Modulated Extended Cumulative Exposure Model with Application to the Step-Up Voltage Test

Takenori Sakumura, and Toshinari Kamakura

***Abstract*—The extended cumulative exposure model (ECEM) includes features of the cumulative exposure model (CEM) and the memoryless model (MM). These often used to express the failure probability model in step-stress accelerated life test (SSALT). The CEM is widely accepted in reliability fields because accumulation of fatigue is considered to be reasonable. The MM is also used in electrical engineering because accumulation of fatigue is not observed in some cases. The ECEM includes features of both models. However, this model is sometimes difficult to estimate their parameters. We propose here a modulated ECEM model based on the time-scale. A simulation study supports the applicability of the proposed model.**

***Index Terms*—step-stress accelerated life test, cumulative exposure model, memoryless model, extended cumulative exposure model, time-scale.**

I. Introduction

In past decades, accelerated life testing (ALT) is one of the most useful methods to find the lifetime of industrial materials (e.g., electrical insulation) in short time [1]. Using failure data from ALT, we can estimate reliability of items such as mean lifetime and some quantile of the lifetime distribution at the service stress. For example, when the major factor of the deterioration of the insulation is the electrical stress, the power law,

$$meantime = K(v - v_{th})^{-n} \tag{1}$$

has been empirically used to estimate the lifetime of the insulation at the service stress, as shown in Figure 1. Here, K is a constant parameter, n is the degradation rate, v is the electrical stress, and v_{th} is the threshold stress below which the failures will not occur [2].

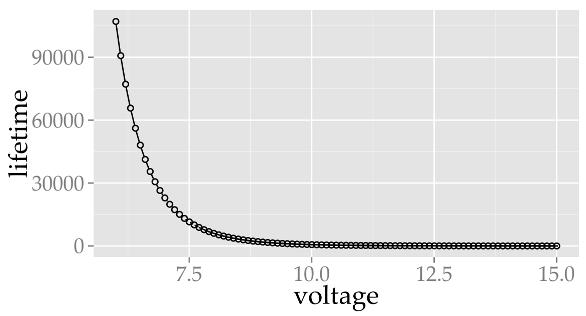

Fig. 1. The Power Law

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A Weibull distribution is often used for the reliability distribution in the power law model. Then, a Weibull power law [1] with the threshold stress can be assumed [3]:

$$F(t; v_i) = 1 - \exp\left\{-\left(K^{-1}(v_i - v_{th})^n t\right)^\beta\right\}. \tag{2}$$

Here, t is the time to failure, v_i is the applied stress at level i, and β is the shape parameter in the Weibull distribution. We can obtain these parameters using maximum likelihood estimation method. If the failure data from ALT is the type II censored data, the likelihood function is expressed as,

$$L = \prod_{i,j} f(t_{i(j)}; v_i)^{\delta i(j)} \times \left\{1 - F(t_{i(j)}; v_i)\right\}^{1-\delta i(j)}, \tag{3}$$

where, f is a pdf of F, $i(j)$ denotes that sample j is broken at level i, and $\delta i(j)$ is a indicator function.

To accelerate ALT much faster, step-stress accelerated life testing (SSALT) is considered to be a special case of ALT. In SSALT, the stress levels are increased during the test period in a specified discrete sequence [4], i.e. the step-stress test as shown in Figure 2.

The first report to use the maximum likelihood estimation method to SSALT combined with the Weibull power law was made Nelson [5], where the cumulative exposure model (CEM) is also proposed. The CEM is often used to express the failure probability models and is widely accepted in reliability fields because accumulation of fatigue to each stress level is considered to be reasonable.

In electrical insulation tests, however, another model is sometimes referred to. That is, the accumulation of fatigue is assumed to vanish each time the applied stress is reduced to zero. This indicates that the insulation materials possess a self-restoring feature when a rest time is given between the consecutive stress imposed. This model is called the memoryless model (MM) [6].

Hirose and Sakumura [6] proposed the extended cumulative exposure model (ECEM), which includes features of the CEM and the MM simultaneously. When there is accumulation of fatigue, the weight on the CEM may be large; however, when no accumulation of fatigue occurs, the weight on the MM may be large.

The fact that the ECEM includes the CEM and the MM deserves special emphasis, but this model is difficult for estimating for parameters owing to recursive calculations. Thus, we proposed an another modulated ECEM model. We assume vanishing of the fatigue accumulation as returning the imposed time.

