Performance Comparison of Bivariate Copulas on the CUSUM and EWMA Control Charts

Sasigarn Kuvattana, Saowanit Sukparungsee, Piyapatr Busababodhin, and Yupaporn Areepong

Abstract— **This article proposes the comparison of control charts for bivariate copulas when observations are exponential distribution. The Monte Carlo simulation was used to investigate the value of Average Run Length (ARL) for in-control and out-of-control process. The dependence of random variables were used and measured by Kendall's tau in each copula. The simulation results show that performance of MCUSUM control chart was similar to MEWMA control chart for almost all shifts.**

Index Terms—**Copula, ARL, MCUSUM, MEWMA, Monte Carlo simulation**

I. INTRODUCTION

CONTROL chart is one of the most widely applied statistical process control (SPC) which is a statistical and visual tool designed to detect shifts in manufacturing process. It is designed and evaluated under the assumption that the observations are from processes which are independent and identically distributed (i.i.d.). Univariate control chart is devised to monitor the quality of a single process characteristic but modern process often monitor more than one quality characteristic. These quality characteristics are clearly correlated and separate univariate control charts for monitoring individual quality characteristic which may not be adequate for detecting changes in the overall quality of the product. Thus, it is desirable to have control charts that can monitor multivariate measurements and they are referred to as multivariate statistical process control charts.

Multivariate statistical process control (MSPC) charts are the most rapidly developing sections of statistical process control [1] and lead to an interest in the simultaneous inspection of several related quality characteristics [2-3]. The three most common multivariate control charts are the multivariate cumulative sum (MCUSUM) [4] control chart, the multivariate exponentially weighted moving average (MEWMA) [5] control chart and the multivariate Shewhart control chart. Multivariate Shewhart control chart is used to detect large shifts in the mean vectors. The MEWMA and MCUSUM are commonly used to detect small or moderate shifts in the mean vectors [6]. MSPC procedures are based on a multi-normality assumption and independence but many processes are often non-normality and correlation. Moreover, multivariate control charts are the lack of the related joint distribution and copula can specify this property. Copulas are functions that join multivariate distribution functions to their one-dimensional margins . It can estimate joint distribution of nonlinear outcomes and explain the dependence structure among variables through the joint distribution by eliminating the effect of univariate marginals. Many researchers have developed the copula on MCUSUM and MEWMA charts (see [5], [7-14]).

This article presents comparison of efficiency between MCUSUM and MEWMA control charts when observations are exponential distribution with the means shifts and use a bivariate copulas function for specifying dependence between random variables.

II. THE MULTIVARIATE CUMULATIVE SUM CONTROL CHART

The multivariate cumulative sum (MCUSUM) control chart is the multivariate extension of the univariate cumulative sum (CUSUM) chart. The MCUSUM chart was initially proposed by Crosier [15]. The MCUSUM chart may be expressed as follows:

$$C_t = [(\mathbf{S}_{t-1} + \mathbf{X}_t - \mathbf{a})' \textstyle\sum^{-1} (\mathbf{S}_{t-1} + \mathbf{X}_t - \mathbf{a})]^{1/2} \; ; \; t = 1, 2, 3, \ldots \qquad (1)$$

where covariance ($\sum$) and $\mathbf{S}_t$ are the cumulative sums expressed as:

$$\mathbf{S}_t = \begin{cases} \mathbf{0}, & \text{if } C_t \le k \\ (\mathbf{S}_{t-1} + \mathbf{X}_t - \mathbf{a})\left(1 - \dfrac{k}{C_t}\right), & \text{if } C_t > k \end{cases} \qquad (2)$$

the reference value $k > 0$ and $\mathbf{a}$ is the aim point or target value for the mean vector [16]. The control chart statistics for MCUSUM chart is

$$Y_t = [\mathbf{S}_t' \textstyle\sum^{-1} \mathbf{S}_t]^{1/2} \; ; \; t = 1, 2, 3, \ldots \qquad (3)$$

The signal gives an out-of-control if $Y_t > h$ where h is the control limit [17].

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III. The Multivariate Exponentially Weighted Moving Average Control Chart

Lowry et.al. [18] have developed a multivariate exponentially weighted moving average (MEWMA) control chart. The MEWMA control chart is a logical extension of the univariate exponentially weighted moving average (EWMA) control chart. The EWMA statistic assigns less and less weight to the past observations than the current observation [6].

