Proceedings of the World Congress on Engineering and Computer Science 2015 Vol II
WCECS 2015, October 21-23, 2015, San Francisco, USA

Portfolio Optimization with Reward-Risk Ratio Measure based on the Conditional Value-at-Risk

Wlodzimierz Ogryczak, Michał Przyłuski, Tomasz Śliwiński

***Abstract*—In several problems of portfolio selection the reward-risk ratio criterion is optimized to search for a risky portfolio offering the maximum increase of the mean return, compared to the risk-free investment opportunities. We analyze such a model with the CVaR type risk measure. Exactly the deviation type of risk measure must be used, i.e. the so-called conditional drawdown measure. We analyze both the theoretical properties (SSD consistency) and the computational complexity (LP models).**

***Index Terms*—portfolio optimization, reward-risk ratio, conditional-value-at-risk, linear programming, stochastic dominance.**

I. Introduction

PORTFOLIO selection problems are usually tackled with the mean-risk models that characterize the uncertain returns by two scalar characteristics: the mean, which is the expected return, and the risk - a scalar measure of the variability of returns. In the original Markowitz model the risk is measured by the standard deviation or variance. Several other risk measures have been later considered thus creating the entire family of mean-risk (Markowitz-type) models. While the original Markowitz model forms a quadratic programming problem [1], many attempts have been made to linearize the portfolio optimization procedure (c.f., [2], [3] and references therein). The LP solvability is very important for applications to real-life financial decisions where the constructed portfolios have to meet numerous side constraints (including the minimum transaction lots, transaction costs and mutual funds characteristics). In order to guarantee that the portfolio takes advantage of diversification, no risk measure can be a linear function of the portfolio weights. Nevertheless, a risk measure can be LP computable in the case of discrete random variables, i.e., in the case of returns defined by their realizations under specified scenarios.

The simplest LP computable risk measures are dispersion measures similar to the variance. The mean absolute deviation was very early considered in portfolio analysis [4] while [5] presented and analyzed the complete portfolio optimization model (the so-called MAD model). Yitzhaki [6] introduced the mean-risk model using Gini's mean (absolute) difference as the risk measure. Both the mean absolute deviation and the Gini's mean difference turn out to be special aggregation techniques of the multiple criteria LP model [7] based on the pointwise comparison of the absolute Lorenz curves. The latter leads the quantile shortfall risk measures which are more commonly used and accepted. Recently, the second order quantile risk measures have been introduced in different ways by many authors [8], [9], [10]. The measure, now commonly called the Conditional Value at Risk (CVaR) (after [10] or Tail VaR, represents the mean shortfall at a specified confidence level. It leads to LP solvable portfolio optimization models in the case of discrete random variables represented by their realizations under specified scenarios. The CVaR has been shown by [11] to satisfy the requirements of the so-called coherent risk measures [8] and is consistent with the second degree stochastic dominance as shown by [12]. Several empirical analyses [13], [14], [15] confirm its applicability to various financial optimization problems.

In this paper we analyze the reward-risk ratio criterion is optimized to search for a risky portfolio offering the maximum increase of the mean return, compared to the risk-free investment opportunities. We analyze such a model with the CVaR type risk measure. Exactly the deviation type of risk measure must be used, i.e. the so-called conditional drawdown measure. Both the theoretical properties and the computational complexity are analyzed. In Section III we show that under natural restriction on the target value the CVaR reward-risk ratio optimization is SSD consistent. Further in Section IV we show that while carefully transforming the CVaR risk-reward ratio optimization to an LP model and taking advantages of the LP duality we are able to get a model formulation providing high computational efficiency.

II. Portfolio optimization and CVaR measures

WE consider a situation where an investor intends to optimally select a portfolio of assets and hold it until the end of a defined investment horizon. Let $J = \{1, 2, \ldots, n\}$ denote a set of assets available for the investment. For each asset $j \in J$, its rate of return is represented by a random variable (r.v.) R_j with a given mean $\mu_j = \mathbb{E}\{R_j\}$. Furthermore, let $\mathbf{x} = (x_j)_{j=1,\ldots,n}$ denote a vector of decision variables x_j representing the shares (weights) that define a portfolio of assets. To represent a portfolio, these weights must satisfy a set of constraints. The basic set of constraints includes the requirement that the weights must sum to one, i.e., $\sum_{j=1}^{n} x_j = 1$, and that short sales are not allowed, i.e., $x_j \geq 0$ for $j = 1, \ldots, n$. An investor usually needs to consider some other requirements expressed as a set of additional side constraints. Most of them can be expressed as linear equations and inequalities. We will assume that the basic set of feasible portfolios Q, i.e. the set of solutions that do not violate the basic set of constraints mentioned above, is a general LP feasible set given in a canonical form as a system of linear equations with nonnegative variables.

