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Optimal Thresholds for Stochastically Deteriorating Systems

A. Raza and V. Ulansky

Abstract— **We consider the analytical modeling of a condition-based monitoring for a system which is subject to degradation over time. The system condition is described by a monotonically increasing stochastic process that can be observed at discrete times by means of imperfect inspections. In addition to the critical threshold, for each time point of inspection a replacement threshold is introduced. The decision rule when checking system suitability for use in the upcoming time interval is considered. The expressions for the probabilities of correct and incorrect decisions when checking system suitability are derived with considering the results of previous inspections. A specific deterioration process is used to illustrate the proposed general expressions for the probabilities of correct and incorrect decisions. To determine the optimal threshold at each time of inspection, it is proposed to use criteria such as the maximum a posteriori probability criterion, minimum Bayes risk criterion and minimum total error probability criterion. A numerical example illustrates the efficiency of the proposed approach.**

***Index Terms*—Condition based maintenance, suitability checking, decision rule, optimal replacement threshold**

I. Introduction

Condition-based maintenance (CBM) is a type of maintenance wherein maintenance decisions depend on the information obtained from the condition monitoring (CM). Obviously, CM is preferred among other maintenance techniques in cases where system deterioration can be measured and where the system enters the failed state when at least one state parameter deteriorates beyond the level of functional failure. Condition-based maintenance allows to assess the system state via continuous monitoring or inspections at discrete times. The growing interest about CM is evident from the large number of studies related to various mathematical models and optimization techniques. Most of the existing mathematical models of CM with inspections at discrete times can be classified into two groups: models of CM with perfect inspections and models of CM with imperfect inspections. The latter is the subject of this study.

Maintenance models with imperfect inspections usually consider two types of errors: “false positives” (false alarms) with probability α and “false negatives” (i.e. non-detecting of failure) with probability β; for example [1]. Such models are not CBM models because in reality the error probabilities are not constant coefficients but depend on time and the parameters of the deterioration process. Moreover, such models depend on the results of previous multiple inspections, as shown in [2, 3]. Therefore, we analyze only those studies in which the probabilistic indicators of the inspection errors depend on the deterioration process parameters. In [2], CBM policies with imperfect operability checks are analyzed. The probabilities of four possible correct and incorrect decisions when checking system operability are considered. The proposed expressions depend on the deterioration process parameters and the results of previous inspections. In [4], the result of a measurement includes the original deterioration process along with a normally distributed measurement error. Based on this model, a decision rule was analyzed and optimal monitoring policies were found. The same approach was used in [5] to include measurement error in a Wiener diffusion process-based degradation model. A similar approach was used in [6] to find the likelihood for more than one inspection. A simple extension to the Bayesian updating model was proposed, such that the model can incorporate the results of inaccurate measurements. In [7], the threshold-type policy introduced for the maintenance action. If the system deterioration stage is less than the minimal threshold, no maintenance is conducted; if the system deterioration stage is found to be between the minimal threshold and the major threshold, than minimal maintenance is carried out; and major maintenance is performed if the system deterioration stage is larger than the major maintenance threshold. The model is based on a stochastic Petri net. In [8], an optimal replacement policy is considered when the state of system is unknown but can be estimated based on the observed condition. A proportional hazards model is used to represent the system's degradation. The optimization of the optimal maintenance policy is formulated as a partially observed Markov decision process, and the problem is solved using dynamic programming. In [9], the analytical model is developed for condition-based imperfect inspections of a stochastically deteriorating single-unit system. The system condition is described by a stochastic process with monotonically decreasing realizations. The analytical expressions for the probabilities of correct and incorrect decisions are derived. However, the proposed model does not consider the results of previous inspections.

