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On Unique Solution of Quantum Stochastic Differential Inclusions

S. A. Bishop, *IAENG Member*, M. C. Agarana, H. I. Okagbue and J. G. Oghonyon

***Abstract*— We investigate the existence of solutions of quantum stochastic differential inclusion (QSDI) with some uniqueness properties as a variant of the results in the literature. We impose some weaker conditions on the coefficients and show that under these conditions, a unique solution can be obtained provided the functions $K^{P}_{\eta\xi}: [0, T] \to \mathbb{R}_+$ are measurable such that their integral is finite.**

***Index Terms*— Uniqueness of solution; Weak Lipschitz conditions; stochastic processes. Successive approximations**

I INTRODUCTION

In this paper, we establish existence and uniqueness of solution of the following quantum stochastic differential inclusion (QSDI):

$$x(t) \in a + \int_0^t E(s, x(s))d\Lambda_\pi + F(s, x(s))dA_g(s)$$
$$+G(s, x(s))dA_f^+(s) + H(s, x(s))ds,\ t \in [0, T] \qquad (1)$$

QSDI (1) is understood in the framework of the Hudson and Parthasarathy [9] formulation of Boson quantum stochastic calculus. The maps f, g, π appearing in (1) lie in some suitable function spaces defined in [7]. The integrators $\Lambda_\pi,\ A_f^+ \ and\ \ A_g$ are the gauge, creation and annihilation processes associated with the basic field operators of quantum field theory defined in [7]. However, in [7] it has been shown that inclusion (1) is equivalent to this first order nonclassical ordinary differential inclusion

$$\frac{d}{dt}\langle \eta, x(t)\xi\rangle \in P(t, x)(\eta, \xi)$$
$$x(0) = a,\ \ t \in [0, t] \qquad (2)$$

The map $(\eta, \xi) \to P(t, x)(\eta, \xi)$ appearing in (2) is defined by

$$P(t, y)(\eta, \xi) = (\mu E)(t, y)(\eta, \xi) + (\gamma F)(t, y)(\eta, \xi)$$
$$+ (\sigma G)(t, y)(\eta, \xi) + H(t, y)(\eta, \xi)$$

In [5, 6], some of the results in [2, 7] were generalized. Results on multifunction associated with a set of solutions of non-Lipschitz quantum stochastic differential inclusion (QSDI), which still admits a continuous selection from some subsets of complex numbers were established.

In [3], results on non-uniqueness of solutions of inclusion (2) were established under some strong conditions. Motivated by the results in [5, 6], we establish existence and uniqueness of solution of inclusion (2) under weaker conditions defined in [5]. Here, the map $x \to P(t, x)(\eta, \xi)$ is not necessarily Lipschitz in the sense of [3]. Hence the results here are weaker than the results in [3]. Inclusion (1) has applications in quantum stochastic control theory and the theory of quantum stochastic differential equations with discontinuous coefficients. See [7] and the references therein.

The rest of this paper is organized as follows; Section 3 of this paper will be devoted to the main results of the work while in section 2 some definitions, preliminary results and notations will be presented.

II PRELIMINARY RESULTS

Some of the notations and definitions used here will come from the references [3, 5-7]. $\mathcal{N}$ is a topological space, while clos($\mathcal{N}$), comp($\mathcal{N}$) denote the collection of all nonempty closed, compact subsets of $\mathcal{N}$ respectively. The space $\tilde{\mathcal{A}}$ (a locally convex space) is generated by the family of seminorms $\{\|x\|_{\eta\xi} = \langle \eta, x\xi\rangle,\ x \in \tilde{\mathcal{A}},\ \eta, \xi \in (\mathbb{D} \underline{\otimes} \mathbb{E})\}$. $(\tilde{\mathcal{A}}, \tau)$ is the completion of $\mathcal{A}$. Here $\tilde{\mathcal{A}}$ consists of linear operators defined in [3]. In what follows, $\mathbb{D}$ is a pre-Hilbert space, $\mathcal{R}$ its completion, γ a fixed Hilbert space and $L^2_\gamma(\mathbb{R}_+)$ is the space of square integrable γ- valued maps on $\mathbb{R}_+$. For the definitions and notations of the Hausdorff topology on $\tilde{\mathcal{A}}$ and more see [3] and the references therein.

