix} = \begin{bmatrix} \langle p_1, u_i\rangle \\ \langle p_2, u_i\rangle \\ \ldots \\ \langle p_m, u_i\rangle \end{bmatrix} \tag{7}$$

$$\widetilde{A} = \begin{bmatrix} \langle p_1, p_1\rangle & 0 & \ldots & 0 \\ 0 & \langle p_2, p_2\rangle & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \langle p_m, p_m\rangle \end{bmatrix} \tag{8}$$

The coefficients $a_i(\mathrm{x})$ are determined from equation (7). Then the approximate solution will take the following form:

$$u^h(\mathrm{x}) = \sum_{i=1}^{n} \widetilde{\varphi}_i(\mathrm{x}) u_i = \widetilde{\varphi}_1 u_1 + \widetilde{\varphi}_2 u_2 + \ldots \widetilde{\varphi}_n u_n \tag{9}$$

The orthogonal moving least squares shape function is given by:

$$\widetilde{\varphi}_i(\mathrm{x}) = \mathrm{P}^{\mathrm{T}}(\mathrm{x}) \widetilde{A}^{-1}(\mathrm{x}) \mathbf{B}_i(\mathrm{x}) \tag{10}$$

The nodal weight function $\mathrm{w}(\mathrm{x} - \mathrm{x}_i)$ is a function which depends on the difference distance $\|\mathrm{x} - \mathrm{x}_i\|$. In the present paper we used the cubic-spline weight function of the following form **[18]**:

$$w(\mathrm{x} - \mathrm{x}_i) = \begin{cases} \frac{2}{3} - 4r^2 + 4r^3 & r \leq \frac{1}{2} \\ \frac{4}{3} - 4r + 4r^3 & \frac{1}{2} < r \leq 1 \\ 0 & r > 1 \end{cases} \tag{12}$$

where r is the normalized radius and is given by $r = \frac{\|\mathrm{x} - \mathrm{x}_i\|}{d_{mi}}$, where d_{mi} is the size of the support domain of i[th] node. In the present paper, some difficulties are encountered when the collocation points coincide with the field points, these difficulties are; the normalized radius and so its derivatives will vanish at the points of coincidence, and result in missing terms in the shape function derivatives. These missing terms cause a degradation of accuracy. Let $\mathrm{x} = \mathrm{x}_L + \varepsilon$, the shape function in equation (10) takes the following form

$$\widetilde{\varphi}_{\mathbf{i}}(\mathrm{x}_L + \varepsilon) = \mathrm{P}^{\mathrm{T}}(\mathrm{x}_L + \varepsilon) \widetilde{A}^{-1}(\mathrm{x}_L + \varepsilon) \mathbf{B}_i(\mathrm{x}_L + \varepsilon) \tag{12}$$

It is simple to prove that there is no missing terms in the shape function derivatives due to this threshold translation of the collocation points, and then the normalized radius $r = \frac{\|(\mathrm{x}_L + \varepsilon) - \mathrm{x}_i\|}{d_{mi}}$, as well as the higher derivatives of the weight function $w(\mathrm{x}_L + \varepsilon - \mathrm{x}_i)$ cannot be zero when $\mathrm{x}_L = \mathrm{x}_i$ and $\varepsilon \neq 0$.

Collocation based orthogonal moving least squares

In this section, mesh-less collocation method based on improved moving least squares shape function has been presented for a general advection diffusion equation as an example and of the following form:

$$\frac{\partial \mathrm{u}(\mathrm{x}, \mathrm{t})}{\partial \mathrm{t}} - \mathbf{L}\mathrm{u}(\mathrm{x}, \mathrm{t}) = \mathrm{f}(\mathrm{x}, \mathrm{t}), \mathrm{x} \in \Omega, \mathrm{t} > 0 \tag{13}$$

$$\beta \mathrm{u}(\mathrm{x}, \mathrm{t}) = \mathrm{g}(\mathrm{x}, \mathrm{t}), \mathrm{x} \in \partial\Omega, t > 0 \tag{14}$$

$$\mathrm{u}(\mathrm{x}, \mathrm{t}) = \mathrm{u}_0(\mathrm{t}) \tag{15}$$

where the differential operator $\mathbf{L} = k\nabla^2 + \nu\nabla$, and the boundary operator β i.e.. The different values of ξ_1, ξ can determine the type of boundary conditions. The physical parameter which describes the relation between the diffusion coefficient k and the advection coefficient ν is the Peclet number. The discretization of equations (13-15) starts with the approximation of time derivatives using forward difference and for the spatial derivatives, the Θ-weighted Crank- Nicholson scheme is used. Therefore, equations (13-14) will take the following form:

(10)

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$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}=\Theta \mathbf{L}u(x,t+\Delta t) \tag{16}$$
$$+(1-\Theta)\mathbf{L}u(x,t)+f(x,t+\Delta t)$$
$$\beta u(x,t+\Delta t)=g(x,t+\Delta t) \tag{17}$$

Re-arrange and simplify the two equations (16-17), leads to:

$$(1-\Delta t\Theta\,\mathbf{L})u^{n+1}=(1+\Delta t(1-\Theta)\mathbf{L})u^{n}+\Delta t f^{n+1} \tag{18}$$
$$\beta u^{n+1}=g^{n+1}$$

(19) where the notation u^{n+1} stands for the potential at time $(t^{n}+\Delta t)$, Δt is the time step size. For a selected number of collocation points x_i defined as $\{x_i\}_{i=1}^{N}$, of which $\{\mathbf{x}_i\}_{i=1}^{N_d}$ are the internal domain nodes and $\{x_i\}_{i=N_d+1}^{N}$ are the boundary nodes, the approximation of the potential functions are given by

$$u(x_i,t^n)=\sum_{j=1}^{N}\tilde{\varphi}(x_i,x_j)\lambda_j^{n}(t)=\Phi\,\boldsymbol{\lambda},\quad i=1,2,..,N \tag{20}$$

The global improved moving least squares shape function $\Phi=\tilde{\varphi}(x_i,x_j)$ is obtained from equation (10) and can be written in the following form:

$$\Phi=\begin{bmatrix}\tilde{\varphi}_{\mathbf{11}} & \tilde{\varphi}_{\mathbf{12}} & \tilde{\varphi}_{\mathbf{12}} & \cdots & \tilde{\varphi}_{\mathbf{1N}}\\ \tilde{\varphi}_{\mathbf{21}} & \tilde{\varphi}_{\mathbf{22}} & \tilde{\varphi}_{\mathbf{23}} & \cdots & \tilde{\varphi}_{\mathbf{2N}}\\ \tilde{\varphi}_{\mathbf{31}} & \tilde{\varphi}_{\mathbf{32}} & \tilde{\varphi}_{\mathbf{33}} & \cdots & \tilde{\varphi}_{\mathbf{3N}}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \tilde{\varphi}_{\mathbf{N1}} & \tilde{\varphi}_{\mathbf{N2}} & \tilde{\varphi}_{\mathbf{N3}} & \cdots & \tilde{\varphi}_{\mathbf{NN}}\end{bmatrix}=\begin{bmatrix}\Phi_{\Omega}\\ \Phi_{\partial\Omega}\end{bmatrix} \tag{21}$$

Where $\Phi_{\partial\Omega}$ and Φ_{Ω} are the improved moving least square shape functions for the boundary and internal domain nodes, respectively and the interpolation coefficients vector $\boldsymbol{\lambda}$ can be defined as:

$$\boldsymbol{\lambda}=[\lambda_1\ \lambda_2\ \lambda_3 \ldots\ldots\lambda_N\ ]^{T} \tag{22}$$

Enforce equations (20) into the two equations (18-19) and then the collocation scheme takes the following form:

$$(\Phi_{\Omega}-\Delta t\Theta\mathbf{L}\Phi_{\Omega})\lambda^{n+1}=(\Phi_{\Omega}+\Delta t(1-\Theta)\mathbf{L}\Phi_{\Omega})\lambda^{n}$$
$$+\Delta t\,f^{n+1} \tag{23}$$
$$\beta\Phi_{\partial\Omega}\boldsymbol{\lambda}^{n+1}=g^{n+1} \tag{24}$$