II. Step-Stress Test

The step-stress test, also called the step-up voltage test, is discribed as follows. 1) A stress $v_1 = \Delta v$ is imposed on the insulation for time t_1. 2) If the insulation is not broken during

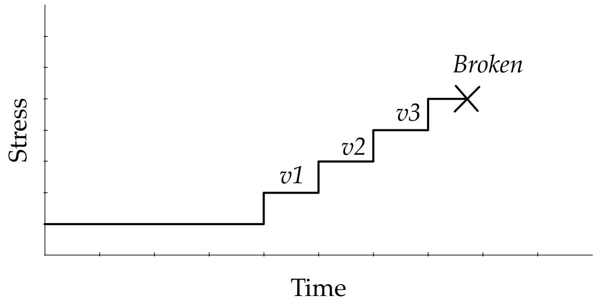

Fig. 2. Step-Stress Test

that period, then $v_2 = v_1 + \Delta v$ is loaded. 3) This procedure is repeated until the insulation is broken, as shown in Figure 2. 4) The resultant breakdown voltage is recorded as v_i when it fails, during that stress loading period between v_i and v_{i+1}.

III. Proposed Modulated Extended Cumulative Exposure Model Based On Time-Scale Changing

We briefly review the cumulative exposure model (CEM), the memoryless model (MM), and the extended cumulative exposure model (ECEM). Then, we propose an another modulated ECEM that would be resulted in time reducing. In this section, we assume the power law under the stress level v_i be expressed as Equation (1).

A. Cumulative Exposure Model, CEM

The CEM [5] connects the cumulative distribution functions by transforming the stress loading durations recursively. For example, at the very first step, we assume,

$$F_2(s_1) = F_1(t_1). \tag{4}$$

This means that the fatigue accumulation, $F_1(t_1)$, succeeds the next stress imposed. Then, s_1 is determined by

$$s_1 = (t_1 - t_0)\left(\frac{v_1 - v_{th}}{v_2 - v_{th}}\right)^n. \tag{5}$$

In general, $F_i(s_{i-1})$ and s_{i-1} are

$$F_i(s_{i-1}) = F_{i-1}(\Delta t_{i-1} + s_{i-2}), \tag{6}$$
$$s_{i-1} = (\Delta t_{i-1} + s_{i-2})\left(\frac{v_{i-1} - v_{th}}{v_i - v_{th}}\right)^n, \tag{7}$$

where, Δt_i expresses the time duration $t_i - t_{i-1}$ at a constant stress application v_i. Thus, a consistent continuous cumulative distribution function (CDF) is defined as,

$$G(t) = 1 - \exp\left\{-\varepsilon(t)^\beta\right\}, \tag{8}$$
$$\varepsilon(t) = K^{-1}\big[(v_i - v_{th})^n (t - t_{i-1}) + \sum_{j=1}^{i-1}\{(v_j - v_{th})^n \Delta t_j\}\big]. \tag{9}$$

This probability model will be shown in Figure 3.

B. Memoryless Model (MM)

We assume that all the fatigue accumulations vanish everytime the applied stress is reduced to zero. This indicates that the insulation materials reveal a self-restoring feature when a rest time is given between the consecutive stress imposed. This is called the memoryless model (MM) here. Considering that all the fatigue accumulations vanish when the applied stress is reduced to zero, the memoryless model is exactly the same model as that presented in Equation (2) after all.

C. Extended Cumulative Exposure Model (ECEM)

The extended cumulative exposure model (ECEM) is assumed partial fatigue accumulation instead of full fatigue accumulation as shown in Equation (10); the rate α expresses the fatigue accumulation rate. For example, at the first step,

$$F_2(s_1) = \alpha F_1(t_1), \quad (0 \le \alpha \le 1). \tag{10}$$

This means that some fraction, $(1-\alpha)F_1(t_1)$, will vanish, and that the remaining fatigue accumulation, $\alpha F_1(t_1)$, will succeed the next stress imposed. Then, s_1 is determined by

$$s_1 = K\frac{[-\log\{1 - \alpha F_1(t_1)\}]^{1/\beta}}{(v_2 - v_{th})^n}. \tag{11}$$

In general, $F_i(s_{i-1})$ and s_{i-1} become

$$F_i(s_{i-1}) = \alpha F_{i-1}(\Delta t_{i-1} + s_{i-2}), \tag{12}$$
$$s_{i-1} = K\frac{[-\log\{1 - \alpha F_{i-1}(\Delta t_{i-1} + s_{i-2})\}]^{\frac{1}{\beta}}}{(v_i - v_{th})^n} \tag{13}$$

and the corresponding continuous CDF is calculated recursively,

$$G(t) = 1 - \exp(-\varepsilon(t)^\beta), \tag{14}$$
$$\varepsilon(t) = [K^{-1}(v_i - v_{th})^n (t - t_{i-1}) + (-\log\{1 - \alpha F_{i-1}(\Delta t_{i-1} + s_{i-2})\})^{\frac{1}{\beta}}]. \tag{15}$$

This probability model will be shown in Figure 4.