Suppose that $\mathbf{X}_i$ is a $p\times 1$ vector of observations at sample $i=1,2,3,...$ with target mean vector equal to the zero vector and known covariance matrix Σ and the vectors are independent over time. The extension of the EWMA control chart to the multivariate case is defined as follows:

$$\mathbf{Z}_i = \lambda\mathbf{X}_i + (1-\lambda)\mathbf{Z}_{i-1} \tag{4}$$

with the scalar charting constant λ, $0\le\lambda\le 1$ which may be adjusted to change the weighting of the past observations and $Z_0 = 0$. The quantity plotted on the control chart of the MEWMA [19] is

$$T_i^2 = \mathbf{Z}_i^{/}\textstyle\sum_{\mathbf{Z}_i}^{-1}\mathbf{Z}_i \tag{5}$$

The control chart signals a shift in the mean vector when $T_i^2 > h$, where h is the control limit chosen to achieve a desired in-control and the covariance matrix for $\mathbf{Z}_i$ is

$$\textstyle\sum_{\mathbf{Z}_i} = \frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2i}\right]\sum \tag{6}$$

which is analogous to the variance of the univariate EWMA. The Average Run Length performance of the MEWMA control chart depends on the off-target mean vector because the process mean vector shifts from the zero vector to a new out-of-control vector . The shift size is reported in terms of a quantity $\delta = (\boldsymbol{\mu}'\sum^{-1}\boldsymbol{\mu})^{1/2}$, it usually called the non-centrality parameter. The value $\delta = 0$ is the in-control state and large values of δ correspond to bigger shifts in the mean.

Note that if λ in equation (4) equal to 1, the MEWMA control chart statistic reduces to $T_i^2 = \mathbf{X}_i^{/}\sum_{\mathbf{Z}_i}^{-1}\mathbf{X}_i$ the statistic used to on the Hotelling T^2 control chart [19].

IV. Copula Function

Copulas introduced by Sklar [20]. According to Sklar's theorem for a bivariate case, let X and Y be continuous random variables with joint distribution function H and marginal cumulative distribution $F(x)$ and $F(\mathrm{y})$, respectively. Then $H(x,y) = C\left(F(x),F(y);\theta\right)$ with a copula $C:\left[0,1\right]^2 \to \left[0,1\right]$ where θ is a parameter of the copula called the dependence parameter, which measures dependence between the marginals. For the purposes of statistical method it is desirable to parameterize the copula function. Let θ denote the association parameter of the bivariate distribution and there exists a copula C. Then $F(x)=u$, $F(y)=v$ where u and v are uniformly distributed variates [21]. This paper focuses on two types of Archimedean copulas which are Clayton and Frank [22].

Archimedean copulas

Let a class Φ of functions $\phi:\left[0,1\right]\to\left[0,\infty\right]$ with continuous, strictly decreasing, such that $\phi(1)=0$, $\phi'(t)<0$ and $\phi''(t)>0$ for all $0<t<1$ [22-24]. Archimedean copulas of two types are generated as follow:

A. Clayton copula

$$C(u,v;\ \theta) = \left[max(u^{-\theta}+v^{-\theta}-1,\ 0)\right]^{-1/\theta}, \tag{7}$$

where $\phi(t) = (t^{-\theta}-1)/\theta$; $\theta\in[-1,\infty)\setminus 0$.

B. Frank copula

$$C(u,v;\theta) = -\frac{1}{\theta}ln(1+\frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}), \tag{8}$$

where $\phi(t) = -ln(\frac{e^{-\theta t}-1}{e^{-\theta}-1})$; $\theta\in(-\infty,\infty)\setminus 0$.

V. Dependence Measures for Data

Generally, a parametric measure of the linear dependence between random variables is correlation coefficient and nonparametric measures of dependence are Spearman's rho and Kendall's tau. According to the earlier literature, the copulas can be used in the study of dependence or association between random variables and the values of Kendall's tau are easy to calculate so this measure is used for observation dependencies.