Definition 1

$\Phi : I \times \tilde{\mathcal{A}} \to clos(\tilde{\mathcal{A}})$ is Lipschitzian if

$$\rho_{\eta\xi}(\Phi(t, x) - \Phi(t, y)) \le K^{\Phi}_{\eta\xi}(t)W(\|x - y\|_{\theta_{\Phi(\eta\xi)}}) \qquad (3)$$

where $W(t) \ne t, I = [0, T] \subseteq \mathbb{R}_+,\ \eta, \xi \in (\mathbb{D} \underline{\otimes} \mathbb{E})$.

Remark 2: (i) If $\eta, \xi \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2$ and $W(t) = t$ then we obtain the results in [3]

In this case, we obtain a class of multivalued maps which are not necessarily Lipschitzian in the sense of definition (3) (b) in [3].

Definition 3 By a solution of (1) or equivalently (2) we mean a stochastic process $\Phi: I \to \tilde{\mathcal{A}}$ lying in $Ad(\tilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\tilde{\mathcal{A}})$ satisfying (1).

The following result established in [3] is modified here. However we refer the reader to [3] for a detailed proof as we will only highlight the major changes due to the conditions in this setting.

Theorem 4 Let $B: \mathbb{R}_+ \to L(\tilde{\mathcal{A}})$. For any $x \in \tilde{\mathcal{A}}, L(t, x)$ defined by

$$L(t, x) = \{\ \|B(t)x\|_{\eta\xi}\}\mathcal{N}$$

is Lipschitzian with $W(t) \ne t$.

Proof: We adopt the method of the proof of Theorem 2.2 in [3] as follows:

Let the function $C^B_{\eta\xi}(t) > 0$, then for $x, y \in \tilde{\mathcal{A}},\ t \in \mathbb{R}_+$ we have

$$\rho_{\eta\xi}(L(t, x) - L(t, y)) =$$
$$\rho_{\eta\xi}(\|B(t)x\|_{\eta\xi}\mathcal{N}, \|B(t)y\|_{\eta\xi}\mathcal{N}\ )$$
$$\le \big|\|B(t)x\|_{\eta\xi} - \|B(t)y\|_{\eta\xi}\big|\rho_{\eta\xi}\mathcal{N}$$
$$\le \|\mathcal{N}\|_{\eta\xi}C^B_{\eta\xi}(t)\|x - y\|_{\theta_{B(\eta,\xi)}}$$
$$\le K^L_{\eta\xi}(t)W(\|x - y\|_{\theta_{B(\eta,\xi)}}),$$

Where $K^L_{\eta\xi}(t)W = \|\mathcal{N}\|_{\eta\xi}C^B_{\eta\xi}(t)$.

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S. A. Bishop, M. C. Agarana, H. I. Okagbue and J. G. Oghonyon are with the Department of Mathematics, Covenant University, Ota, Nigeria.
sheila.bishop@covenantuniversity.edu.ng
michael.agarana@covenantuniversity.edu.ng
hilary.okagbue@covenantuniversity.edu.ng
godwin.oghonyon@covenantuniversity.edu.ng

Let $\theta_B: (\mathbb{D} \underline{\otimes} \mathbb{E}) \to (\mathbb{D} \underline{\otimes} \mathbb{E})$. So that (2.2) in [3] becomes

$$\rho_{\eta\xi}((\Phi(t,x) - \Phi(t,y)) \leq K^{\Phi}_{\eta\xi}(t) W(\|x-y\|_{\eta\cdot\xi\cdot}) \quad (4)$$

$$\eta, \xi \in (\mathbb{D} \underline{\otimes} \mathbb{E}),\ x,\ y \in \tilde{\mathcal{A}},\ W(t) \neq t \text{ and } t \in I.$$

Similarly, (2.5) in [3] becomes

$$\rho_{\eta\xi}(\mathrm{P}(t,x) - \mathrm{P}(t,y)) \leq K^{P}_{\eta\xi}(t) W(\|x-y\|_{\eta\cdot\xi\cdot}) \quad (5)$$

For the map $(t,x) \to P(t,x)(\eta,\xi)$. Hence we conclude that the given map is also Lipschitzian.