We can see that this model is an extension of the CEM and the MM. When $\alpha = 0$, the model is reduced to the MM. When $\alpha = 1$, the model is reduced to the CEM. Thus, this model includes the CEM and the MM together. When $0 < \alpha < 1$, we can control the partial fatigue accumulation by a constant α.

D. Modulated Extended Cumulative Exposure Model

Here, we propose the modulated ECEM based on time scale changing. We consider that vanishings of the fatigue accumulations are defined as the imposed time reversing. The rate γ expresses the imposed time rate. For example, at the very first step, we assume,

$$F_2(s_1) = F_1(\gamma t_1), \quad (0 \le \gamma \le 1). \tag{16}$$

The above Equation (16) means that the imposed time will be $t_1 - (1-\gamma)t_1$ back to the time points $(1-\gamma)t_1$ and that the fatigue accumulation corresponding to the imposed time $F_1(\gamma t_1)$ is taken over to the next stress imposed. Then, s_1 is determined by

$$s_1 = \gamma t_1 \left(\frac{v_1 - v_{th}}{v_2 - v_{th}}\right)^n, \tag{17}$$

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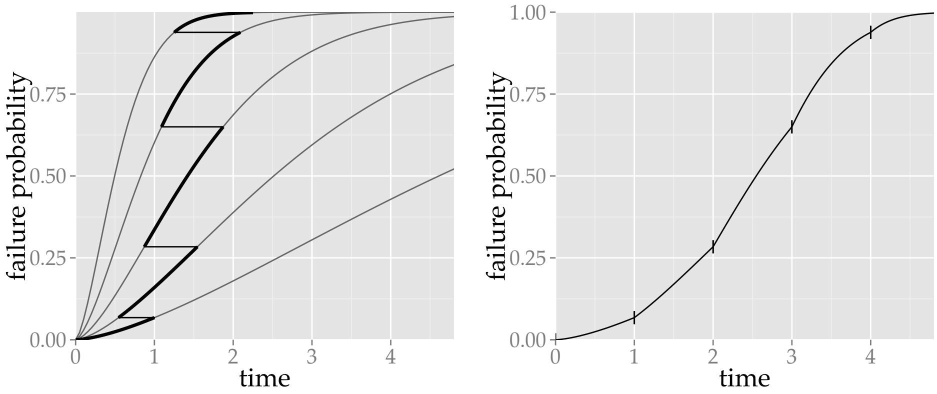

Fig. 3. Cumulative exposure model (CEM); left is PDF $F(t)$ and right is CDF $G(t)$

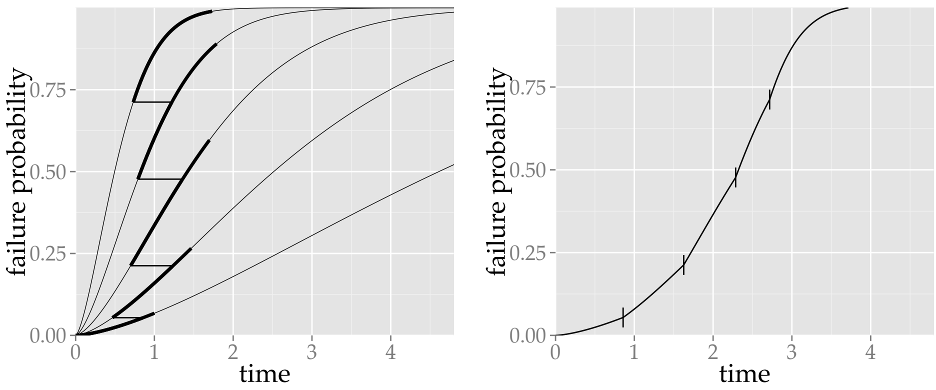

Fig. 4. Extended Cumulative Exposure Model (ECEM) at $\alpha = 0.8$; left is PDF $F(t)$ and right is CDF $G(t)$

and the corresponding continuous CDF becomes

$$\begin{aligned} G(t) &= F_2(t - t_1 + s_1) \\ &= 1 - \exp\big[- \big\{ K^{-1}\big[(v_2 - v_{th})^n (t - t_1) + \\ &\quad \gamma (v_1 - v_{th})^n \Delta t_1 \big] \big\}^{\beta} \big], \end{aligned} \tag{18}$$

where, Δt_i expresses the time duration $t_i - t_{i-1}$ at a constant stress application v_i. In general, $F_i(s_{i-1})$, and s_{i-1} become

$$F_i(s_{i-1}) = F_{i-1}(\gamma(\Delta t_{i-1} + s_{i-2})), \tag{19}$$

$$s_{i-1} = \sum_{j=1}^{i-1} \gamma^j \Delta t_{i-j} \left(\frac{v_{i-j} - v_{th}}{v_i - v_{th}} \right)^n . \tag{20}$$