The algebraic equations (23-24) can be solved for $\boldsymbol{\lambda}^{\mathbf{n+1}}$ to obtain the following solution:

$$\boldsymbol{\lambda}^{n+1}=H_{\alpha}^{-1}H_{\gamma}\boldsymbol{\lambda}^{n}+H_{\alpha}^{-1}F^{n+1} \tag{25}$$

where

$$H_{\alpha}=\begin{bmatrix}\Phi_{\Omega}-\Delta t\Theta\mathbf{L}\Phi_{\Omega}\\ \beta\Phi_{\partial\Omega}\end{bmatrix},H_{\gamma}=\begin{bmatrix}\Phi_{\Omega}+\Delta t(1-\Theta)\mathbf{L}\Phi_{\Omega}\\ 0\end{bmatrix}$$
$$,F^{n+1}=\begin{bmatrix}\Delta t\,f^{n+1}\\ g^{n+1}\end{bmatrix} \tag{26}$$

Using equation (25) and equation (20), the potential u^{n+1} takes the following form:

$$u^{n+1}=\Phi^{-1}H_{\alpha}^{-1}H_{\gamma}u^{n}+\Phi^{-1}H_{\alpha}^{-1}F^{n+1} \tag{27}$$

Numerical results

Advection-Diffusion equation

Consider the following advection-diffusion equation **[19]**:

$$\frac{\partial u(x,t)}{\partial t}=k\frac{\partial^2 u(x,t)}{\partial x^2}+v\frac{\partial u(x,t)}{\partial x},\quad 0\le x\le 1,\ t>0 \quad (4.1\text{-}1)$$

The boundary and initial conditions are:

$$u(0,t)=u^{ex}(0,t)$$
$$u(1,t)=u^{ex}(1,t)$$
$$u(x,0)=u^{ex}(x,0)$$

Diffusion coefficient $k=1$
Advection coefficient $v=0.1$

The computational domain is discretized using Crank-Nicholson scheme, i.e. $\Theta=0.5$, is implemented with a time step $\Delta t=0.05$ sec, and a quadratic Schmidt basis function with cubic spline weight function is considered. The variation of temperature against space variable is plotted in figures (1-2) for two different arrangement of nodes 1×10 and 1×26 at different times $t=0.25\ \&\ 1$. As regards the computational speed, it is shown that the collocation method based improved moving least squares scheme consumed less time at the same number of collocation points compared with collocation based moving least squares, this is because, there are fewer coefficients have to be determined in the improved moving least squares than in the moving least squares. The effect of increasing number of collocation points is examined in the convergence plot, figure (3) and the relative error norm is defined by:

$$R.E.=\frac{\sqrt{\sum_{i=1}^{N}\left|u_{num}-u_{exact}\right|^2}}{\sqrt{\sum_{i=1}^{N}\left|u_{exact}\right|^2}}$$

It is evident that the collocation method based on improved moving least squares scheme can achieve higher convergence speed with fewer collocation points and the more increasing of collocation points have a slight effect on accuracy as shown in the convergence plot figure (3). Further, figures (4-5) are added to show that the present method is a powerful tool in the treatment of the advection-diffusion problems for different peclet numbers i.e., $P_e=1$ and $P_e=10$

Heat equation (pure diffusion)

Consider a 2-D diffusion problem over a unit square slab is considered **[5]**. The government differential equation is:

$$\frac{\partial u(x,y,t)}{\partial t}=k\left(\frac{\partial^2 u(x,y,t)}{\partial x^2}+\frac{\partial^2 u(x,y,t)}{\partial y^2}\right),\quad \mathbf{0\le x,y\le 1}$$

The analytic solution is:

$$u^{ex}(x,y,t)=(\sin\pi x+\sin\pi y)\exp(-k\pi^2 t)$$

The initial and boundary conditions are obtained from the analytic solution as follows:

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$$u(x,y,0)=u^{ex}(x,y,0)$$
$$u(0,y,t)=u^{ex}(0,y,t)$$
$$u(1,y,t)=u^{ex}(1,y,t)$$
$$u(x,0,t)=u^{ex}(x,0,t)$$
$$u(x,1,t)=u^{ex}(x,1,t)$$