III MAJOR RESULTS

To establish the major result in this section, we use the methods used in [3, 7]. In the sequel, except otherwise stated, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ and $t \in [0,1]$ is arbitrary. In line with [5, 6], we make the following assumptions

Let $Z : [0,1] \to \tilde{\mathcal{A}} \in Ad(\tilde{\mathcal{A}})_{wac}$ and for almost all $t \in [0,T]$, there exists $S_{\eta\xi} \in L^1_{loc}([0,T])$ such that

$$\left(\frac{d}{dt}\langle \eta, Z(t)\xi\rangle, P\big(t, Z(t)\big)(\eta,\xi)\right) \leq S_{\eta\xi}(t)$$

Fix $\gamma > 0$ and define the set $Q_{Z,\gamma}$ by,

$$Q_{Z,\gamma} = \{(t,x) \in t \times \tilde{\mathcal{A}} : \|x - Z(t)\|_{\eta\xi} \leq \gamma\}.$$

Since the coefficients E, F, G, H in (1) are Lipschitzian with $Q_{Z,\gamma} \to (clos(\tilde{\mathcal{A}}), \tau_h)$, (5) above holds for a. e.

$$P : \tilde{\mathcal{A}}(\eta,\xi) \to \tilde{\mathcal{A}}(\eta,\xi)$$

I. $\delta_{\eta\xi} \equiv \|x_0 - Z(0)\|_{\eta\xi}$ *and* $\delta_{\eta\xi} \leq \gamma$

II. $R_{\eta\xi} :=$ max $(\delta_{\eta\xi}, S_{\eta\xi})$ where

$$N_{\eta\xi} = ess\ sup_{[0,1]} S_{\eta\xi}(t)$$

III. Given the sequence $\{(\eta_n, \xi_n) \subseteq (\mathbb{D} \underline{\otimes} \mathbb{E}), n = 1, 2, \ldots\}$, we have $\mathrm{W}\big(sup_{n\in\mathbb{N}}\{ess\ sup_{t\in[0,1]} K^P_{\eta\xi}(t)\}\big) < \infty$

IV. Define $L_{\eta_j\xi_j} := W\left(ess\ sup_{[0,1]} K^P_{\eta_j\xi_j}(t)\right), j \geq 2$

V. Define

$$\varepsilon_{\eta\xi}(t) = 2L_{\eta\xi} + W\left(2L_{\eta\xi}\int_0^t (K^P_{\eta\xi}(s) e^{L_{\eta\xi,s}}) ds\right)$$

VI. Define $J \subset [0, 1]$ by

$$\mathrm{J} = \{t \in [0,1]: \varepsilon_{\eta\xi}(t) \leq \gamma\}.$$

We adopt the definitions of following; $L_{\eta\xi}, L_{\eta\xi,n}, R_{\eta_1\xi_1}, L_{\eta\cdot\xi\cdot}$ from the reference [3].

Proposition 5. Let $\{\Phi\}_{i=1}^{\infty} \in Ad(\tilde{\mathcal{A}})_{wac}$ be a sequence satisfying;

(i) $(t, \Phi_i(t)) \in Q_{Z,\gamma}, i \geq 1$ for a.e. $t \in$ J.

(ii) There exists $\{V_i\}_{i=1}^{\infty}$ and a constant $L_{\eta\xi} > 0$, such that

(a) $\Phi_i(t) = x_0 + \int_0^t V_{i-1}(s) ds,\ i \geq 1$

(b)

$$\left|\frac{d}{dt}\langle \eta, \Phi_i(t)\xi\rangle - \frac{d}{dt}\langle \eta, \Phi_{i-1}(t)\xi\rangle\right| \leq W\left(2L^{i-1}_{\eta\xi} K^P_{\eta\xi}(t)\frac{t^{i-2}}{(i-2)!}\right), \text{ for a. e. } t \in \mathrm{J}.$$

Then,

(c) $\|\Phi_i(t) - \Phi_{i-1}(t)\|_{\eta\xi} \leq W\left(2L_{\eta\xi}\int_0^t K^P_{\eta\xi}(s)\frac{L^{i-2}_{\eta\xi}}{(i-2)!} ds\right)$,

where $t \in$ J, $i \geq 2$.

Proof: The proof is an adaptation of the arguments employed in [3], Proposition 3.1.