Each portfolio $\mathbf{x}$ defines a corresponding r.v. $R_{\mathbf{x}} = \sum_{j=1}^{n} R_j x_j$ that represents the portfolio rate of return. The mean rate of return for portfolio $\mathbf{x}$ is given as $\mu(R_{\mathbf{x}}) =$

Manuscript received July 8, 2015; revised July 28, 2015. This work was supported in part by the National Science Centre (Poland) under the grant DEC-2012/07/B/HS4/03076.

W. Ogryczak, M. Przyłuski and T. Śliwiński are with the Institute of Control and Computation Engineering, Warsaw University of Technology, Poland, e-mail: {w.ogryczak, m.przyluski, t.sliwinski}@elka.pw.edu.pl.

ISBN: 978-988-14047-2-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$\mathbb{E}\{R_{\mathbf{x}}\} = \sum_{j=1}^{n} \mu_j x_j$. We consider T scenarios, each one with probability p_t, where $t = 1, \dots, T$. We assume that, for each r.v. R_j, its realization r_{jt} under scenario t is known and that, for each asset j, $j = 1, \dots, n$, its mean rate of return is computed as $\mu_j = \sum_{t=1}^{T} r_{jt} p_t$. The realization of the portfolio rate of return $R_{\mathbf{x}}$ under scenario t is given by $y_t = \sum_{j=1}^{n} r_{jt} x_j$.

The portfolio optimization problem considered in this paper follows the original Markowitz' formulation and is based on a single period model of investment. At the beginning of a period, an investor allocates the capital among various assets, thus assigning a nonnegative weight (share of the capital) to each asset. Let $J = \{1, 2, \dots, n\}$ denote a set of assets considered for an investment. For each asset $j \in J$, its rate of return is represented by a random variable R_j with a given mean $\mu_j = \mathbb{E}\{R_j\}$. Further, let $\mathbf{x} = (x_j)_{j=1,2,\dots,n}$ denote a vector of decision variables x_j expressing the weights defining a portfolio. The weights must satisfy a set of constraints to represent a portfolio. The simplest way of defining a feasible set Q is by a requirement that the weights must sum to one and they are nonnegative (short sales are not allowed), i.e.

$$Q = \{\mathbf{x} \,:\, \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad \text{for } j = 1, \dots, n\} \quad (1)$$

Hereafter, we perform detailed analysis for the set Q given with constraints (1). Nevertheless, the presented results can easily be adapted to a general LP feasible set given as a system of linear equations and inequalities.

Each portfolio $\mathbf{x}$ defines a corresponding random variable $R_{\mathbf{x}} = \sum_{j=1}^{n} R_j x_j$ that represents the portfolio rate of return while the expected value can be computed as $\mu(\mathbf{x}) = \sum_{j=1}^{n} \mu_j x_j$. We consider T scenarios with probabilities p_t (where $t = 1, \dots, T$). We assume that for each random variable R_j its realization r_{jt} under the scenario t is known. Typically, the realizations are derived from historical data treating T historical periods as equally probable scenarios ($p_t = 1/T$). Although the models we analyze do not take advantages of this simplification. The realizations of the portfolio return $R_{\mathbf{x}}$ are given as $y_t = \sum_{j=1}^{n} r_{jt} x_j$.

The portfolio optimization problem is modeled as a mean-risk bicriteria optimization problem where the mean $\mu(\mathbf{x})$ is maximized and the risk measure $\varrho(\mathbf{x})$ is minimized. In the original Markowitz model, the standard deviation was used as the risk measure. Several other risk measures have been later considered thus creating the entire family of mean-risk models (c.f., [15], [16]). These risk measures, similar to the standard deviation, are law-invariant (are not affected by any shift of the outcome scale) and are risk relevant (equal to 0 in the case of a risk-free portfolio while taking positive values for any risky portfolio). Unfortunately, such risk measures are not consistent with the stochastic dominance order [17] or other axiomatic models of risk-averse preferences [18] and coherent risk measurement [8].

