

# Continued Fractions versus Decimal Expansions in Diagonalization

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**Abstract**— When a representation of real numbers, such as decimal expansions, allows us to use the diagonalization argument to prove that the set of real numbers is uncountable, can't we similarly apply the diagonalization argument to rational numbers in the same representation? Doesn't the diagonalization argument similarly prove that the set of rational numbers is uncountable too? This doubt concerning the diagonalization argument often arises in discrete mathematics and theory of computation classes. We consider two answers: one based on the familiar decimal expansions and the other based on continued fractions. The continued fraction-based answer circumvents a difficulty that is associated with the decimal expansion-based answer. A brief introduction to continued fractions is included and some related algorithms are presented in Java code.

**Index Terms**—diagonalization, continued fraction, decimal expansion, countable

## I. INTRODUCTION

IN discrete mathematics and theory of computation classes, the set of real numbers, henceforth denoted by  $\mathbf{R}$ , is commonly shown to be uncountable by a contradiction proof using the diagonalization argument. For the purpose of contradiction, such a proof assumes that  $\mathbf{R}$  is countable and thus there is a one-to-one correspondence between the set of natural numbers and  $\mathbf{R}$ . The assumption is then contradicted by showing the existence of a real number that is not included in the assumed one-to-one correspondence. More specifically, by the assumption that  $\mathbf{R}$  is countable, all of the real numbers in  $\mathbf{R}$  can be enumerated in a countably infinite sequence  $L$ ,

$$L = r_1, r_2, r_3, \dots$$

where each  $r_i$  is a real number in  $\mathbf{R}$ . Let each real number  $r_i$  be represented by its decimal expansion and let  $d_{ij}$  denote the digit in the  $j$ th decimal place of the decimal expansion of  $r_i$ . That is, each real number  $r_i$  in the sequence  $L$  is represented by the following expression:

$$w_i \cdot d_{i1} d_{i2} d_{i3} \dots$$

where  $w_i$  is the integer portion of  $r_i$ , and  $d_{i1} d_{i2} d_{i3} \dots$  is an infinite sequence of digits that represents the fractional part of  $r_i$  and possibly includes trailing 0's. The fractional parts of all of the real numbers  $r_1, r_2, r_3, \dots$  in the sequence  $L$ , that is, the digits  $d_{ij}$  for all  $i \geq 1$  and for all  $j \geq 1$ , form a matrix as

shown below in Fig. 1.

$$\begin{aligned} r_1 &= w_1 \cdot \mathbf{d}_{11} d_{12} d_{13} d_{14} d_{15} d_{16} \dots \\ r_2 &= w_2 \cdot d_{21} \mathbf{d}_{22} d_{23} d_{24} d_{25} d_{26} \dots \\ r_3 &= w_3 \cdot d_{31} d_{32} \mathbf{d}_{33} d_{34} d_{35} d_{36} \dots \\ r_4 &= w_4 \cdot d_{41} d_{42} d_{43} \mathbf{d}_{44} d_{45} d_{46} \dots \\ r_5 &= w_5 \cdot d_{51} d_{52} d_{53} d_{54} \mathbf{d}_{55} d_{56} \dots \\ r_6 &= w_6 \cdot d_{61} d_{62} d_{63} d_{64} d_{65} \mathbf{d}_{66} \dots \\ &\dots \\ &\dots \\ &\dots \end{aligned}$$

Fig. 1 Matrix and Diagonal

The diagonal of the matrix is the sequence of digits  $d_{kk}$  for all  $k=1,2,3,\dots$ , as shown in bold face in Fig. 1. From the diagonal, a new number

$$r_0 = w_0 \cdot d_{01} d_{02} d_{03} \dots$$

can be derived such that for all  $k=1, 2, 3 \dots$ ,  $d_{0k}$  differs from  $d_{kk}$  and is neither 0 nor 9. In other words, the new number  $r_0$  differs from the real number  $r_1$  in the 1<sup>st</sup> decimal place, from  $r_2$  in the 2<sup>nd</sup> decimal place and, in general, from each real number  $r_k$  in the  $k^{\text{th}}$  decimal place. Although a real number can have an alternative representation in decimal form, for example 5.26999999... is the alternative decimal representation of 5.27000000... and vice versa,  $r_0$  cannot be the alternative representation of any real number since the digits  $d_{01} d_{02} d_{03} \dots$  can neither be 0 nor 9. Therefore, the new number  $r_0$  cannot be equal to any real number in the sequence  $L$ . The existence of  $r_0$  contradicts the assumption that the sequence  $L$  includes all real numbers in  $\mathbf{R}$ . Hence,  $\mathbf{R}$  is not countable. This argument is based on the existence of the new number  $r_0$ , which is derived from the diagonal of the matrix in Fig. 1, hence the name *diagonalization argument*.