In this study, we consider a stochastically and continuously deteriorating system whose state is described by one parameter and monitored through imperfect inspections. The system state parameter is assumed to be a

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stochastic process with monotonically increasing realizations. When the system state parameter exceeds its functional failure level FF, the system passes into the failed state and corrective replacement is necessary. The system state is inspected at discrete instants of time. When checking the system state parameter, errors are possible due to the imperfection of the measuring equipment. Each system rejected by the results of inspection is replaced by a new one. Currently, when checking the operability of a one-parameter system the following decision rule is used: if $z(t_k) < FF$, the system is judged operable and allowed for intended use in the interval (t_k, t_{k+1}), $k = 1, 2, \ldots$, otherwise (i.e. when $z(t_k) \geq FF$) the system is judged inoperable and beyond repair, where $z(t_k)$ is the measured value of the system state parameter at time t_k. When optimizing this decision rule, different criteria such as, criterion of minimum Bayes risk, criterion of maximum a posteriori probability and criterion of minimum total error probability are used. Each of these criteria is expressed through the probabilities of a "false failure" $\alpha(t_k)$ and an "undetected failure" $\beta(t_k)$, which are computed for the time point t_k by using equations described in previous studies, for example, in [10]. Therefore, when optimizing the decision rule by using the probabilities $\alpha(t_k)$ and $\beta(t_k)$ the behavior of the system state parameter in the interval (t_k, t_{k+1}) is not considered. Indeed, if the operable system was falsely rejected at time point t_k, then this decision would be correct if the system further failed in the interval (t_k, t_{k+1}). Analogically, the decision that the operable system at time point t_k was judged as operable would be wrong if this system further failed in the interval (t_k, t_{k+1}). Thus, when determining the decision rule and probabilities $\alpha(t_k)$ and $\beta(t_k)$, considering the behavior of the system state parameter in the coming interval of operation is necessary.

In this study, we propose a mathematical model for calculating the probabilities of correct and incorrect decisions while considering the behavior of the system in the interval between inspections and the results of previous inspections. The proposed approach allows determining the optimal threshold PF_k ($PF_k < FF$) at time of inspection t_k ($k = 1, 2, \ldots$). This will obviously reduce the probability of a functional failure in the intervals between inspections and improve the economic efficiency of the maintenance policy.

II. The Space of Events

Let the state of a system be determined by the values of one parameter $X(t)$, which is a stochastic process with monotonically increasing realizations. A system is inspected at successive times t_k ($k = 1, 2, \ldots$) under an infinite horizon planning, where $t_0 = 0$. Denote the result of measuring the parameter $X(t)$ at time t_k as

$$Z(t_k) = X(t_k) + Y(t_k), \tag{1}$$

where $Y(t_k)$ is the measurement error of the system state parameter at time t_k. We assume further that $X(t_k)$ and $Y(t_k)$ are independent random variables.

A typical realization of the stochastic process $X(t)$ measured at time t_k is shown in Fig. 1.

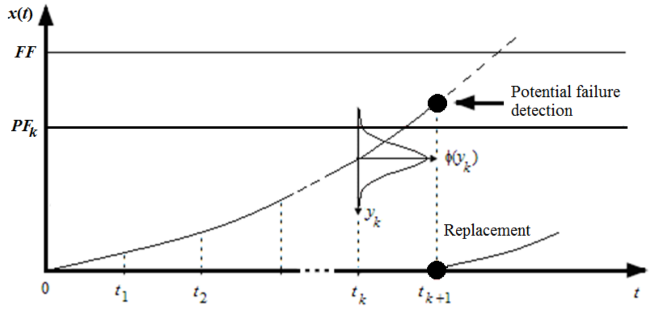

Fig. 1. Realisation of the stochastic process $X(t)$ measured at time point t_k with error y_k having the probability density function $\varphi(y_k)$.

We introduce the following decision rule when checking the system condition at time t_k. If $z(t_k) < PF_k$, the system is judged as suitable over the interval (t_k, t_{k+1}). If $z(t_k) \geq PF_k$, the system is judged as unsuitable and not allowed to be used over the interval (t_k, t_{k+1}). Thus, this decision rule is aimed at rejection of any system that is unsuitable for use in the next interval of operation.

Based on this decision rule two maintenance policies are possible. If $PF_k \leq Z(t_k) < FF$, the preventive replacement or repair is conducted. If $Z(t_k) \geq FF$, the corrective replacement of the system is performed. Any type of replacement leads to a complete renewal of the system, i.e. after replacement the system becomes as good as new.