In stochastic dominance, uncertain returns (modeled as random variables) are compared by pointwise comparison of some performance functions constructed from their distribution functions. The first performance function $F_{\mathbf{x}}^{(1)}$ is defined as the right-continuous cumulative distribution function: $F_{\mathbf{x}}^{(1)}(\eta) = F_{\mathbf{x}}(\eta) = \mathbb{P}\{R_{\mathbf{x}} \leq \eta\}$ and it defines the first degree stochastic dominance (FSD). The second function is derived from the first as $F_{\mathbf{x}}^{(2)}(\eta) = \int_{-\infty}^{\eta} F_{\mathbf{x}}(\xi)\, d\xi$ and it defines the second degree stochastic dominance (SSD). We say that portfolio $\mathbf{x}'$ dominates $\mathbf{x}''$ under the SSD ($R_{\mathbf{x}'} \succ_{_{SSD}} R_{\mathbf{x}''}$), if $F_{\mathbf{x}'}^{(2)}(\eta) \leq F_{\mathbf{x}''}^{(2)}(\eta)$ for all η, with at least one strict inequality. A feasible portfolio $\mathbf{x}^0 \in Q$ is called SSD efficient if there is no $\mathbf{x} \in Q$ such that $R_{\mathbf{x}} \succ_{_{SSD}} R_{\mathbf{x}^0}$. Stochastic dominance relates the notion of risk to a possible failure of achieving some targets. As shown by [19], function $F_{\mathbf{x}}^{(2)}$, used to define the SSD relation, can also be presented as follows: $F_{\mathbf{x}}^{(2)}(\eta) = \mathbb{E}\{\max\{\eta - R_{\mathbf{x}}, 0\}\}$ and thereby its values are LP computable for returns represented by their realizations y_t.

An alternative characterization of the SSD relation can be achieved with the so-called Absolute Lorenz Curves (ALC) [9] which represent the second quantile functions defined as $F_{\mathbf{x}}^{(-2)}(0) = 0$ and

$$F_{\mathbf{x}}^{(-2)}(p) = \int_0^p F_{\mathbf{x}}^{(-1)}(\alpha) d\alpha \quad \text{for } 0 < p \leq 1, \qquad (2)$$

where $F_{\mathbf{x}}^{(-1)}(p) = \inf\ \{\eta : F_{\mathbf{x}}(\eta) \geq p\}$ is the left-continuous inverse of the cumulative distribution function $F_{\mathbf{x}}$. The pointwise comparison of ALCs is equivalent to the SSD relation [12] in the sense that $R_{\mathbf{x}'} \succeq_{_{SSD}} R_{\mathbf{x}''}$ if and only if $F_{\mathbf{x}'}^{(-2)}(\beta) \geq F_{\mathbf{x}''}^{(-2)}(\beta)$ for all $0 < \beta \leq 1$. Moreover,

$$\begin{array}{rcl} F_{\mathbf{x}}^{(-2)}(\beta) & = & \max\limits_{\eta \in R} \left[\beta\eta - F_{\mathbf{x}}^{(2)}(\eta) \right] \\ & = & \max\limits_{\eta \in R} \left[\beta\eta - \mathbb{E}\{\max\{\eta - R_{\mathbf{x}}, 0\}\} \right] \end{array} \qquad (3)$$

where η is a real variable taking the value of β-quantile $Q_\beta(\mathbf{x})$ at the optimum. For a discrete r.v. represented by its realizations y_t problem (3) becomes an LP.

For any real tolerance level $0 < \beta \leq 1$, the normalized value of the ALC defined as

$$M_\beta(\mathbf{x}) = F_{\mathbf{x}}^{(-2)}(\beta)/\beta \qquad (4)$$

is called the *Conditional Value-at-Risk (CVaR)* or Tail VaR or Average VaR. The CVaR measure is an increasing function of the tolerance level β, with $M_1(\mathbf{x}) = \mu(\mathbf{x})$. For any $0 < \beta < 1$, the CVaR measure is SSD consistent [12] and coherent [11]. Opposite to deviation type risk measures, for coherent measures larger values are preferred and therefore the measures are sometimes called safety measures [15]. Due to (3), for a discrete random variable represented by its realizations y_t the CVaR measures are LP computable. It is important to notice that although the quantile risk measures (VaR and CVaR) were introduced in banking as extreme risk measures for very small tolerance levels (like $\beta = 0.05$), for the portfolio optimization good results have been provided by rather larger tolerance levels [15].

For β approaching 0, the CVaR measure tends to the Minimax measure

$$M(\mathbf{x}) = \min_{t=1,\dots,T} y_t \qquad (5)$$

introduced to portfolio optimization by Young [20]. Note that the maximum (downside) semideviation

$$\Delta(\mathbf{x}) = \mu(\mathbf{x}) - M(\mathbf{x}) = \max_{t=1,\dots,T} (\mu(\mathbf{x}) - y_t) \qquad (6)$$

and the conditional β-deviation

$$\Delta_\beta(\mathbf{x}) = \mu(\mathbf{x}) - M_\beta(\mathbf{x}) \quad \text{for } 0 < \beta \leq 1, \qquad (7)$$

ISBN: 978-988-14047-2-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

respectively, represent the corresponding deviation risk measures. They may be interpreted as the drawdown measures [21]. For $\beta = 0.5$ the measure $\Delta_{0.5}(\mathbf{x})$ represents the mean absolute deviation from the median [16].