When such a proof is discussed in discrete mathematics or theory of computation classes, the following doubt often arises. When a representation of real numbers allows us to use the diagonalization argument to prove that the set of real numbers is uncountable, can't we apply the argument similarly to rational numbers in the same representation? Doesn't the diagonalization argument similarly prove that the set of rational numbers is uncountable too? For example, when rational numbers are represented by their decimal expansions, can't we similarly apply the diagonalization argument to their decimal expansions and prove that the set of rational numbers is uncountable, as we do in the case of real numbers?

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The next section discusses an answer that is based on the familiar decimal expansions, and section III provides an unconventional answer based on continued fractions. Section III also includes a brief introduction to continued fractions and presents some related algorithms in Java code.

The diagonalization argument is well known and is often discussed in textbooks, e.g., in [3,4]. The ideas used in the decimal expansion-based answer, to be presented in the next section, are also widely known, e.g. [2]. Continued fractions are a well studied subject and have been used to enumerate rational numbers, e.g., [1]. A salient feature of this article is our use of continued fractions in diagonalization to circumvent a difficulty that is associated with the conventional decimal expansion-based answer. This difficulty will be discussed in detail in the next section.

## II. AN ANSWER BASED ON DECIMAL EXPANSIONS

Suppose that we wish to use the diagonalization argument to prove that the set of rational numbers, henceforth denoted by  $\mathbf{Q}$ , is uncountable. *For the purpose of contradiction, assume that  $\mathbf{Q}$  is countable* and let  $L_{\mathbf{Q}} = f_1, f_2, f_3, \dots$  be an enumeration of  $\mathbf{Q}$ . Let each rational number  $f_i$  in  $L_{\mathbf{Q}}$  be represented by its decimal expansion  $w_i.d_{i1} d_{i2} d_{i3} \dots$ , with trailing 0's if  $f_i$  has a finite decimal expansion. The digits  $d_{ij}$  for all  $i \geq 1$  and for all  $j \geq 1$  then form a matrix similar to that shown in Fig. 1. A new number  $f_0$  can then be derived from the diagonal of this matrix, such that  $f_0$  differs from every rational number in the enumeration  $L_{\mathbf{Q}}$ . Does the existence of  $f_0$  not contradict the assumption that  $\mathbf{Q}$  is countable, as the existence of  $r_0$  does in the previous proof that  $\mathbf{R}$  is uncountable? Relying on our prior knowledge that  $\mathbf{Q}$  is countable, the answer is obvious: no, because  $f_0$  **cannot** be rational. Based on the knowledge that  $\mathbf{Q}$  is countable, the argument that  $f_0$  cannot be rational is straightforward: since  $\mathbf{Q}$  is countable,  $L_{\mathbf{Q}}$  includes all rational numbers and thus  $f_0$  cannot be rational because if it is, it must be included in  $L_{\mathbf{Q}}$  and since  $f_0$  differs from every number in  $L_{\mathbf{Q}}$ ,  $f_0$  must differ from itself! This argument, however, begs the question that we wish to decide by using the diagonalization argument: whether  $\mathbf{Q}$  is countable, i.e., whether  $L_{\mathbf{Q}}$  includes all of the rational numbers. The following argument, which appears to be widely known, does not rely on the knowledge that  $\mathbf{Q}$  is countable.

It is well known that the decimal expansion of any rational number, after a number of decimal places, infinitely repeats some finite sequence of digits. For example,  $1/2$  is  $0.5\ 0\ 0\ 0\dots$  and so on,  $1/6$  is  $0.1\ 6\ 6\ 6\dots$  and so on, and  $169/550$  is  $0.23\ 45\ 45\ 45\dots$  and so on. Such a decimal expansion is said to be *periodic* or *recurring*. The repeated sequence in a periodic decimal expansion is known as its *period*, such as the sequence 6 in  $0.1\ 6\ 6\ 6\dots$  and the sequence 45 in  $0.23\ 45\ 45\ 45\dots$ . The number of digits in a period is its *period length*. For example, the period length of  $0.1\ 6\ 6\ 6\dots$  is 1 and that of  $0.23\ 45\ 45\ 45\dots$  is 2. For the new number  $f_0$  to be rational, it must have a finite period length. Since every rational number has a finite period length, it is natural to ask whether  $f_0$  can have a finite period length that is the least common multiple of the period lengths of the rational numbers in the enumeration  $L_{\mathbf{Q}}$ . However, there is not an upper bound on the period lengths of rational numbers, and hence there is not an upper bound

on the least common multiple of the period lengths of the rational numbers in the enumeration  $L_{\mathbf{Q}}$ .