From the perspective of the system suitability for use in the interval (t_k, t_{k+1}), when checking the parameter $X(t)$ at time $t = t_k$, one of the following mutually exclusive events may appear:

$$\begin{cases} H_1(\overline{t_1, t_k}; t_{k+1}) = \left\{ X(t_{k+1}) < FF \cap \left[\bigcap_{i=1}^{k} Z(t_i) < PF_i \right] \right\}, \\ H_2(\overline{t_1, t_k}; t_{k+1}) = \left\{ X(t_{k+1}) < FF \cap Z(t_k) \geq PF_k \cap \left[\bigcap_{i=1}^{k-1} Z(t_i) < PF_i \right] \right\}, \\ H_3(\overline{t_1, t_k}; t_{k+1}) = \left\{ X(t_k) < FF \cap X(t_{k+1}) \geq FF \cap \left[\bigcap_{i=1}^{k} Z(t_i) < PF_i \right] \right\}, \\ H_4(\overline{t_1, t_k}; t_{k+1}) = \{ X(t_k) < FF \cap X(t_{k+1}) \geq FF \cap Z(t_k) \geq PF_k \\ \qquad \cap \left[\bigcap_{i=1}^{k-1} Z(t_i) < PF_i \right] \}, \\ H_5(\overline{t_1, t_k}; t_{k+1}) = \left\{ X(t_k) \geq FF \cap \left[\bigcap_{i=1}^{k} Z(t_i) < PF_i \right] \right\}, \\ H_6(\overline{t_1, t_k}; t_{k+1}) = \left\{ X(t_k) \geq FF \cap Z(t_k) \geq PF_k \cap \left[\bigcap_{i=1}^{k-1} Z(t_i) < PF_i \right] \right\}, \end{cases} \tag{2}$$

where $H_1(\overline{t_1, t_k}; t_{k+1})$ is the joint occurrence of the following events: the system is suitable for use over the interval (t_k, t_{k+1}) and judged to be suitable when checking at time points $t_1, \ldots, t_k$; $H_2(\overline{t_1, t_k}; t_{k+1})$ is the joint occurrence of the following events: the system is suitable for use over the interval (t_k, t_{k+1}), judged as suitable at time points $t_1, \ldots, t_{k-1}$ and judged as unsuitable when checking at time point t_k; $H_3(\overline{t_1, t_k}; t_{k+1})$ is the joint occurrence of the following events: the operable at time t_k system fails up to the time t_{k+1} and when checking the system at time points t_1,..., t_k it is judged as suitable; $H_4\left(\overline{t_1, t_k}; t_{k+1}\right)$ is the joint occurrence of the following events: the operable at time t_k system fails up to the time t_{k+1}; when checking the system at time points t_1,..., t_{k-1} it is judged as suitable and at time point t_k the system is judged as unsuitable; $H_5\left(\overline{t_1, t_k}; t_{k+1}\right)$ is the joint occurrence of the following events: at time point t_k the system is inoperable and judged as suitable when checking suitability at time points t_1,..., t_k; $H_6\left(\overline{t_1, t_k}; t_{k+1}\right)$ is the joint occurrence of the following events: at time point t_k the system is inoperable; when checking suitability at time points t_1,..., t_{k-1} the system is judged as suitable and at time point t_k the system is judged as unsuitable.

Returning to Fig. 1, we find the following sequence of events occurred at times $\overline{t_1, t_{k+1}};$, respectively: $H_1(t_1; t_2)$, …, $H_1\left(\overline{t_1, t_k}; t_{k+1}\right)$, $H_4\left(\overline{t_1, t_{k+1}}; t_{k+2}\right)$.

III. The Probabilities of Correct and Incorrect Decisions

Let us determine the probabilities of the events $H_i\left(\overline{t_1, t_k}; t_{k+1}\right)$, $i = \overline{1,6}$. By the theorem of multiplication of probabilities for the event $H_1\left(\overline{t_1, t_k}; t_{k+1}\right)$, we have

$$P\left\{H_1\left(\overline{t_1, t_k}; t_{k+1}\right)\right\} = P\left\{X\left(t_{k+1}\right) < FF\right\}\times P\left\{\bigcap_{i=1}^{k} Z\left(t_i\right) < PF_i \middle| X\left(t_{k+1}\right) < FF\right\}, \tag{3}$$

where $P\left\{X\left(t_{k+1}\right) < FF\right\}$ is the a priori probability of an operable state of the system at time point t_{k+1} and $P\left\{\bigcap_{i=1}^{k} Z\left(t_i\right) < PF_i \middle| X\left(t_{k+1}\right) < FF\right\}$ is the conditional probability of judging the system suitable at time points t_1, ..., t_k under the condition that the system will not fail up to time t_{k+1}.