Assume (i) and (ii) above hold. Then

$\|\Phi_i(t) - \Phi_{i-1}(t)\|_{\eta\xi} \leq \left|\int_0^t \langle \eta, (\mathrm{V}_{i-1}(s) - \mathrm{V}_{i-2}(s))\xi\rangle ds\right|$ by (ii) in (a)

$$= \left|\int_0^t \frac{d}{dt}\langle \eta, \Phi_i(t)\xi\rangle - \frac{d}{dt}\langle \eta, \Phi_{i-1}(t)\xi\rangle ds\right|$$

$$\leq \int_0^t \left|\frac{d}{dt}\langle \eta, \Phi_i(t)\xi\rangle - \frac{d}{dt}\langle \eta, \Phi_{i-1}(t)\xi\rangle\right| ds$$

$$\leq W(2L^{i-1}_{\eta\xi}\int_0^t K^P_{\eta\xi}(s)\frac{s^{i-2}}{(i-2)!} ds)$$

$$= W\left(2L_{\eta\xi}\int_0^t K^P_{\eta\xi}(s)\frac{(L_{\eta\xi}s)^{i-2}}{(i-2)!} ds\right), \qquad t \in J, i \geq 2.$$

The next Theorem is a major result.

Theorem 6. Suppose conditions I- VI hold, $\{E, F, G, H\}: [0,1] \times \tilde{\mathcal{A}} \to (clos(\tilde{\mathcal{A}}), \tau_h)$ is continuous. Then $\exists$ a unique solution such that

$$\|\Phi(\mathrm{t}) - \mathrm{Z}(\mathrm{t})\|_{\eta\xi} \leq \varepsilon_{\eta\xi}(t), t \in J, \quad (6)$$

$$\left|\frac{d}{dt}\langle \eta, \Phi(t)\xi\rangle - \frac{d}{dt}\langle \eta, Z(t)\xi\rangle\right| \leq L_{\eta\xi}(1 + W(2K^P_{\eta\xi}(t) e^{L_{\eta\xi}t}) \quad (7)$$

Proof. From the references [3, 7, 8], we construct a τ_W-Cauchy sequence $\Phi_n(t)$ of successively approximates $\Phi(t)$. We make the following assumptions:

$\left\{\frac{d}{dt}\langle \eta, \Phi_\mathrm{n}(t)\xi\rangle\right\}_{n\geq 0}$ is Cauchy in $\mathbb{C}$ for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. $\Phi_0(t) = Z$ is adapted.

By Theorem 1.14.2 in [1], there exists a measurable selection $\mathrm{V}_0(.)(\eta,\xi) \in P(., \Phi_0(.))(\eta,\xi)$ so that (3.3) in [3] holds and since $\mathrm{V}_0(.)(\eta,\xi)$ is locally absolutely integrable, then $\mathrm{V}_0 \in L^1_{loc}(\tilde{\mathcal{A}})$.

Let $\Phi_1(t)$ be defined as

$$\Phi_1(t) = X_0 + \int_0^t V_0(s) ds \quad (8)$$

If $V_0(t) \in \tilde{\mathcal{A}}$, then $\Phi_1(t) \in \tilde{\mathcal{A}}_t$. It implies that (3.4) and (3.5) in [3] hold in this case.

Again $\exists$ a measurable selection

$$V_1(.)(\eta,\xi) \in P(., V_1(.))(\eta,\xi)$$

which yields

$$\left|V_1(t)(\eta,\xi) - \frac{d}{dt}\langle \eta, \Phi_1(t)\xi\rangle\right|$$

$$= d\left(\frac{d}{dt}\langle \eta, \Phi_1(t)\xi\rangle, P(t, \Phi_1(\mathrm{t}))(\eta,\xi)\right)$$

$$\leq \rho\left(P\big(t, \Phi_1(t)\big)(\eta,\xi), P\big(t, \Phi_0(t)\big)(\eta,\xi)\right)$$

$$\leq W\left(K^P_{\eta\xi}(t)\|\Phi_1(t) - \Phi_0(t)\|_{\eta_1\xi_1}\right)$$

$$\leq W\left(K^P_{\eta\xi}(t)(\delta_{\eta_1\xi_1} + \int_0^t S_{\eta_1\xi_1}(s) ds)\right) \quad (9)$$

For some $(\eta,\xi), (\eta_1,\xi_1) \in \mathbb{D} \underline{\otimes} \mathbb{E}$. Similarly, for $\mathrm{V}_0(.)$, $\exists\ \mathrm{V}_1 \in L^1_{loc}(\tilde{\mathcal{A}})$ resulting in;

$$\mathrm{V}_1(\mathrm{t})(\eta,\xi) = \langle \eta, \mathrm{V}_1(t)\xi\rangle, t \in J \quad (10)$$

Define $\Phi_2(t)$ as $\Phi_1(t)$ in (8) above. Then $\Phi_2(t) \in \tilde{\mathcal{A}}_t$ since $\mathrm{V}_1 \in \tilde{\mathcal{A}}$ and hence $\Phi_2(t)$ is adapted.