The decimal expansion-based argument presented above, though intuitive, **does not** actually provide a proof that  $f_0$  cannot have a finite period length. Without such a proof, and given the flexibility that is permitted in choosing the individual digits for the new number  $f_0$ , it is natural to wonder whether it is possible to choose the digits for  $f_0$  in such a manner as to derive a periodic decimal expansion for  $f_0$ .

The next section presents an answer that is based on continued fractions. This answer circumvents the difficulty in justifying that the new number  $f_0$  must be irrational, without begging the question.

## III. AN UNCONVENTIONAL ANSWER

This section presents a continued fraction-based answer. Since this answer uses continued fractions, a brief introduction to continued fractions is given in section III.A, and the answer itself is provided in section III.B.

### A. Continued Fractions

Any positive real number, rational or irrational, can be represented in the following *staircase notation* (Fig. 2), where  $a$  is a nonnegative integer and  $b, c, d, e, f, \dots$  are positive integers. This representation of a real number is known as a *continued fraction*.

$$a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{e + \frac{1}{f \dots}}}}}$$

Fig. 2 Staircase Notation

For ease of presentation, henceforth a list representation will be used instead of the staircase notation. In the list representation, the above continued fraction is written as

$$[a; b, c, d, e, f, \dots]$$

Hence, as a continued fraction, any nonnegative real number can be represented by a sequence of nonnegative integers. For example,  $6/7$  is  $[0; 1, 6]$  as a continued fraction and  $0.857142\ 857142\dots$  in decimal form; and the square root of 2 (an irrational number) is  $[1; 2, 2, 2, 2, 2, \dots]$  as a continued fraction and  $1.414213562373095\dots$  in decimal form. It is interesting to note the regularity that is present in the continued fraction  $[1; 2, 2, 2, 2, 2, \dots]$  and the lack of regularity in the decimal expansion  $1.414213562373095\dots$  of the same number.

Algorithms for converting a rational number or an irrational number to an equivalent continued fraction are well known and are presented below as static methods in Java.

```
public static String cf(int num, int den)
{ String result="";
  int temp;

  do {
    result=result+num / den + " ";
    temp=num % den;
    num=den;
    den=temp;
  } while(den != 0);
  return result;
} //end cf
```

The method `cf` in the above Java code implements an algorithm to convert a positive rational number to an equivalent continued fraction. Given two positive integers as the numerator (`num`) and the denominator (`den`) of a fraction representing a rational number, the method returns the equivalent continued fraction in the list representation - as a string consisting of a list of space-separated integers. For example, `cf(6,7)` returns the string "0 1 6". The algorithm is similar to the Euclidean algorithm for finding the greatest common divisor and terminates similarly. In other words, every rational number can be represented by a finite list of integers as a continued fraction.

```
public static String cfIr(double x)
{ String result="";
  double temp=x;

  do {
    result=result + (int) temp+" ";
    temp=1.0/(temp - (int) temp);
  } while (Math.abs(x-eval(
    new Scanner(result)))>EPS);

  return result;
} //end cfIr
```

The method `cfIr` in the above Java code implements a similar algorithm to convert a positive irrational number (approximated as a `double` value in the above Java code) to its equivalent continued fraction. Given an irrational number, the method returns an (approximate) continued fraction as a list. For example, the method call `cfIr(Math.sqrt(2))` (to find the continued fraction of the square root of 2) returns the list "1 2 2 2 2...". In the above code, the method call `eval(new Scanner(result))` returns the value of the continued fraction represented by the string parameter `result`, and `EPS` is a constant, usually a very small value, that specifies the accuracy of the resultant continued fraction, as described below. Every irrational number, when represented as a continued fraction, is an infinite sequence of integers. Of course, the method `cfIr` can only return a finite sequence that approximates a given irrational number (which is also approximated as a `double` in the above Java code). The constant `EPS` specifies how closely the returned continued fraction should approximate the given irrational number - a small `EPS` leads to a close approximation and a long list representing the continued fraction. The Java code for the method `eval` is not included here as the code is not

essential to an understanding of the algorithms presented above.