The probability $P\left\{X\left(t_{k+1}\right) < FF\right\}$ is determined as follows:

$$P\left\{X\left(t_{k+1}\right)\right\} = \int_{-\infty}^{FF} f\left(x_{k+1}\right) dx_{k+1}, \tag{4}$$

where $f(x_{k+1})$ is the a priori probability density function (PDF) of the system state parameter $X(t)$ at time $t = t_{k+1}$. From the monotonicity property of $X(t)$, it follows that the probability $P\{X(t_{k+1})\}$ is the reliability function.

The conditional probability of the event $\left\{\bigcap_{i=1}^{k} Z\left(t_i\right) < PF_i \middle| X\left(t_{k+1}\right) < FF\right\}$ is determined by integrating the conditional PDF $q\left\{\overline{z_1, z_k} \middle| X\left(t_{k+1}\right) < FF\right\}$ of the random variables $Z(t_1)$, …, $Z(t_k)$ on the area of the system suitability, i.e.:

$$P\left\{\bigcap_{i=1}^{k} Z\left(t_i\right) < PF_i \middle| X\left(t_{k+1}\right) < FF\right\} = \int_{-\infty}^{PF_1} \cdots \int_{-\infty}^{PF_k} q\left\{\overline{z_1, z_k} \middle| X\left(t_{k+1}\right) < FF\right\} \overline{dz_1 dz_k}. \tag{5}$$

Since $X(t_k)$ and $Y(t_k)$ are independent random variables, the conditional PDF $q\left\{\overline{z_1, z_k} \middle| X\left(t_{k+1}\right) < FF\right\}$ is the convolution of functions $f\left\{\overline{x_1, x_k} \middle| X\left(t_{k+1}\right) < FF\right\}$ and $\Phi\left(\overline{y_1, y_k}\right)$, where $f\left\{\overline{x_1, x_k} \middle| X\left(t_{k+1}\right) < FF\right\}$ is the conditional PDF of $X(t)$ at times t_1, …, t_k on condition that $X\left(t_{k+1}\right) < FF$ and $\Phi\left(\overline{y_1, y_k}\right)$ is the joint PDF of the random variables $Y(t_1)$, …, $Y(t_k)$.

Further, we assume that the measurement errors are independent. In practice, the condition of independence of random variables $Y(t_1)$, …, $Y(t_k)$ is usually adopted because the correlation intervals of the measurement errors are considerably smaller than the intervals between inspections. Therefore,

$$\Phi\left(\overline{y_1, y_k}\right) = \prod_{i=1}^{k} \varphi\left(y_i\right) \tag{6}$$

and

$$q\left\{\overline{z_1, z_k} \middle| X\left(t_{k+1}\right) < FF\right\} = \int_{-\infty}^{FF} \cdots \int_{-\infty}^{FF} f\left\{\overline{x_1, x_k} \middle| X\left(t_{k+1}\right) < FF\right\} \times \prod_{i=1}^{k} \varphi\left(z_i - x_i\right) \overline{dx_1 dx_k}. \tag{7}$$

Substituting (7) in (5) gives

$$P\left\{\bigcap_{i=1}^{k} Z\left(t_i\right) < PF_i\} \middle| X\left(t_{k+1}\right) < FF\right\} = \int_{-\infty}^{FF} \cdots \int_{-\infty}^{FF} f\left\{\overline{x_1, x_k} \middle| X\left(t_{k+1}\right) < FF\right\} \left[\prod_{i=1}^{k} \int_{-\infty}^{PF_i} \varphi\left(z_i - x_i\right) dz_i\right] \times \overline{dx_1 dx_k}. \tag{8}$$

Further, by making the change of variables $y_k = z_k - x_k$ in (8), we obtain

$$P\left\{\bigcap_{i=1}^{k} Z(t_i) < PF_i \middle| X(t_{k+1}) < FF\right\} = \tag{9}$$

$$\int_{-\infty}^{FF}\ldots\int_{-\infty}^{FF} f\left\{\overline{x_1, x_k}\middle| X(t_{k+1}) < FF\right\}\left[\prod_{i=1}^{k}\int_{-\infty}^{PF_i - x_i}\varphi(y_i)dy_i\right]\overline{dx_1 dx_k}.$$