The commonly accepted approach to implementation of the Markowitz-type mean-risk model (with deviation type risk measures) is based on the use of a specified lower bound μ_0 on expected returns while optimizing the risk measure. This bounding approach provides a clear understanding of investor preferences and a clear definition of optimal portfolio to be sought. For deviation type risk measures ϱ the approach results in the following minimum risk problem:

$$\min\{\varrho(\mathbf{x}) : \quad \mu(\mathbf{x}) \geq \mu_0, \quad \mathbf{x} \in Q\} \tag{8}$$

While using the coherent and SSD consistent risk measures μ_ϱ one may focus on the measure maximization without additional constraints

$$\max\{\mu_\varrho(\mathbf{x}) : \quad \mathbf{x} \in Q\} \tag{9}$$

or still consider some preferential constraints on the mean expectation

$$\max\{\mu_\varrho(\mathbf{x}) : \quad \mu(\mathbf{x}) \geq \mu_0, \quad \mathbf{x} \in Q\}. \tag{10}$$

In the case of CVaR measure both models can be effectively solved for large numbers of scenarios while taking advantages of appropriate dual formulations [22].

III. Reward-risk ratio optimization

An alternative specific approach to portfolio optimization looks for a risky portfolio offering the maximum increase of the mean return, compared to the risk-free target τ. Namely, given the risk-free rate of return τ, a risky portfolio $\mathbf{x}$ that maximizes the ratio $(\mu(\mathbf{x}) - \tau)/\varrho(\mathbf{x})$ is sought. This leads us to the following ratio optimization problem:

$$\max\left\{\frac{\mu(\mathbf{x}) - \tau}{\varrho(\mathbf{x})} \ : \ \mathbf{x} \in Q\right\}. \tag{11}$$

The approach is well appealing with respect to the preferences modeling and applied to standard portfolio selection or (extended) index tracking problems (with a benchmark as the target). We illustrate ratio optimization (11) in Fig. 1. For the LP computable risk measures the reward-risk ratio optimization problem can be converted into an LP form [16].

When the risk-free return r_0 is used instead of the target τ than the ratio optimization (11) corresponds to the classical Tobin's model [23] of the modern portfolio theory (MPT) where the capital market line (CML) is the line is drawn from the risk-free rate at the intercept that passes tangent to the mean-risk efficient frontier. Any point on this line provides the maximum return for each level of risk. The tangency (tangent, super-efficient) portfolio is the portfolio of risky assets on the efficient frontier at the point where the CML is tangent to the efficiency frontier. It is a risky portfolio offering the maximum increase of the mean return while comparing to the risk-free investment opportunities. Namely having given the risk-free rate of return r_0 one seeks a risky portfolio $\mathbf{x}$ that maximizes the ratio $(\mu(\mathbf{x}) - r_0)/\varrho(\mathbf{x})$.

Instead of the reward-risk ratio maximization one may consider an equivalent model of the risk-reward ratio minimization (see Fig. 2):

$$\min\left\{\frac{\varrho(\mathbf{x})}{\mu(\mathbf{x}) - \tau} \ : \ \mathbf{x} \in Q\right\}. \tag{12}$$

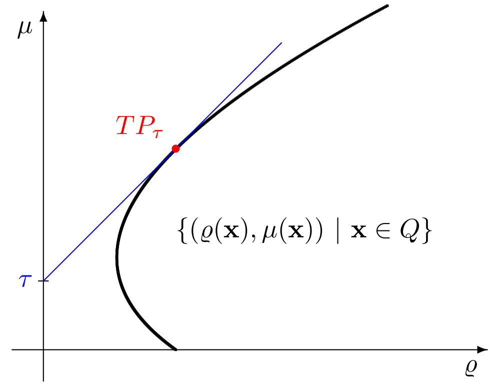

Fig. 1. Reward-risk ratio optimization

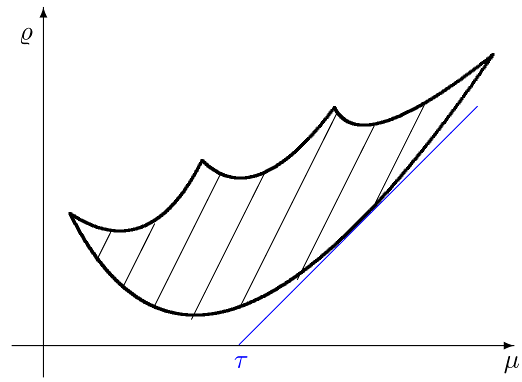

Fig. 2. Risk-reward ratio optimization

Actually, this is a classical model for the tangency portfolio as considered by Markowitz [1] and used in statistics books [24].