The corresponding continuous CDF is

$$\begin{aligned} G(t) &= F_i(t - t_{i-1} + s_{i-1}) \\ &= 1 - \exp[-\varepsilon(t)^{\beta}] \end{aligned} \tag{21}$$

$$\begin{aligned} \varepsilon(t) &= K^{-1} \Bigg[(v_i - v_{th})^n (t - t_{i-1}) + \\ &\quad \sum_{j=1}^{i-1} \gamma^j (v_{i-j} - v_{th})^n \Delta t_{i-j} \Bigg]. \end{aligned} \tag{22}$$

Here, for numerical stability in computation, we transformed $K^{-1} = k^n$. Then Equation (22) becomes

$$\begin{aligned} \varepsilon(t) &= \Bigg[\{k(v_i - v_{th})\}^n (t - t_{i-1}) + \\ &\quad \sum_{j=1}^{i-1} \gamma^j \{k(v_{i-j} - v_{th})\}^n \Delta t_{i-j} \Bigg]. \end{aligned} \tag{23}$$

This probability model will be shown in Figure 5.

We can see that this model is another extension of the CEM and the MM as the ECEM is. When $\gamma = 0$, the model is equivalent to the MM, that is, Weibull distribution with power law, Equation (2). When $\gamma = 1$, the model is equivalent to the CEM. It is clear from the Equation (2), (8), (9) (21) and (22). Thus, this model includes the CEM and the MM together like the ECEM.

IV. Simulation Study

To check if we can estimate the parameters well, we perform a simulation study. The simulation condition is $v_{th} = 0$, $n = 10$, $K = 1/(0.632122 \times 20^{10})$, and $\beta = 0.5,\ 1,\ 1.5$, which is a mimicked case in typical solid electrical insulation [7]. The starting stress is 0 and the stress time duration to each stress is 1 (unit time). We set three cases $\gamma = 0,\ 0.5,\ 1$. The number of items, N, is 200 which is sufficiently large enough for estimation. The number of replications in simulation is 10,000 for each case.

The basic statistics (mean, standard deviation and the root mean square error (RMSE)) for the estimated parameters are shown in Table 1.

As a typical case, we present Figure 6 which shows the estimated parameter frequency distributions for $\hat{\gamma}$, $\hat{n}$, and $\hat{k}$, and $\hat{\beta}$ when $\gamma = 0.5$, $n = 10$, $k = K^{-1/n} = 0.05234677$, $\beta = 1.0$. From Table 1 and Figure 6, we can see that the parameters are well estimated.

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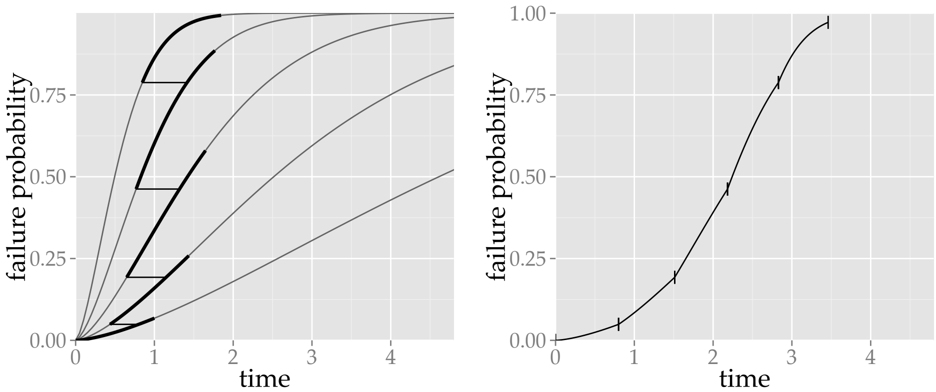

Fig. 5. Modulated Extended Cumulative Exposure Model Based On Time Scale Changing at $\gamma = 0.8$; left is PDF $F(t)$ and right is CDF $G(t)$