The conditional PDF $f\left\{\overline{x_1, x_k}\middle| X(t_{k+1}) < FF\right\}$ is determined by the Bayes formula for continuous random variables

$$f\left\{\overline{x_1, x_k}\middle| X(t_{k+1}) < FF\right\} = \frac{\int_{-\infty}^{FF} f\left(\overline{x_1, x_{k+1}}\right)dx_{k+1}}{\int_{-\infty}^{FF} f\left(x_{k+1}\right)dx_{k+1}}, \tag{10}$$

where $f\left(\overline{x_1, x_{k+1}}\right)$ is the joint PDF of random variables $X(t_1), \ldots, X(t_{k+1})$.

Substituting (10) in (9) results in

$$P\left\{\bigcap_{i=1}^{k} Z(t_i) < PF_i\}\middle| X(t_{k+1}) < FF\right\} = \tag{11}$$

$$\frac{\int_{-\infty}^{FF}\ldots\int_{-\infty}^{FF} f\left(\overline{x_1, x_{k+1}}\right)\left[\prod_{i=1}^{k}\int_{-\infty}^{PF_i - x_i}\varphi(y_i)dy_i\right]\overline{dx_1 dx_{k+1}}}{\int_{-\infty}^{FF} f\left(x_{k+1}\right)dx_{k+1}}.$$

Finally, substituting (4) and (11) in (3), we find the probability of the event $H_1\left(\overline{t_1, t_k}; t_{k+1}\right)$

$$P\left\{H_1\left(\overline{t_1, t_k}; t_{k+1}\right)\right\} = \tag{12}$$

$$\int_{-\infty}^{FF}\ldots\int_{-\infty}^{FF} f\left(\overline{x_1, x_{k+1}}\right)\left[\prod_{i=1}^{k}\int_{-\infty}^{PF_i - x_i}\varphi(y_i)dy_i\right]\overline{dx_1 dx_{k+1}}.$$

The probabilities of the events $H_2\left(\overline{t_1, t_k}; t_{k+1}\right)$, …, $H_6\left(\overline{t_1, t_k}; t_{k+1}\right)$ are derived analogically to the probability $P\left\{H_1\left(\overline{t_1, t_k}; t_{k+1}\right)\right\}$. After long mathematical manipulations, we obtain the following expressions:

$$P\left\{H_2\left(\overline{t_1, t_k}; t_{k+1}\right)\right\} = \int_{-\infty}^{FF}\ldots\int_{-\infty}^{FF} f\left(\overline{x_1, x_{k+1}}\right)\int_{PF_k - x_k}^{\infty}\varphi(y_k)dy_k \times$$

$$\left[\prod_{i=1}^{k-1}\int_{-\infty}^{PF_i - x_i}\varphi(y_i)dy_i\right]\overline{dx_1 dx_{k+1}}, \tag{13}$$

$$P\left\{H_3\left(\overline{t_1, t_k}; t_{k+1}\right)\right\} = \tag{14}$$

$$\int_{FF}^{\infty}\int_{-\infty}^{FF}\ldots\int_{-\infty}^{FF} f\left(\overline{x_1, x_{k+1}}\right)\left[\prod_{i=1}^{k}\int_{-\infty}^{PF_i - x_i}\varphi(y_i)dy_i\right]\overline{dx_1 dx_{k+1}},$$

$$P\left\{H_4\left(\overline{t_1, t_k}; t_{k+1}\right)\right\} = \int_{FF}^{\infty}\int_{-\infty}^{FF}\ldots\int_{-\infty}^{FF} f\left(\overline{x_1, x_{k+1}}\right)\times$$

$$\int_{PF_k - x_k}^{\infty}\varphi(y_k)dy_k\left[\prod_{i=1}^{k-1}\int_{-\infty}^{PF_i - x_i}\varphi(y_i)dy_i\right]\overline{dx_1 dx_{k+1}}, \tag{15}$$

$$P\left\{H_5\left(\overline{t_1, t_k}; t_{k+1}\right)\right\} = \sum_{j=0}^{k-1}\int_{-\infty}^{FF}{}_{(j)}\int_{-\infty}^{FF}\int_{FF}^{\infty}{}_{(k-j)}\int_{FF}^{\infty} f\left(\overline{x_1, x_k}\right)\times \tag{16}$$