Now if we consider t $\in$ J, we get,

$$\|\Phi_2(t) - \Phi_1(t)\|_{\eta\xi} = \left\|\int_0^t (\mathrm{V}_1(s) - \mathrm{V}_0(s)) ds\right\|_{\eta\xi}$$

$$= |\int_0^t \langle \eta, (\mathrm{V}_1(s) - \mathrm{V}_0(s))\xi\rangle ds|$$

$$\leq \int_0^t \rho(\mathrm{P}(\mathrm{t}, \Phi_1(s))(\eta,\xi), P(t, \Phi_0(s))(\eta,\xi) ds$$

$$\leq W(\int_0^t \mathrm{K}^\mathrm{P}_{\eta\xi}(s)\|\Phi_1(s) - \Phi_0(s)\|_{\eta_1\xi_1} ds) \quad (11)$$

By (3.4) in [3], we obtain

$$\|\Phi_2(t) - \Phi_1(t)\|_{\eta\xi} \mathrm{W}(\int_0^t \mathrm{K}^\mathrm{P}_{\eta\xi}(s)[\delta_{\eta_1\xi_1} + \int_0^t + \mathrm{S}_{\eta_1\xi_1}(\mathrm{r})\, \mathrm{dr}])\mathrm{ds}) \quad (12)$$

Continuing in this manner and replacing 1 with 2 in $\mathrm{V}_1(.)$ for $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ we get

$$\left|\mathrm{V}_2(t)(\eta,\xi) - \frac{d}{dt}\langle \eta, \Phi_2(t)\xi\rangle\right|$$

$$= d\left(\frac{d}{dt}\langle \eta, \Phi_2(t)\xi\rangle, P(t, \Phi_2(t))(\eta,\xi)\right)$$

$\leq \rho(P(t,\Phi_2(t))(\eta,\xi), P(t,\Phi_1(t))$
$\leq W(K^P_{\eta\xi}(t)\|\Phi_2(t)-\Phi_1(t)\|_{\eta_2\xi_2}$
$\leq W^2(K^P_{\eta\xi}(t)\int_0^t(K^P_{\eta_2\xi_2}(s)[\delta_{\eta_1\xi_1}+\int_0^s S_{\eta_1\xi_1}(r)dr])ds)$ (13)
by (12).

In a similar way we can show that since $V_2(t), V_3(t) \in L^1_{loc}(\tilde{\mathcal{A}})$ there exist $\Phi_3(t), \Phi_4(t)$ defined by

$$\Phi_3(t) = X_0 + \int_0^t V_2(s)ds, \quad t \in J$$

and

$$\Phi_4(t) = X_0 + \int_0^t V_3(s)ds, \quad t \in J \qquad (14)$$

satisfying

$$\|\Phi_3(t)-\Phi_2(t)\|_{\eta\xi} = \left\|\int_0^t (V_2(s)-V_1(s))ds\right\|_{\eta\xi}$$
$$\leq W^2(\int_0^t K^P_{\eta\xi}(s)[\int_0^s K^P_{\eta_2\xi_2}(s')[\delta_{\eta_1\xi_1} + \int_0^{s'} S_{\eta_1\xi_1}(r)dr]ds'])ds)$$
$$= W^2(\int_0^t K^P_{\eta\xi}(s)\int_0^s \delta_{\eta_1\xi_1}K^P_{\eta_2\xi_2}(s')\,ds'ds$$
$$+ W^2(\int_0^t K^P_{\eta\xi}(s)\int_0^s K^P_{\eta_2\xi_2}(s')\int_0^{s'} S_{\eta_1\xi_1}(r)dr\,ds'ds \quad (15)$$

and

$$\|\Phi_4(t)-\Phi_3(t)\|_{\eta\xi} = \left\|\int_0^t (V_3(s)-V_2(s))ds\right\|_{\eta\xi}$$

$$\leq W^3\left(\int_0^t K^P_{\eta\xi}(s)\int_0^s K^P_{\eta_3\xi_3}(s')\int_0^{s'}\delta_{\eta_1\xi_1}K^P_{\eta_2\xi_2}(s'')\,ds''ds'ds\right)$$
$$+ W^3\left(\int_0^t K^P_{\eta\xi}(s)\int_0^s K^P_{\eta_3\xi_3}(s')\int_0^{s'}\delta_{\eta_1\xi_1}K^P_{\eta_2\xi_2}(s'')\right)$$
$$\times\left(\int_0^{s''} S_{\eta_1\xi_1}(r)dr\,ds''ds'ds\right) \qquad (16)$$