The computational domain is discretized with different arrangement of collocation points and Crank-Nicholson scheme, i.e. $\Theta = 0.5$, is implemented with a time step $\Delta t = 0.1$ the diffusion coefficient $k = 0.1$, $\varepsilon = 0.05$, a linear Schmidt basis function with cubic spline weight function is considered and all the collocation points in the models were assumed to have the same support domain. The variation of temperature against space variables for a uniform 400 (20×20) collocation points after 10 iterations is shown in figure (6), which showing excellent agreement of the collocation method based improved moving least squares scheme to capture all points of the analytic solution, as well as a good surface properties, in other meaning the surface is smooth and has no oscillations as shown in figure (6) right. Two different uniform arrangements of nodes with 100 (10×10) and 169 (13×13) collocation points are used in the computations, as shown in figures (7-8) respectively. It is found that the collocation method based on IMLS scheme consumed less time with higher accuracy at the same number of collocation points compared with the collocation method based moving least squares as shown in figures (7-8) right. The effect of increasing number of collocation points is examined in the convergence plot, figure (9), which is plotted for different arrangements of nodes i.e. 7×7, 10×10, 12×12, 14×14. It is observed that in order to capture a relative error norm of $\approx 1.5\times10^{-4}$ the collocation based improved moving least squares required $\approx$ 90 collocation points, whereas the collocation based moving least squares required 130 collocation points. It is evident that the collocation method based improved moving least squares scheme can achieve higher convergence speed with fewer numbers of nodes. As shown in the convergence plot, figure (9).

III. Conclusion

In the present paper, the direct collocation method based orthogonal moving least squares shape function is developed for the solution of a class of partial differential equations. A translation of the collocation points by a threshold value ε is adopted to improve the degradation of accuracy associated with the arrangement of field nodes. There are many advantages of the proposed method such that, it is a truly mesh–less method, simple computations. A comparison with collocation based moving least squares shows that the present method is accurate and it can achieve higher convergence speed with fewer nodes as well as lesser computing time than the collocation method based on moving least squares. Finally, the collocation method based the improved moving least squares scheme is very useful and practical scheme.

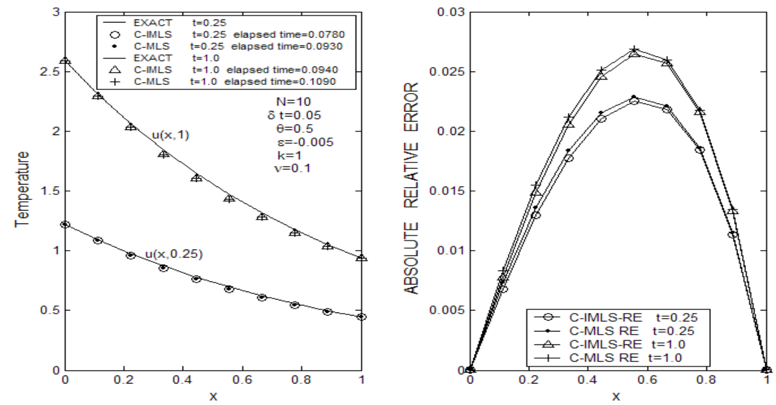

Figure (1) variation of temperature against space variable (left) and a corresponding absolute relative error for 1×10 nodes (right)

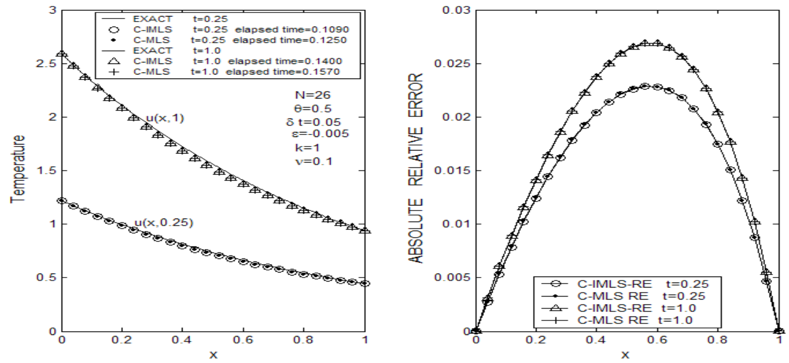

Figure (2) variation of temperature against space variable (left) and a corresponding absolute relative error for 1×26 nodes **(**right)