$$\left[\prod_{i=1}^{k}\int_{-\infty}^{PF_i - x_i}\varphi(y_i)dy_i\right]\overline{dx_1 dx_k},$$

$$P\left\{H_6\left(\overline{t_1, t_k}; t_{k+1}\right)\right\} = \sum_{j=0}^{k-1}\int_{-\infty}^{FF}{}_{(j)}\int_{-\infty}^{FF}\int_{FF}^{\infty}{}_{(k-j)}\int_{FF}^{\infty} f\left(\overline{x_1, x_k}\right)\times \tag{17}$$

$$\int_{PF_k - x_k}^{\infty}\varphi(y_k)dy_k\left[\prod_{i=1}^{k-1}\int_{-\infty}^{PF_i - x_i}\varphi(y_i)dy_i\right]\overline{dx_1 dx_k}.$$

IV. Determination of Optimal Thresholds

The problem of determining the optimum value of the replacement threshold PF_k (k = 1, 2, …) depends on the selected optimization criterion.

Consider some optimization criteria. The maximum a posteriori probability criterion, when deciding on the system suitability in the interval (t_k, t_{k+1}), is formalized as follows:

$$PF_k^{opt} \Rightarrow \max_{PF_k} P\left\{X(t_{k+1}) < FF\middle|\bigcap_{i=1}^{k} Z(t_i) < PF_i\right\}, \tag{18}$$

where PF_k^{opt} is the optimal value of the replacement threshold PF_k when checking system suitability at time point t_k and $P\left\{X(t_{k+1}) < FF\middle|\bigcap_{i=1}^{k} Z(t_i) < PF_i\right\}$ is the a posteriori probability of the system suitability in the interval (t_k, t_{k+1}), which is determined as

$$P\left\{X\left(t_{k+1}\right)<FF\left|\bigcap_{i=1}^{k} Z\left(t_i\right)<PF_i\right.\right\}= \tag{19}$$

$$\frac{P\left\{H_1\left(\overline{t_1,t_k};t_{k+1}\right)\right\}}{\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} f\left(\overline{x_1,x_k}\right)\left[\prod_{i=1}^{k}\int_{-\infty}^{PF_i-x_i}\varphi\left(y_k\right)dy_k\right]\overline{dx_1 dx_k}}.$$

The criterion of minimum Bayes risk can be formulated as follows:

$$PF_k^{opt}\Rightarrow\min_{PF_k}\left\{C_1\alpha\left(\overline{t_1,t_k};t_{k+1}\right)+C_2\beta\left(\overline{t_1,t_k};t_{k+1}\right)\right\}, \tag{20}$$

where $\alpha\left(\overline{t_1,t_k};t_{k+1}\right)$ and $\beta\left(\overline{t_1,t_k};t_{k+1}\right)$ are the probabilities of the "false failure" and "undetected failure" when checking system suitability at time t_k, respectively, and C_1 and C_2 are the losses due to the "false failure" and "undetected failure", respectively. The probabilities $\alpha\left(\overline{t_1,t_k};t_{k+1}\right)$ and $\beta\left(\overline{t_1,t_k};t_{k+1}\right)$ are found as

$$\alpha\left(\overline{t_1,t_k};t_{k+1}\right)=P\left\{H_2\left(\overline{t_1,t_k};t_{k+1}\right)\right\}, \tag{21}$$

$$\beta\left(\overline{t_1,t_k};t_{k+1}\right)=P\left\{H_3\left(\overline{t_1,t_k};t_{k+1}\right)\right\}+P\left\{H_5\left(\overline{t_1,t_k};t_{k+1}\right)\right\}. \tag{22}$$

The criterion of minimum total error probability is represented as

$$PF_k^{opt}\Rightarrow\min_{PF_k}\left\{\alpha\left(\overline{t_1,t_k};t_{k+1}\right)+\beta\left(\overline{t_1,t_k};t_{k+1}\right)\right\}. \tag{23}$$

V. Example of Deterioration Process Modeling

Assume that the deterioration process of a one-parameter system is described by the following monotonic stochastic function:

$$X\left(t\right)=a_0+A_1 t^{\gamma}, \tag{24}$$

where a_0 and γ are the deterministic parameters of the system deterioration process, and A_1 is the random rate of degradation defined in the interval from 0 to ∞ with known PDF $\psi(a_1)$. It should be pointed out that (24) represents a wide class of degradation models. For example, a linear regressive model analyzed in [11] is a special case of (24).