Let X and Y be continuous random variables whose copula is C then Kendall's tau for X and Y is given by $\tau_c = 4\iint_{\mathbf{I}^2} C(u,v)\ dC(u,v)$ - 1 where τ_c is Kendall's tau of copula C and the unit square I^2 is the product $\mathrm{I}\times\mathrm{I}$ where $\mathrm{I}=\left[0,1\right]$ and the expected value of the function $C(u,v)$ of uniform (0,1) random variables U and V whose joint distribution function is C, i.e., $\tau_c = 4E[C(U,V)]-1$ [23].

Genest and McKay [22] considered Archimedean copula C generated by ϕ, then $\tau_{Arch} = 4\int_0^1 \frac{\phi(t)}{\phi'(t)}\ dt + 1$ where τ_{Arch} is Kendall's tau of Archimedean copula C.

A. Clayton copula

$$\tau = \theta/(\theta+2)\ ;\ \theta = [-1,\ \infty)\setminus\{0\} \tag{9}$$

B. Frank copula

$$\tau = 1+4\left(\frac{1}{\theta}\int_0^{\theta}\frac{\mathrm{t}}{\mathrm{e}^{\mathrm{t}}-1}\ \mathrm{dt}\ \text{-1}\right)/\theta\ ;\ (-\infty,\infty)\setminus\{0\} \tag{10}$$

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VI. AVERAGE RUN LENGTH AND SIMULATION RESULTS

The popular performance measure for control charts is the Average Run Length (ARL). ARL is classified into ARL_0 and ARL_1, where ARL_0 is the Average Run Length when the process is in-control and ARL_1 is the Average Run Length when the process is out-of-control [25]. The copula approach focuses on Clayton and Frank. This article use Monte Carlo simulation in R statistical software [26-28] with the number of simulation runs 50,000 and sample size is 1,000. Observations were from exponential distribution with parameter (α) equal to 1 for in-control process ($\mu_0 = 1$) and the shifts of the process level (δ) by $\mu = \mu_0 + \delta$. The process means are equal to 1, 1.25, 1.5, 1.75, 2, 2.25 and 2.5.

Copula estimations are restricted to the cases of dependence (positive and negative dependence) and all copula models, setting θ correspondes with Kendall's tau. The level of dependence is measured by Kendall's tau values $(-1 \le \tau \le 1)$ which are defined to 0.8 and -0.8, respectively.

The results of simulation experiments are showed in Table I - IV for the different values of Kendall's tau and denote μ_1 for the variables X and μ_2 for the variables Y. The control chart was chosen by setting the desired ARL_0 = 370 for each copula. Table I-II show strong positive dependence $(\tau = 0.8)$ and Table III -IV show strong negative dependence $(\tau = -0.8)$. For example, Table I shows positive dependence $(\tau > 0)$ when the shifts in one exponential parameter. In the case of $\mu_1 = 1$ and μ_2 is changed, for all shifts $(\mu_1 = 1,\ 1.25 \le \mu_2 \le 2.5)$, the ARL_1 values of Frank copula on MCUSUM control chart are less than MEWMA control chart. In the case of $\mu_2 = 1$ and μ_1 is changed, for small shift $(\mu_1 = 1.25,\ \mu_2 = 1)$, the ARL_1 value of Clayton copula on MCUSUM control chart is less than MEWMA control chart, for moderate and large shifts $(1.5 \le \mu_1 \le 2.5,\ \mu_2 = 1)$, the ARL_1 values of Frank copula on MCUSUM control chart are less than MEWMA control chart. Table II shows positive dependence $(\tau > 0)$ when the shifts in both exponential parameters. For small shifts $(1.25 \le \mu_1 \le 1.75,\ 1.25 \le \mu_2 \le 1.75)$, the ARL_1 values of Clayton copula on MEWMA control chart are less than MCUSUM control chart , for moderate and large shifts $(2 \le \mu_1 \le 2.5,\ 2 \le \mu_2 \le 2.5)$, the ARL_1 values of Clayton copula on MCUSUM control chart are less than MEWMA control chart.

TABLE I
COMPARISON OF ARL ON CONTROL CHARTS WITH KENDALL'S TAU VALUE EQUAL TO 0.8 WHEN THE SHIFTS IN ONE EXPONENTIAL PARAMETER.