### B. An Answer Based on Continued Fractions

We now provide a continued fraction-based answer to the doubt described in section 1: when a representation of real numbers allows us to use the diagonalization argument to prove that the set of real numbers is uncountable, can we not apply the argument similarly to rational numbers in the same representation? Doesn't the diagonalization argument similarly prove that the set of rational numbers is uncountable too? Indeed, the continued fraction representation allows us to use the diagonalization argument to prove that the set of irrational numbers, and hence the set of real numbers, is uncountable, as outlined below. For the purpose of contradiction, let us assume that the set of positive irrational numbers can be enumerated in some sequence  $r_1, r_2, r_3, \dots$  with each irrational number  $r_i$  represented as a continued fraction by an infinite list  $[a_{i0}; a_{i1}, a_{i2}, a_{i3}, \dots]$ , where  $a_{ik}$  for each  $k \geq 0$  is the  $k^{\text{th}}$  integer in the list representation of the irrational number  $r_i$ . The integers  $a_{ik}$  for all  $i \geq 1$  and for all  $k \geq 1$  then form a matrix of integers, similar to that shown in Fig. 1, with the sequence  $a_{11}, a_{22}, a_{33}, \dots$ , that is, the sequence  $a_{kk}$  for all  $k \geq 1$ , as the diagonal of this matrix. A *new* irrational number  $r_0 = [a_{00}; a_{01}, a_{02}, a_{03}, \dots]$ , where  $a_{00}$  is any non-negative integer, can be derived from the diagonal of this matrix such that for all  $k \geq 1$ ,  $a_{0k}$  differs from  $a_{kk}$  and is not 0. That is,  $r_0$  differs from every irrational number in the enumeration  $r_1, r_2, r_3, \dots$ . This contradicts the assumption that the set of irrational numbers can be enumerated in a sequence. Therefore, the set of irrational numbers, and hence the set of real numbers, is uncountable.

Similar to the proof in section I, this diagonalization proof depends on the existence of a new number  $r_0 = [a_{00}; a_{01}, a_{02}, a_{03}, \dots]$  that is derived from the diagonal  $a_{11}, a_{22}, a_{33}, \dots$  such that  $a_{01}$  differs from  $a_{11}$ ,  $a_{02}$  from  $a_{22}$ ,  $a_{03}$  from  $a_{33}$ , and so on. While such a diagonal exists in any countably infinite sequence of irrational numbers represented as continued fractions, such a diagonal **does not exist** in any enumeration of rational numbers represented as continued fractions: when a rational number appears as the  $k^{\text{th}}$  element in the enumeration but has fewer than  $k$  integers representing its fractional part, the  $k^{\text{th}}$  position of the diagonal will not have a value, and this occurs infinitely many times in the diagonal. Hence, the diagonal as a sequence of integers that the diagonalization argument depends on does not exist in any enumeration of rational numbers represented as continued fractions. Thus, with the continued fraction representation, the diagonalization argument is not appropriate for rational numbers as it is for irrational numbers.

## IV. SUMMARY

Two answers to a doubt concerning diagonalization have been discussed: one based on the familiar decimal expansions and the other based on continued fractions. The ideas used in the first answer are well known. A salient feature of this article is our use of continued fractions in diagonalization to avoid a difficulty associated with the decimal expansion-based answer. When both rational and

real numbers are represented as decimal expansions, diagonalization can be applied to both rational and real numbers – however, diagonalization proves that real numbers are uncountable but **does not** prove the same for rational numbers – this is what often raises doubt. In contrast, with the continued fraction representation, the diagonalization argument is not appropriate for rational numbers as it is for irrational numbers.

#### REFERENCES

- [1] Czyz, J. and Self, W., “The Rationals are Countable: Euclid's Proof,” *The College Mathematics Journal*, Vol. 34, No. 5. (Nov., 2003), pp. 367-369.
- [2] MathPages, <http://www.mathpages.com/home/kmath371.htm>.
- [3] Rosen, K. H., *Discrete Mathematics and Its Applications*. McGraw-Hill, 6<sup>th</sup> ed., 2007.
- [4] Sipser, M., *Introduction to the Theory of Computation*. Thomson Course Technology, 2<sup>nd</sup> ed., 2006.