Using the change of variables method described in previous studies, for example, in [12-14], we derive the PDF $f\left(\overline{x_1,x_{k+1}}\right)$ as follows:

$$f\left(\overline{x_1,x_{k+1}}\right)=\frac{1}{t_1^{\gamma}}\psi\left(\frac{x_1-a_0}{t_1^{\gamma}}\right)\times$$

$$\prod_{i=1}^{k}\delta\left\{x_{i+1}-\left[a_0+\frac{\left(x_i-a_0\right)t_{i+1}^{\gamma}}{t_i^{\gamma}}\right]\right\}, \tag{25}$$

where $\delta(\cdot)$ is the delta function.

Substituting (25) in (12)-(17), after certain mathematical manipulations we obtain the following analytical formulas for calculating the probabilities of correct and incorrect decisions when checking system suitability at time t_k:

$$P\left\{H_1\left(\overline{t_1,t_k};t_{k+1}\right)\right\}=\int_0^{(FF-a_0)/t_{k+1}^{\gamma}}\psi\left(\lambda\right)\left[\prod_{i=1}^{k}\int_{-\infty}^{PF_i-(a_0+\lambda t_i^{\gamma})}\varphi\left(y_i\right)dy_i\right]d\lambda, \tag{26}$$

$$P\left\{H_2\left(\overline{t_1,t_k};t_{k+1}\right)\right\}=\int_0^{(FF-a_0)/t_{k+1}^{\gamma}}\psi\left(\lambda\right)\left[\prod_{i=1}^{k-1}\int_{-\infty}^{PF_i-(a_0+\lambda t_i^{\gamma})}\varphi\left(y_i\right)dy_i\right]\times \tag{27}$$

$$\int_{PF_k-(a_0+\lambda t_k^{\gamma})}^{\infty}\varphi\left(y_k\right)dy_k d\lambda,$$

$$P\left\{H_3\left(\overline{t_1,t_k};t_{k+1}\right)\right\}=\int_{(FF-a_0)/t_{k+1}^{\gamma}}^{(FF-a_0)/t_k^{\gamma}}\psi\left(\lambda\right)\left[\prod_{i=1}^{k}\int_{-\infty}^{PF_i-(a_0+\lambda t_i^{\gamma})}\varphi\left(y_i\right)dy_i\right]d\lambda, \tag{28}$$

$$P\left\{H_4\left(\overline{t_1,t_k};t_{k+1}\right)\right\}=\int_{(FF-a_0)/t_{k+1}^{\gamma}}^{(FF-a_0)/t_k^{\gamma}}\psi\left(\lambda\right)\left[\prod_{i=1}^{k-1}\int_{-\infty}^{PF_i-(a_0+\lambda t_i^{\gamma})}\varphi\left(y_i\right)dy_i\right]\times \tag{29}$$

$$\int_{PF_k-(a_0+\lambda t_k^{\gamma})}^{\infty}\varphi\left(y_k\right)dy_k d\lambda,$$

$$P\left\{H_5\left(\overline{t_1,t_k};t_{k+1}\right)\right\}=\int_{(FF-a_0)/t_k^{\gamma}}^{\infty}\psi\left(\lambda\right)\left[\prod_{i=1}^{k}\int_{-\infty}^{PF_i-(a_0+\lambda t_i^{\gamma})}\varphi\left(y_i\right)dy_i\right]d\lambda, \tag{30}$$

$$P\left\{H_6\left(\overline{t_1,t_k};t_{k+1}\right)\right\}=\int_{(FF-a_0)/t_k^{\gamma}}^{\infty}\psi\left(\lambda\right)\left[\prod_{i=1}^{k-1}\int_{-\infty}^{PF_i-(a_0+\lambda t_i^{\gamma})}\varphi\left(y_i\right)dy_i\right]\times \tag{31}$$

$$\int_{PF_k-(a_0+\lambda t_k^{\gamma})}^{\infty}\varphi\left(y_k\right)dy_k d\lambda.$$