Both the ratio optimization models (11) and (12) are theoretically equivalent. However the risk-reward ratio optimization (12) enables easy control of the denominator positivity by simple inequality $\mu(\mathbf{x}) \geq \tau + \varepsilon$ added to the problem constraints. The model may also be additionally regularized for the case of multiple risk-free solutions. Regularization $(\varrho(\mathbf{x}) + \varepsilon)/(\mu(\mathbf{x}) - \tau)$ guarantees that the risk-free portfolio with the highest mean return will be selected then.

Theorem 1: If risk measure $\varrho(\mathbf{x})$ is mean-complementary SSD consistent, i.e.

$$R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''} \quad \Rightarrow \quad \mu(\mathbf{x}') - \varrho(\mathbf{x}') \geq \mu(\mathbf{x}'') - \varrho(\mathbf{x}'')$$

then the reward-risk ratio optimization (11) or equivalently (12) is SSD consistent provided that $\mu(\mathbf{x}) > \tau > \mu(\mathbf{x}) - \varrho(\mathbf{x})$.

Proof: Note that

$$-1 + \frac{\varrho(\mathbf{x})}{\mu(\mathbf{x}) - \tau} = \frac{\tau - (\mu(\mathbf{x}) - \varrho(\mathbf{x}))}{\mu(\mathbf{x}) - \tau}.$$

If $R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''}$, then $\mu(\mathbf{x}') - \varrho(\mathbf{x}') \geq \mu(\mathbf{x}'') - \varrho(\mathbf{x}'')$ and $\mu(\mathbf{x}') \geq \mu(\mathbf{x}'')$. Hence,

$$\frac{\tau - (\mu(\mathbf{x}') - \varrho(\mathbf{x}'))}{\mu(\mathbf{x}') - \tau} \geq \frac{\tau - (\mu(\mathbf{x}'') - \varrho(\mathbf{x}''))}{\mu(\mathbf{x}'') - \tau}$$

provided that both numerators and denominators remains positive. ■

The reward-risk ratio is well defined for the deviation type risk measures. Therefore while dealing with the CVaR risk

ISBN: 978-988-14047-2-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

model we must replace this performance measure (coherent risk measure) with its complementary deviation representation. The deviation type risk measure complementary to the $CVaR_\beta$ representing the tail mean within the β-quantile takes the form of $\Delta_\beta(\mathbf{x}) = \mu(\mathbf{x}) - CVaR_\beta(\mathbf{x})$ (conditional semideviation or drawdown measure) thus leading to the ratio optimization [16]:

$$\frac{\mu(\mathbf{x}) - \tau}{\Delta_\beta(\mathbf{x})} \quad \to \quad \max \tag{13}$$

Taking advantages of possible inverse formulation of the risk-reward ratio optimization (12) as ratio

$$\frac{\Delta_\beta(\mathbf{x})}{\mu(\mathbf{x}) - \tau} \quad \to \quad \min \tag{14}$$

we get a model well defined for $\mu(\mathbf{x}) > \tau$ and SSD consistent for $\tau - M_\beta(\mathbf{x}) \geq 0$. Thus, this CVaR ratio optimization is consistent with the SSD rules (similar to the standard CVaR optimization [12]), despite that the ratio does not represent a coherent risk measure [8].

Theorem 2: For any target level τ such that there exists portfolio $\mathbf{x} \in Q$ satisfying requirements $\tau \geq M_\beta(\mathbf{x})$ and $\mu(\mathbf{x}) - \tau \geq \varepsilon > 0$, except for portfolios with identical values of the corresponding CVaR risk-reward ratio, every optimal solution of the problem

$$\min\left\{\frac{\Delta_\beta(\mathbf{x})}{\mu(\mathbf{x}) - \tau} : \mathbf{x} \in Q, \tau \geq M_\beta(\mathbf{x}), \mu(\mathbf{x}) - \tau \geq \varepsilon\right\} \tag{15}$$

is an SSD efficient portfolio.

Proof: Let $\mathbf{x}^0$ be an optimal portfolio for ratio optimization (15). If there exists portfolio $\mathbf{x} \in Q$ satisfying requirements $\tau \geq M_\beta(\mathbf{x})$ and $\mu(\mathbf{x}) - \tau \geq \varepsilon$ such that $R_\mathbf{x} \succeq_{SSD} R_{\mathbf{x}^0}$, then following Theorem 1

$$\frac{\Delta_\beta(\mathbf{x})}{\mu(\mathbf{x}) - \tau} \leq \frac{\Delta_\beta(\mathbf{x}^0)}{\mu(\mathbf{x}^0) - \tau}.$$