Then

$$\left|\frac{d}{dt}\langle\eta,\Phi_4(t)\xi\rangle - \frac{d}{dt}\langle\eta,\Phi_3(t)\xi\rangle\right|$$
$$\leq W^3\left(K^P_{\eta\xi}(t)\int_0^t K^P_{\eta_3\xi_3}(s)\int_0^s \delta_{\eta_1\xi_1}K^P_{\eta_2\xi_2}(s')\,ds'ds\right)$$
$$\leq W^3\left(K^P_{\eta\xi}(t)\int_0^t K^P_{\eta_3\xi_3}(s)\int_0^s K^P_{\eta_2\xi_2}(s')\int_0^{s'} S_{\eta_1\xi_1}\,drds'ds\right) \qquad (17)$$

and by (16) and (17) we get

$$\|\Phi_4(t)-\Phi_3(t)\|_{\eta\xi}$$
$$\leq W\left(\int_0^t K^P_{\eta\xi}(s)\int_0^s L_{\eta_3\xi_3}\int_0^{s'}\delta_{\eta_1\xi_1}L_{\eta_2\xi_2}\,ds''ds'ds\right)$$
$$+W\left(\int_0^t K^P_{\eta\xi}(s)\int_0^s L_{\eta_3\xi_3}\int_0^{s'} L_{\eta_2\xi_2}\int_0^{s''} S_{\eta_1\xi_1}(r)drds''ds'ds\right)$$
$$2L^3_{\eta\xi}W(\int_0^t K^P_{\eta\xi}(s)\frac{s^2}{2}ds)$$

Also for $t \in [0,T]$,

$$\left|\frac{d}{dt}\langle\eta,\Phi_4(t)\xi\rangle - \frac{d}{dt}\langle\eta,\Phi_3(t)\xi\rangle\right|$$
$$\leq W\left(K^P_{\eta\xi}(t)[\delta_{\eta_1\xi_1}L_{\eta_2\xi_2}L_{\eta_3\xi_3}\int_0^t\int_0^s ds'ds]\right)$$
$$+W\left(K^P_{\eta\xi}(t)[S_{\eta_1\xi_1}L_{\eta_2\xi_2}L_{\eta_3\xi_3}\int_0^t\int_0^s\int_0^{s'} drds'ds]\right)$$
$$= W\left(K^P_{\eta\xi}(t)[\delta_{\eta_1\xi_1}L_{\eta_2\xi_2}L_{\eta_3\xi_3}\frac{t^2}{2} + S_{\eta_1\xi_1}L_{\eta_2\xi_2}L_{\eta_3\xi_3}\frac{t^3}{6}]\right)$$
$$\leq W\left(K^P_{\eta\xi}(t)[L^3_{\eta\xi,3}\frac{t^2}{2} + L^3_{\eta\xi,3}\frac{t^3}{6}]\right)$$
$$\leq 2W\left(K^P_{\eta\xi}(t)L^3_{\eta\xi,3}\frac{t^2}{2}\right) \qquad (18)$$

Next, we claim that the sequence $\{\Phi_i(t)\}_{i\geq 1}$ exists. To prove this claim, we let $\Phi_{n+i} : J \to \tilde{\mathcal{A}}$ and by Theorem (1.14.2) in [1], $\exists$ a computable selection $V_n(.)(\eta,\xi) \in P(.,\Phi_n(.))(\eta,\xi)$ which yields

$$\left|V_n(t)(\eta,\xi) - \frac{d}{dt}\langle\eta,\Phi_n(t)\xi\rangle\right|$$
$$= d\left(\frac{d}{dt}\langle\eta,\Phi_n(t)\xi\rangle, P(t,\Phi_n(t))(\eta,\xi)\right)$$