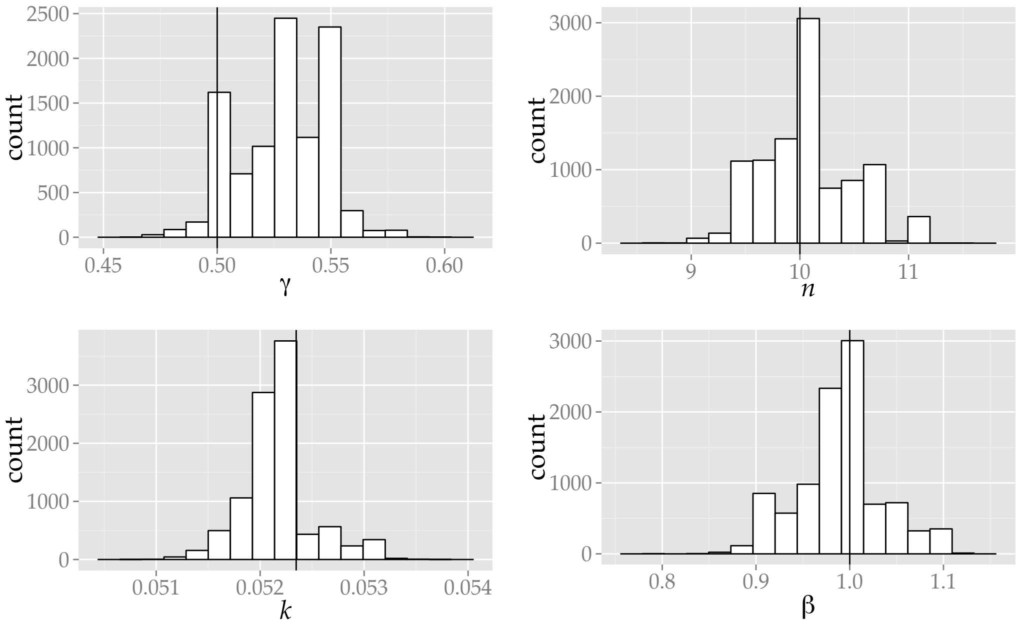

Fig. 6. Frequency distributions for estimated parameters in a simulation study. $\gamma = 0.5, n = 10, k = 0.05234677, \beta = 1.0$.

V. Discussion

Our proposed model, called the modulated ECEM, is similar to the ECEM, but our model has the good property that the Jacobian and Hessian Matrices can be obtained, thus, we can use the Newton-Raphson Method for estimating parameters. Therefore, the calculation speed is much faster.

In this section, we discuss about the relationship to the fatigue accumulation rate α and the imposed time rate γ. Now, we assume that Equation (10) and Equation (16) is equal, that is,

$$\begin{aligned} \alpha F(t_1) &= F(\gamma t_1) \\ &= 1 - \exp[-\{(k(v_1 - v_{th}))^n \gamma t_1\}^\beta] \\ &= 1 - \exp[-\{(k(v_1 - v_{th}))^n t_1\}^\beta \gamma^\beta] \\ &= 1 - \{1 - F(t_1)\}^{\gamma^\beta}. \end{aligned}$$

Therefore,

$$\alpha = F(t_1)^{-1} \left[1 - \{1 - F(t_1)\}^{\gamma^\beta}\right]. \tag{24}$$

Otherwise,

$$\gamma = \left[\frac{\log(1 - \alpha F(t_1))}{\log(1 - F(t_1))}\right]^{1/\beta} \tag{25}$$

We can confirm from Equation (24) and (25); if $\gamma = 1$, then $\alpha = 1$ and if $\gamma = 0$, then $\alpha = 0$.

VI. Conclusion

Failure data obtained from ALT are used to estimate reliability of items. In this paper, we have dealt with the case where the stress is discretely (stepwise) increases, i.e., the step-stress test. The cumulative exposure model (CEM) is often used to express the failure probability model in step-up accelerated life test. Contrary to this, the memoryless model (MM) is also used in electrical engineering because accumulation of fatigue is not observed in some cases. In general, the extended cumulative exposure model (ECEM) includes features of both models. The CDF of ECEM is defined recursively, thus, parameter estimation is difficult.

In this paper, we have proposed a new model, the another modulated extended cumulative exposure model based on

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TABLE I
BASIC STATISTICS FOR THE ESTIMATED PARAMETERS IN SIMULATION STUDY.