Hence, due to optimality of $\mathbf{x}^0$

$$\frac{\Delta_\beta(\mathbf{x})}{\mu(\mathbf{x}) - \tau} = \frac{\Delta_\beta(\mathbf{x}^0)}{\mu(\mathbf{x}^0) - \tau}.$$

which completes the proof. ■

IV. COMPUTATIONAL LP MODELS

IN this section we will show that while transforming the CVaR risk-reward ratio optimization (14) to an LP model, we can take advantages of the LP duality to get a model formulation providing higher computational efficiency. In the introduced model, similar to the direct CVaR optimization [25], the number of structural constraints is proportional to the number of instruments while only the number of variables is proportional to the number of scenarios, thus not affecting so seriously the simplex method efficiency. The model can effectively be solved with general LP solvers even for very large numbers of scenarios (like the case of fifty thousand scenarios and one hundred instruments solved less than a minute). On the other hand such efficiency cannot be achieved with model (13).

Let us consider portfolio optimization problem with asset returns given by discrete random variables with realization r_{jt} thus leading to LP models for coherent risk measures we consider. Let us focus first on measures maximization without additional (preferential) constraints thus considering the optimization models of type (9).

Following (3) and (4), the CVaR portfolio optimization model can be formulated as the following LP problem:

$$\begin{aligned} \max \quad & y - \frac{1}{\beta}\sum_{t=1}^{T} p_t d_t \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad j = 1, \ldots, n \\ & d_t \geq y - \sum_{j=1}^{n} r_{jt} x_j, \;\; d_t \geq 0 \quad t = 1, \ldots, T \end{aligned} \tag{16}$$

where y is unbounded variable. Except from the core portfolio constraints (1), model (16) contains T nonnegative variables d_t plus single η variable and T corresponding linear inequalities. Hence, its dimensionality is proportional to the number of scenarios T. Exactly, the LP model contains $T + n + 1$ variables and $T + 1$ constraints. It does not cause any computational difficulties for a few hundreds scenarios as in several computational analysis based on historical data [26]. However, in the case of more advanced simulation models employed for scenario generation one may get several thousands scenarios. This may lead to the LP model (16) with huge number of variables and constraints thus decreasing the computational efficiency of the model. As shown in [25], the computational efficiency can easily be achieved by taking advantages of the LP dual to model (16). The LP dual model takes the form:

$$\begin{aligned} \min \quad & q \\ \text{s.t.} \quad & q - \sum_{t=1}^{T} r_{jt} u_t \geq 0 \quad j = 1, \ldots, n \\ & \sum_{t=1}^{T} u_t = 1 \\ & 0 \leq u_t \leq \frac{p_t}{\beta} \quad t = 1, \ldots, T \end{aligned} \tag{17}$$

containing T variables u_t, but the T constraints corresponding to variables d_t from (16) take the form of simple upper bounds (SUB) on u_t thus not affecting the problem complexity. Actually, the number of constraints in (17) is proportional to the total of portfolio size n, thus it is independent from the number of scenarios. Exactly, there are $T + 1$ variables and $n + 1$ constraints. This guarantees a high computational efficiency of the dual model even for very large number of scenarios.

For an LP computable risk measure $\varrho(\mathbf{x})$, the ratio optimization problem (11) can be converted into an LP form by the following transformation: introduce variables $v = \mu(\mathbf{x})/\varrho(\mathbf{x})$ and $v_0 = 1/\varrho(\mathbf{x})$, then replace the original decision variables x_j with $\tilde{x}_j = x_j/\varrho(\mathbf{x})$, getting the linear criterion $\max \;\; v - \tau v_0$ and an LP feasible set. Once the transformed problem is solved the values of the portfolio variables x_j can be found by dividing $\tilde{x}_j$ by v_0 while $\varrho(\mathbf{x}) = 1/v_0$ and $\mu(\mathbf{x}) = v/v_0$.

In the CVaR model, risk measure $\varrho(\mathbf{x}) = \Delta_\beta(\mathbf{x})$ is not directly represented. We can introduce, however, the equation:

$$z - y + \frac{1}{\beta}\sum_{t=1}^{T} p_t d_t = z_0$$

ISBN: 978-988-14047-2-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

allowing us to represent $\Delta_\beta(\mathbf{x})$ with variable z_0. Hence, the ratio model takes the form:

$$\begin{array}{ll} \max & \dfrac{z-\tau}{z_0} \\ \text{s.t.} & \\ & z - y + \dfrac{1}{\beta}\sum_{t=1}^{T} p_t d_t = z_0 \\ & d_t + y_t \geq y, \; d_t \geq 0 \quad t = 1,\ldots,T \\ & \sum_{j=1}^{n} \mu_j x_j = z \\ & \sum_{j=1}^{n} r_{jt} x_j = y_t \quad t = 1,\ldots,T \\ & \sum_{j=1}^{n} x_j = 1, \; x_j \geq 0 \quad j = 1,\ldots,n \end{array} \tag{18}$$