It is easily seen that the sum of the probabilities (26)-(31) is equal to

$$\int_0^{\infty}\psi\left(\lambda\right)d\lambda=1. \tag{32}$$

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VI. Numerical Example

Radar transmitter is the most expensive part in a radar system. In practice, it is very important to provide failure prediction of radar transmitter using CBM. According to [11], if the output voltage of radar transmitter exceeds the threshold FF = 25 kV, a corrective maintenance is required. Based on the data given in [11], the radar voltage is well approximated by the stochastic deterioration process (24) with the following parameter values: a_0=19.645 kV; γ = 0.8; $E[A_1]$=0.025 kV/h; $\sigma[A_1]$=0.012 kV/h, where $E[A_1]$ and $\sigma[A_1]$ are the mathematical expectation and standard deviation of the random variable A_1. Assume that A_1 and Y are normal random variables. Moreover, $E[Y] = 0$ and $\sigma[Y]$ = 0.1 kV.

Let us determine the optimal thresholds PF_k (k = 1, 2, …) by the criterion of minimum total error probability. The plot of optimal threshold value versus time of inspection is shown in Fig. 2. As seen, the optimal threshold value increases with increasing inspection time, which is due to the increase of mathematical expectation of the random process (24) with time. Assuming k = 4, t_4 = 400 h and t_5 = 500 h, the plot of the total error probability versus threshold PF_4 is shown in Fig. 3. As seen, the optimal threshold value is 24.13 kV and $(\alpha + \beta)_{\min}$ = 0.013. Note that if $PF_i = FF$ = 25 kV (i = 1, …, 4), the total error probability is 0.103. Thus, the use of the optimal replacement thresholds significantly reduces the total error probability.

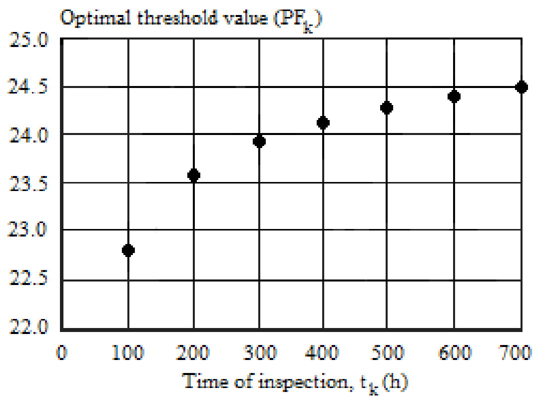

Fig. 2. Optimal threshold value versus time of inspection t_k (k = 1, …, 7).

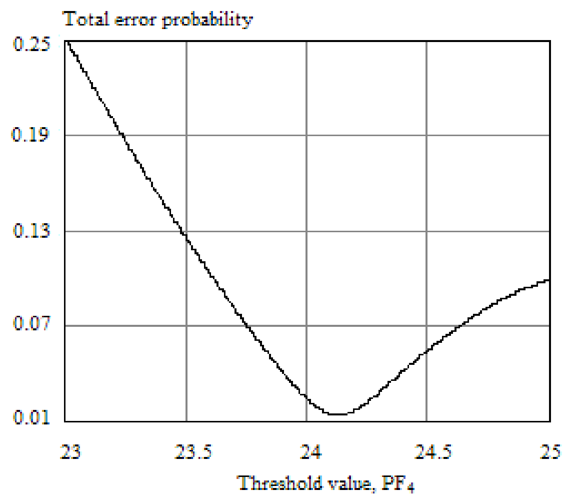

Fig. 3. Total error probability versus threshold PF_4 when t_4 = 400 h, t_5 = 500 h, PF_1 = 22.75 kV, PF_2 = 23.6 kV, and PF_3 = 23.9 kV.

VII. Conclusion

In this study, we have derived the equations for the probabilities of correct and incorrect decisions when checking suitability of a stochastically deteriorating system, which is periodically inspected by imperfect measuring equipment. Proposed expressions also consider the decisions taken at the previous inspections. The problems have been formulated for determining the optimal replacement thresholds by the criteria of maximum a posteriori probability, minimum Bayes risk and minimum total error probability. The proposed general expressions for the probabilities of correct and incorrect decisions have been illustrated by the derivation of the probabilities for a monotonically increasing stochastic process. In the numerical example, the effectiveness of the proposed approach to the determination of the optimal replacement thresholds has been illustrated.

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