β	γ		$\hat{\gamma}$	$\hat{n}$	$\hat{k}$	$\hat{\beta}$
1	0	mean	1.06e-05	10.1	0.0524	0.997
		sd	3.64e-07	0.521	0.000442	0.0557
		rmse	7.18e-07	0.525	0.000445	0.0558
	0.5	mean	0.529	10	0.0522	0.989
		sd	0.0195	0.41	0.000338	0.045
		rmse	0.0352	0.413	0.000373	0.0463
	1	mean	0.949	10.1	0.0554	0.949
		sd	0.0296	0.264	0.00128	0.0474
		rmse	0.0587	0.286	0.00335	0.0694
0.5	0	mean	1.06e-05	9.94	0.0525	0.505
		sd	3.53e-07	0.547	0.000893	0.0265
		rmse	7.35e-07	0.55	0.000903	0.0271
	0.5	mean	0.549	9.92	0.0522	0.5
		sd	0.0441	0.48	0.000993	0.0235
		rmse	0.066	0.487	0.001	0.0235
	1	mean	0.964	10	0.0551	0.504
		sd	0.0259	0.307	0.00114	0.0264
		rmse	0.0445	0.307	0.00302	0.0266
1.5	0	mean	1.06e-05	10.1	0.0524	1.49
		sd	4.01e-07	0.501	0.000284	0.0828
		rmse	7.09e-07	0.509	0.000285	0.0834
	0.5	mean	0.521	10.1	0.0522	1.47
		sd	0.0178	0.324	0.000225	0.0688
		rmse	0.0279	0.341	0.000249	0.074
	1	mean	0.944	10.2	0.0557	1.36
		sd	0.0303	0.257	0.00123	0.0662
		rmse	0.0634	0.3	0.00353	0.156

The upper values are presented as mean, the middle as standard deviation, and the lower as root mean squared error (RMSE).

time scale changing. This model is similar to the ECEM, but we can obtain Jacobian and Hessian matrices for parameter estimation easily. To investigate whether we can estimate the parameters well, we have conducted a simulation study, and we have successfully obtained the reasonable parameters. A simulation study supports the validity of the proposed model.

APPENDIX
PARAMETER ESTIMATION

Figure 7 shows a profile log-likelihood parameterized by four parameter in the modulated ECEM. By the figure, the optimum point can be found. In searching for the optimum point of the log-likelihood function, it is necessary to use iterative methods because of its nonlinearity. Optimization schemes without derivatives are many proposed, but the Newton-Raphson method is finally indispensable for problems appeared in this paper. Because, the likelihood function is extraordinary flat.

A. Likelihood Function

Here, N is the sample size, and $i(j)$ denotes that sample j is broken at level i. To obtain the parameters, we pursue the maximizer of the parameters in the likelihood function,

$$L = \prod_{j=1}^{N} g(t_{i(j)})^{\delta i(j)} \times \{1 - G(t_T)\}^{1-\delta i(j)}, \tag{26}$$

when time of breakdown is continuously observed or unobserved by censoring to the right (type II); in the latter case, t_T is the truncation time. When we only observe the number of failures at each stress level, the likelihood function becomes

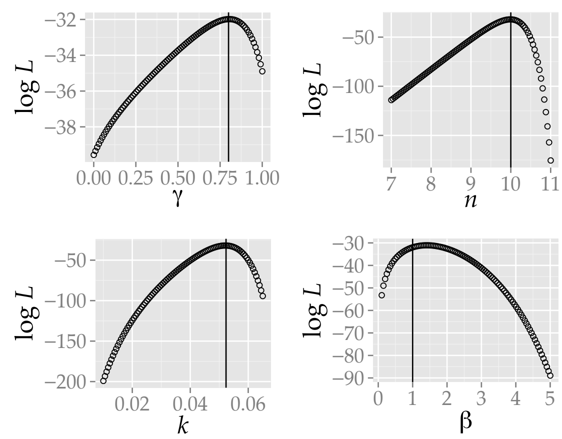

Fig. 7. Profile log-likelihood function for the modulated ECEM; model parameters are $\gamma = 0.8$, $n = 10$, $k = 0.05234677$, and $\beta = 1$, the sample size $N = 50$, and condition values $\Delta t = 1$, $\Delta v = 0.05$, and $v_{th} = 0$

to

$$L = \prod_{j=1}^{N} \left[G(t_{i(j)}) - G(t_{i(j)-1}) \right], \tag{27}$$

which provides the case for grouped data. Furthermore, if the truncation time $t_T(j)$ is provided with r failures, the likelihood function is

$$L = \prod_{j=1}^{N} \left[G(t_{i(j)}) - G(t_{i(j)-1}) \right] / G(t_T(j)), \tag{28}$$

In this paper, we deal with data from step-stress test for Equation (27).