Introducing variables $v = z/z_0$ and $v_0 = 1/z_0$ we get linear criterion $v - \tau v_0$ of the corresponding ratio model. Further, we divide all the constraints by z_0 and make the substitutions: $\tilde{d}_t = d_t/z_0$, $\tilde{y}_t = y_t/z_0$ for $t = 1,\ldots,T$, as well as $\tilde{x}_j = x_j/z_0$, for $j = 1,\ldots,n$ and $\tilde{y} = y/z_0$. Finally, we get the following LP formulation:

$$\begin{array}{ll} \max & v - \tau v_0 \\ \text{s.t.} & \\ & v - \tilde{y} + \dfrac{1}{\beta}\sum_{t=1}^{T} p_t \tilde{d}_t = 1 \\ & \tilde{d}_t + \tilde{y}_t \geq \tilde{y}, \; \tilde{d}_t \geq 0 \quad t = 1,\ldots,T \\ & \sum_{j=1}^{n} \mu_j \tilde{x}_j = v \\ & \sum_{j=1}^{n} r_{jt} \tilde{x}_j = \tilde{y}_t \quad t = 1,\ldots,T \\ & \sum_{j=1}^{n} \tilde{x}_j = v_0, \; \tilde{x}_j \geq 0 \quad j = 1,\ldots,n \end{array} \tag{19}$$

After eliminating defined by equations variables v, v_0 and y_t, one gets the most compact formulation:

$$\begin{array}{ll} \max & \sum_{j=1}^{n} \mu_j \tilde{x}_j - \tau \sum_{j=1}^{n} \tilde{x}_j \\ \text{s.t.} & -\tilde{y} + \sum_{j=1}^{n} \mu_j \tilde{x}_j + \dfrac{1}{\beta}\sum_{t=1}^{T} p_t \tilde{d}_t = 1 \\ & \tilde{y} - \sum_{j=1}^{n} r_{jt} \tilde{x}_j - \tilde{d}_t \leq 0 \quad t = 1,\ldots,T \\ & \tilde{d}_t \geq 0 \quad t = 1,\ldots,T \\ & \tilde{x}_j \geq 0 \quad j = 1,\ldots,n \end{array} \tag{20}$$

that contains $T+n+1$ variables and $T+1$ constraints. Even taking advantages of the LP dual formulation

$$\begin{array}{ll} \min & q \\ \text{s.t.} & -q + \sum_{t=1}^{T} u_t = 0 \\ & \mu_j q - \sum_{t=1}^{T} r_{jt} u_t \geq \mu_j - \tau \quad j = 1,\ldots,n \\ & \dfrac{p_t}{\beta} q - u_t \geq 0 \quad t = 1,\ldots,T \\ & u_t \geq 0 \quad t = 1,\ldots,T \end{array} \tag{21}$$

one cannot get any model that contains less than $T+n+1$ constraints and $T+1$ variables.

The complexity can be reduced however while using the risk-reward ratio optimization (12). The corresponding CVaR model takes the following form:

$$\begin{array}{ll} \min & \dfrac{z - y + \frac{1}{\beta}\sum_{t=1}^{T} p_t d_t}{z-\tau} \\ \text{s.t.} & d_t \geq y - \sum_{j=1}^{n} r_{jt} x_j, \; d_t \geq 0 \quad t = 1,\ldots,T \\ & z = \sum_{j=1}^{n} \mu_j x_j, \; \sum_{j=1}^{n} x_j = 1 \\ & x_j \geq 0, \quad j = 1,\ldots,n \end{array} \tag{22}$$

It can be linearized by substitutions: $\tilde{d}_t = d_t/(z-\tau)$, $\tilde{y} = y/(z-\tau)$, $\tilde{x}_j = x_j/(z-\tau)$, $v = z/(z-\tau)$ and $v_0 = 1/(z-\tau)$ leading to the following LP formulation:

$$\begin{array}{ll} \min & v - \tilde{y} + \dfrac{1}{\beta}\sum_{t=1}^{T} p_t \tilde{d}_t \\ \text{s.t.} & \tilde{d}_t \geq \tilde{y} - \sum_{j=1}^{n} r_{jt} \tilde{x}_j, \; \tilde{d}_t \geq 0 \quad t = 1,\ldots,T \\ & v - v_0 \tau = 1 \\ & v = \sum_{j=1}^{n} \mu_j \tilde{x}_j, \; \sum_{j=1}^{n} \tilde{x}_j = v_0 \\ & \tilde{x}_j \geq 0 \quad j = 1,\ldots,n \end{array} \tag{23}$$