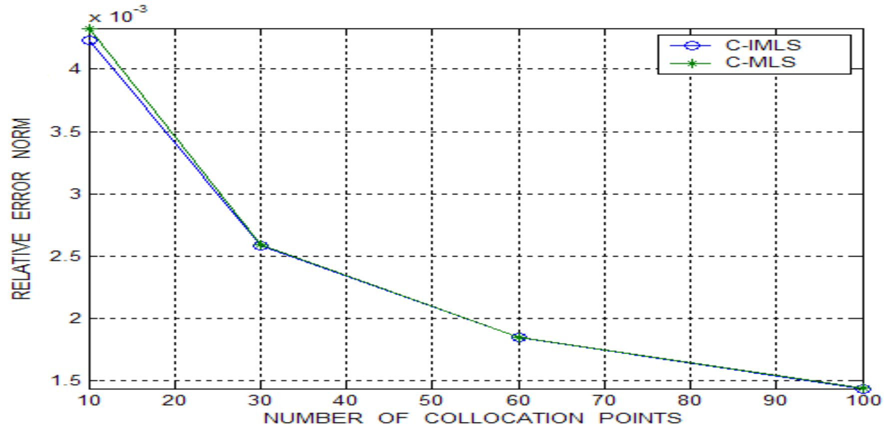

Figure (3) convergence plot for the 1D advection convection equation

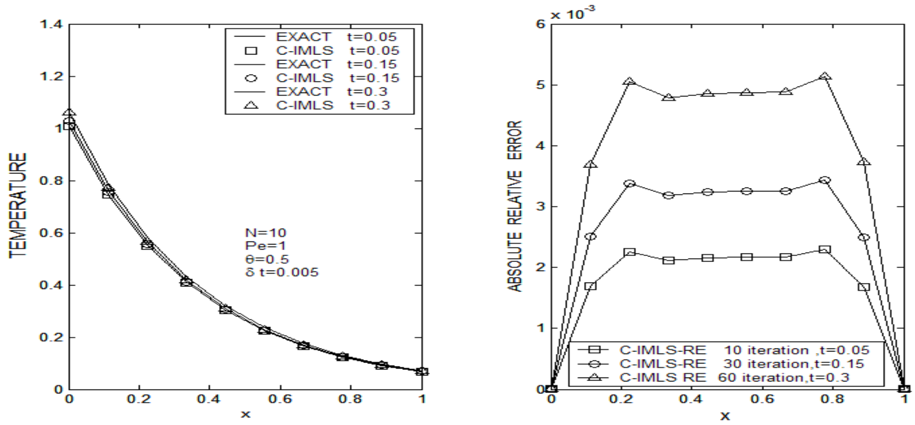

Figure (4) variation of temperature against space variable (left) and a corresponding absolute relative error at $P_e = 1$ (right)

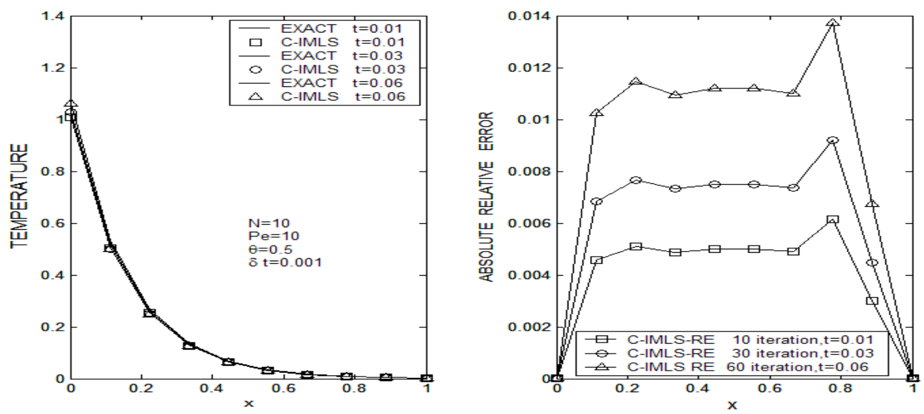

Figure (5) variation of temperature against space variable (left) and a corresponding absolute relative error at $P_e = 10$ (right)