B. The 1st and 2nd Derivatives

We denote Equation (27) as follows,

$$\begin{aligned} L &= \prod_{j=1}^{N} \left[G(t_{i(j)}) - G(t_{i(j)-1}) \right] \\ &= \prod_{j=1}^{N} \left[\exp\{-\varepsilon(t_{i(j)-1})^{\beta}\} - \exp\{-\varepsilon(t_{i(j)})^{\beta}\} \right] \\ &= \prod_{j=1}^{N} x_j, \end{aligned} \tag{29}$$

then the derivatives of the log-likelihood function are,

$$\frac{\partial \log L}{\partial \theta_1} = \sum_{j=1}^{N} \frac{1}{x_j} \frac{\partial x_j}{\partial \theta_1}, \tag{30}$$

where, θ_1 denotes γ, n, k, and β. Note that $k = K^{-1/n}$. And,

$$\begin{aligned} \frac{\partial x_j}{\partial \theta_1} = \exp(-\varepsilon_{i(j)-1}^{\beta}) \left(-\frac{\partial \varepsilon_{i(j)-1}^{\beta}}{\partial \theta_1} \right) \\ - \exp(-\varepsilon_{i(j)}^{\beta}) \left(-\frac{\partial \varepsilon_{i(j)}^{\beta}}{\partial \theta_1} \right), \end{aligned} \tag{31}$$

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where, $\varepsilon_{i(j)} = \varepsilon(t_{i(j)})$. The second derivatives are,

$$\begin{aligned}\frac{\partial^2 \log L}{\partial\theta_1\partial\theta_2} &= \sum_{j=1}^{N}\frac{\partial}{\partial\theta_1}\left(\frac{1}{x_j}\frac{\partial x_j}{\partial\theta_2}\right)\\ &= \sum_{j=1}^{N}\left\{-\frac{1}{x_j^2}\frac{\partial x_j}{\partial\theta_1}\frac{\partial x_j}{\partial\theta_2}+\frac{1}{x_j}\frac{\partial^2 x_j}{\partial\theta_1\partial\theta_2}\right\},\end{aligned} \tag{32}$$

where,

$$\begin{aligned}&\frac{\partial^2 x_j}{\partial\theta_1\partial\theta_2} = \exp(\varepsilon^\beta_{i(j)-1})\\ &\times\left\{\left(-\frac{\partial\varepsilon^\beta_{i(j)-1}}{\partial\theta_1}\right)\left(-\frac{\partial\varepsilon^\beta_{i(j)-1}}{\partial\theta_2}\right)+\left(-\frac{\partial^2\varepsilon^\beta_{i(j)-1}}{\partial\theta_1\partial\theta_2}\right)\right\}\\ &-\exp(\varepsilon^\beta_{i(j)})\\ &\times\left\{\left(-\frac{\partial\varepsilon^\beta_{i(j)}}{\partial\theta_1}\right)\left(-\frac{\partial\varepsilon^\beta_{i(j)}}{\partial\theta_2}\right)+\left(-\frac{\partial^2\varepsilon^\beta_{i(j)}}{\partial\theta_1\partial\theta_2}\right)\right\}.\end{aligned} \tag{33}$$

Here, we express $\varepsilon^\beta_{i(j)}$ and $\varepsilon^\beta_{i(j)-1}$ as ε^β for simplicity and its derivatives for parameter θ as follows;

$$\begin{aligned}\varepsilon_{\theta_1} &= \frac{\partial\varepsilon}{\partial\theta_1},\\ \varepsilon_{\theta_1\theta_2} &= \frac{\partial^2\varepsilon}{\partial\theta_1\partial\theta_2},\end{aligned}$$

then, the 1st derivatives are

$$\frac{\partial\varepsilon^\beta}{\partial\gamma} = \beta\varepsilon^{\beta-1}\varepsilon_\gamma, \tag{34}$$
$$\frac{\partial\varepsilon^\beta}{\partial n} = \beta\varepsilon^{\beta-1}\varepsilon_n, \tag{35}$$
$$\frac{\partial\varepsilon^\beta}{\partial k} = \beta\varepsilon^{\beta-1}\varepsilon_k, \tag{36}$$
$$\frac{\partial\varepsilon^\beta}{\partial\beta} = \varepsilon^\beta\log\varepsilon, \tag{37}$$