After eliminating defined by equations variables v and v_0, one gets the most compact formulation:

$$\begin{array}{ll} \min & \sum_{j=1}^{n} \mu_j \tilde{x}_j - \tilde{y} + \dfrac{1}{\beta}\sum_{t=1}^{T} p_t \tilde{d}_t \\ \text{s.t.} & \tilde{d}_t \geq \tilde{y} - \sum_{j=1}^{n} r_{jt} \tilde{x}_j, \; \tilde{d}_t \geq 0 \quad t = 1,\ldots,T \\ & \sum_{j=1}^{n} (\mu_j - \tau)\tilde{x}_j = 1, \; \tilde{x}_j \geq 0 \quad j = 1,\ldots,n \end{array} \tag{24}$$

The original values of x_j can be then recovered dividing $\tilde{x}_j$ by v_0.

Taking the LP dual to model (24) ones get the model:

$$\begin{array}{ll} \max & q \\ \text{s.t.} & \sum_{t=1}^{T} u_t = 1 \\ & \sum_{t=1}^{T} r_{jt} u_t + (\mu_j - \tau) q \leq \mu_j \quad j = 1,\ldots,n \\ & 0 \leq u_t \leq \dfrac{p_t}{\beta} \quad t = 1,\ldots,T \end{array} \tag{25}$$

containing T variables u_t, but the T constraints corresponding to variables d_t from (24) take the form of simple upper bounds (SUB) on u_t thus not affecting the problem complexity. Thus similar to the standard CVaR optimization (17), the number of constraints in (25) is proportional to the total of portfolio size n, thus it is independent from the number of scenarios. Exactly, there are $T+1$ variables and $n+1$ constraints. This guarantees a high computational efficiency of the dual model even for very large number of scenarios.

ISBN: 978-988-14047-2-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Similarly to experiments with CVaR computational models efficiency [25], we have run computational tests on large scale instances developed by Lim et al. [27]. They were originally generated from a multivariate normal distribution for 50, 100 or 200 assets with the number of scenarios 50,000. All computations were performed on a PC with the Intel Core i7 2.66GHz processor and 6GB RAM employing the simplex code of the CPLEX 12.5 package. An attempt to solve the CVaR reward-risk ratio model in its primal (20) or dual (21) forms with $\beta = 0.05$ resulted in similar high computations times of 620, 1487, 5102 seconds and of 656, 1544, 5347 seconds on average, for problems with 50, 100 and 200 assets, respectively. For the CVaR risk-reward ratio model in its primal form (24) the computation time were remarkably higher than those for the reward-risk ratio, resulting in 864, 1749, 5273 seconds on average. On the other hand, solving the dual models (25) directly by the primal method (standard CPLEX settings) resulted in dramatically shorter computation times 5.8, 14.2 and 39.9 CPU seconds, respectively. Thus, similar to the standard CVaR optimization [25], the dual model for the CVaR risk-reward ratio optimization allows one to solve effectively large scale problems. Moreover, the computation times remain very low for various tolerance levels β as shown in Table I.

TABLE I
COMPUTATIONAL TIMES (IN SECONDS) FOR THE DUAL LP MODEL (25) OF THE CVAR RISK-REWARD RATIO OPTIMIZATION (AVERAGES OF 10 INSTANCES WITH 50,000 SCENARIOS)

n	$\beta = 0.05$	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.3$	$\beta = 0.4$
50	5.8	7.5	9.2	9.5	10.9
100	14.2	18.3	23.1	24.2	26.1
200	39.9	53.1	66.8	76.1	77.3

V. CONCLUSION

We have presented the reward-risk ratio optimization model for the CVaR risk measure and analyzed its properties. Taking advantages of possible inverse formulation of the risk-reward ratio optimization (14) we get a model well defined and SSD consistent under natural restriction on the target value selection. Thus, this CVaR ratio optimization is consistent with the SSD rules (similar to the standard CVaR optimization [12]), despite that the ratio does not represent a coherent risk measure [8].

We show that while transforming the CVaR risk-reward ratio optimization (14) to an LP model, we can take advantages of the LP duality to get a model formulation providing higher computational efficiency. In the introduced dual model, similar to the direct CVaR optimization [25], the number of structural constraints is proportional to the number of assets while only the number of variables is proportional to the number of scenarios, thus not affecting so seriously the simplex method efficiency. The model can effectively be solved with general LP solvers even for very large numbers of scenarios. Actually, the dual CVaR ratio portfolio optimization problems of fifty thousand scenarios and two hundred instruments can be solved with the general purpose LP solvers in less than a minute. On the other hand, such efficiency cannot be achieved with model the standard CVaR reward-risk ratio model (13).

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ISBN: 978-988-14047-2-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)