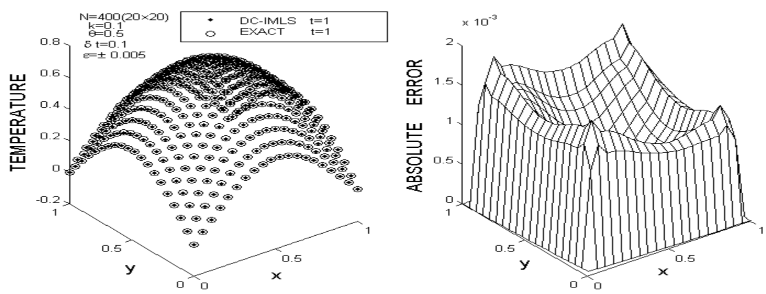

Figure (6) Collocation based IMLS solution of heat equation at t=1 (left) and a corresponding relative error for 400 (20×20) nodes (right)

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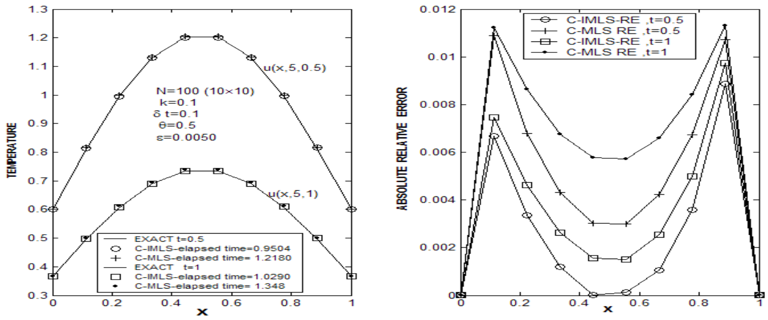

Figure (7) comparison of elapsed time results at t=0.5 and t=1 (left), the corresponding absolute relative error for 100 (10×10) (right)

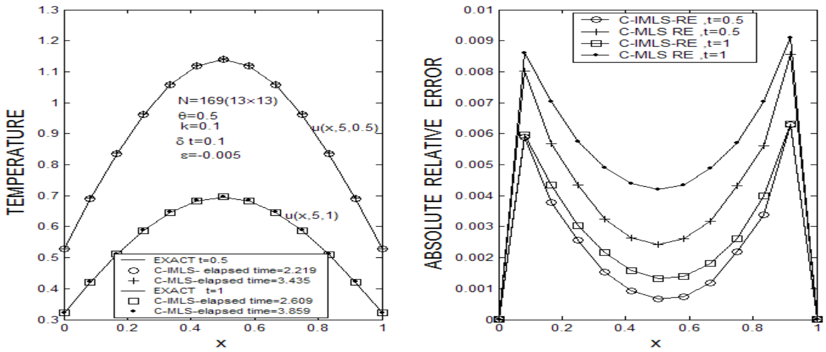

Figure (8) comparison for elapsed time results at t=0.5 and t=1 (left) and corresponding absolute relative error for 169 (13×13) nodes (right)

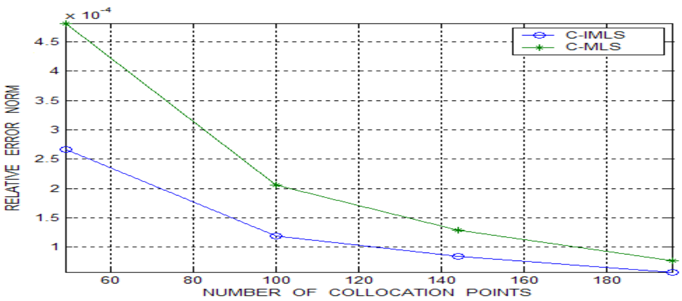

Figure (9) Convergence plot for 2D heat equation

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