and the 2nd derivatives are

$$\frac{\partial^2\varepsilon^\beta}{\partial\gamma^2} = \beta(\beta-1)\varepsilon^{\beta-2}\varepsilon_\gamma^2+\beta\varepsilon^{\beta-1}\varepsilon_{\gamma\gamma}, \tag{38}$$
$$\frac{\partial^2\varepsilon^\beta}{\partial n^2} = \beta(\beta-1)\varepsilon^{\beta-2}\varepsilon_n^2+\beta\varepsilon^{\beta-1}\varepsilon_{nn}, \tag{39}$$
$$\frac{\partial^2\varepsilon^\beta}{\partial k^2} = \beta(\beta-1)\varepsilon^{\beta-2}\varepsilon_k^2+\beta\varepsilon^{\beta-1}\varepsilon_{kk}, \tag{40}$$
$$\frac{\partial^2\varepsilon^\beta}{\partial\beta^2} = \varepsilon^\beta\left\{\log\varepsilon\right\}^2, \tag{41}$$
$$\frac{\partial^2\varepsilon^\beta}{\partial\gamma\partial n} = \beta(\beta-1)\varepsilon^{\beta-2}\varepsilon_\gamma\varepsilon_n+\beta\varepsilon^{\beta-1}\varepsilon_{\gamma n}, \tag{42}$$
$$\frac{\partial^2\varepsilon^\beta}{\partial\gamma\partial k} = \beta(\beta-1)\varepsilon^{\beta-2}\varepsilon_\gamma\varepsilon_k+\beta\varepsilon^{\beta-1}\varepsilon_{\gamma k}, \tag{43}$$
$$\frac{\partial^2\varepsilon^\beta}{\partial n\partial k} = \beta(\beta-1)\varepsilon^{\beta-2}\varepsilon_n\varepsilon_k+\beta\varepsilon^{\beta-1}\varepsilon_{nk}, \tag{44}$$
$$\frac{\partial^2\varepsilon^\beta}{\partial\gamma\partial\beta} = \varepsilon^{\beta-1}\varepsilon_\gamma\left\{1+\beta\log\varepsilon\right\}, \tag{45}$$
$$\frac{\partial^2\varepsilon^\beta}{\partial n\partial\beta} = \varepsilon^{\beta-1}\varepsilon_n\left\{1+\beta\log\varepsilon\right\}, \tag{46}$$
$$\frac{\partial^2\varepsilon^\beta}{\partial k\partial\beta} = \varepsilon^{\beta-1}\varepsilon_k\left\{1+\beta\log\varepsilon\right\}, \tag{47}$$

Now, $\xi_{ij} = k(v_{i-j}-v_{th})$ and Equation (23), then

$$\varepsilon(t_i) = \sum_{j=0}^{i-1}\left[\gamma^j\xi_{ij}^n\Delta t_{i-j}\right]. \tag{48}$$

Finally, we can obtain the 1st and 2nd derivatives as follows;

$$\varepsilon_\gamma = \sum_{j=0}^{i-1}\left[j\gamma^{j-1}\xi_{ij}^n\Delta t_{i-j}\right], \tag{49}$$
$$\varepsilon_n = \sum_{j=0}^{i-1}\left[\gamma^j\xi_{ij}^n(\log\xi_{ij})\Delta t_{i-j}\right], \tag{50}$$
$$\varepsilon_k = \sum_{j=0}^{i-1}\left[\gamma^j nk^{n-1}\xi_{ij}^n\Delta t_{i-j}\right], \tag{51}$$
$$\varepsilon_{\gamma\gamma} = \sum_{j=0}^{i-1}\left[j(j-1)\gamma^{j-2}\xi_{ij}^n\Delta t_{i-j}\right], \tag{52}$$
$$\varepsilon_{nn} = \sum_{j=0}^{i-1}\left[\gamma^j\xi_{ij}^n(\log\xi_{ij})^2\Delta t_{i-j}\right], \tag{53}$$
$$\varepsilon_{kk} = \sum_{j=0}^{i-1}\left[\gamma^j n(n-1)k^{n-2}\xi_{ij}^n\Delta t_{i-j}\right], \tag{54}$$
$$\varepsilon_{\gamma n} = \sum_{j=0}^{i-1}\left[j\gamma^{j-1}\xi_{ij}^n(\log\xi_{ij})\Delta t_{i-j}\right], \tag{55}$$
$$\varepsilon_{\gamma k} = \sum_{j=0}^{i-1}\left[j\gamma^{j-1}nk^{n-1}\xi_{ij}^n\Delta t_{i-j}\right], \tag{56}$$
$$\begin{aligned}\varepsilon_{nk} = \sum_{j=0}^{i-1}&\left[\gamma^j nk^{n-1}\xi_{ij}^n(\log\xi_{ij})\Delta t_{i-j}\right.\\ &\left.+\gamma^j k^{-1}\xi_{ij}^n\Delta t_{i-j}\right].\end{aligned} \tag{57}$$

We can obtain the Jacobian and Hessian metrices from Equation (30)-(57).

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