Parameters		ARL_0 and ARL_1			
		MCUSUM		MEWMA	
μ_1	μ_2	Clayton	Frank	Clayton	Frank
1	1	370.041	370.129	369.984	370.042
1	1.25	195.156	*195.096*	200.143	197.932
1	1.5	107.211	*105.377*	109.134	106.283
1	1.75	62.768	*59.581*	68.107	66.309
1	2	38.588	*36.230*	47.712	44.407
1	2.25	25.877	*23.634*	34.206	29.761
1	2.5	18.506	*16.566*	23.184	19.036
1	1	370.041	370.129	369.984	370.042
1.25	1	*195.710*	196.190	199.332	197.433
1.5	1	108.657	*105.015*	109.455	106.719
1.75	1	63.247	*60.205*	67.890	66.209
2	1	38.869	*36.091*	47.970	44.633
2.25	1	25.919	*23.573*	34.067	29.834
2.5	1	18.501	*16.441*	23.432	19.060

TABLE II
COMPARISON OF ARL ON CONTROL CHARTS WITH KENDALL'S TAU VALUE EQUAL TO 0.8 WHEN THE SHIFTS IN BOTH EXPONENTIAL PARAMETERS.

Parameters		ARL_0 and ARL_1			
		MCUSUM		MEWMA	
μ_1	μ_2	Clayton	Frank	Clayton	Frank
1	1	370.041	370.129	369.984	370.042
1.25	1.25	145.866	149.487	*145.435*	150.129
1.5	1.5	76.576	80.001	*74.421*	78.224
1.75	1.75	46.467	48.708	*45.555*	49.047
2	2	*30.540*	31.957	31.957	34.814
2.25	2.25	*21.532*	22.423	23.649	25.791
2.5	2.5	*15.978*	16.533	18.390	19.461

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TABLE III
COMPARISON OF *ARL* ON CONTROL CHARTS WITH KENDALL'S TAU VALUE EQUAL TO -0.8 WHEN THE SHIFTS IN ONE EXPONENTIAL PARAMETER.

Parameters		ARL_0 and ARL_1			
		MCUSUM		MEWMA	
μ_1	μ_2	Clayton	Frank	Clayton	Frank
1	1	370.073	369.898	370.048	369.860
1	1.25	217.642	213.616	206.189	*203.742*
1	1.5	125.222	122.585	116.565	*114.836*
1	1.75	73.649	*72.141*	75.813	73.019
1	2	47.062	*45.447*	53.299	51.502
1	2.25	32.383	*30.621*	40.395	38.061
1	2.5	23.563	*22.106*	29.449	27.579
1	1	370.073	369.898	370.048	369.860
1.25	1	217.844	211.881	205.465	*203.404*
1.5	1	125.629	121.631	116.712	*114.335*
1.75	1	74.104	*71.510*	74.983	72.594
2	1	47.452	*44.831*	53.409	51.980
2.25	1	32.486	*30.440*	40.168	38.269
2.5	1	23.635	*22.082*	29.611	27.783

TABLE IV
COMPARISON OF *ARL* ON CONTROL CHARTS WITH KENDALL'S TAU VALUE EQUAL TO -0.8 WHEN THE SHIFTS IN BOTH EXPONENTIAL PARAMETERS.

Parameters		ARL_0 and ARL_1			
		MCUSUM		MEWMA	
μ_1	μ_2	Clayton	Frank	Clayton	Frank
1	1	370.073	369.898	370.048	369.860
1.25	1.25	158.821	159.226	*148.415*	149.336
1.5	1.5	80.217	81.915	*78.387*	78.738
1.75	1.75	*45.524*	46.497	49.877	49.953
2	2	*28.945*	29.362	35.369	34.948
2.25	2.25	*19.808*	19.990	25.562	25.497
2.5	2.5	*14.329*	14.522	18.411	18.236

VII. CONCLUSION

The authors compared efficiency between MCUSUM and MEWMA control charts for bivariate copulas when observations are exponential distribution using the Monte Carlo simulation approach. The results found that MCUSUM control chart performs better than MEWMA control chart when one exponential parameter changes but the performance of MCUSUM control chart was found to be similar to the MEWMA control chart referring to the shift in both exponential parameters for all shifts.

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