Since $(\eta,\xi) \to V_n(t)(\eta,\xi)$, $\exists\ V_n \in L^1_{loc}(\tilde{\mathcal{A}})$ defined by (10) a.e. on J and define $\Phi_{n+1}(t)$ as in (8). Then for arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, we get,

$$\left|\frac{d}{dt}\langle\eta,\Phi_{n+1}(t)\xi\rangle - \frac{d}{dt}\langle\eta,\Phi_n(t)\xi\rangle\right|$$
$$= \langle\eta,\Phi_n(t)\xi\rangle - \langle\eta,\Phi_{n-1}(t)\xi\rangle$$
$$\leq \rho(P(t,\Phi_n(t))(\eta,\xi), P(t,\Phi_{n-1}(t))(\eta,\xi))$$
$$\leq W(K^P_{\eta\xi}(t)\|\Phi_n(t)-\Phi_{n-1}(t)\|_{\eta\xi})$$
$$\leq 2W\left(K^P_{\eta\xi}(t)[L_{\eta\xi}\int_0^t K^P_{\eta_n\xi_n}(s)\frac{(L_{\eta\xi}s)^{n-2}}{(n-2)!}ds]\right)$$
$$\leq W\left(L^n_{\eta\xi}(t)K^P_{\eta\xi}(t)\frac{(t)^{n-1}}{(n-1)!}\right)$$

This establishes (ii)(b) of Prop. 5. Now, for $t \in J$, we get

$$\|\Phi_{n+1}(t)-\Phi_0(t)\|_{\eta\xi} \leq \|\Phi_1(t)-\Phi_0(t)\|_{\eta\xi}$$
$$+\|\Phi_2(t)-\Phi_1(t)\|_{\eta\xi}$$
$$+\cdots+\|\Phi_{n+1}(t)-\Phi_n(t)\|_{\eta\xi}$$
$$\leq 2L_{\eta\xi}[1+W(\sum_{k=0}^{n-1}\int_0^t K^P_{\eta\xi}(s)\frac{(L_{\eta\xi}s)^k}{k!}ds)]$$
$$\leq 2L_{\eta\xi}(1+W(\int_0^t K^P_{\eta\xi}(s)e^{L_{\eta\xi}s}ds) \leq \gamma \qquad (19)$$

So that (6) of theorem 6 and (i) of proposition 5 follows. (ii)(b) of proposition 5 yields

$$\left|\frac{d}{dt}\langle\eta,\Phi_{n+1}(t)\xi\rangle - \frac{d}{dt}\langle\eta,\Phi_0(t)\xi\rangle\right|$$
$$\leq \left|\frac{d}{dt}\langle\eta,\Phi_1(t)\xi\rangle - \frac{d}{dt}\langle\eta,\Phi_0(t)\xi\rangle\right|$$
$$+W(\sum_{k=0}^{n-1}2L_{\eta\xi}K^P_{\eta\xi}(t)\frac{(L_{\eta\xi}t)^k}{k!})$$
$$+2L_{\eta\xi}(1+W(K^P_{\eta\xi}(t)e^{L_{\eta\xi}t})$$

Taking the limit as $n\to\infty$, (7) of Theorem 6 follows. Hence $\{\Phi_n(t)\}$ is a Cauchy sequence in $\tilde{\mathcal{A}}$ and converges to $\Phi(t)$. Since $\Phi_n(t) \in Ad(\tilde{\mathcal{A}})_{wac}$, it implies that $\Phi(t) \in Ad(\tilde{\mathcal{A}})_{wac}$.

Remark 7. The result of Corollary 3.4 and Theorem 3.4 in [3] fails in this case since the Lipschitz function is independent of t and $W(t) \neq t$ will not be applicable. Hence we establish our result on uniqueness.

Uniqueness of solution

To establish this result, we assume that $\overline{\Phi}(t) \in Ad(\tilde{\mathcal{A}})_{wac}, t \in [0,T]$ is another solution with $\overline{\Phi}(0) = X_0$. By using equation (8) and hypothesis (iii) in Theorem 6 (see also equation (2.1) in [4]), we obtain

$$\|\Phi(t)-\overline{\Phi}(t)\|_{\eta\xi} = \left\|\int_0^t\left(V(s)-\overline{V}(s)\right)ds\right\|_{\eta\xi}$$
$$= \left|\int_0^t\langle\eta,\left(V(s)-\overline{V}(s)\right)\xi\rangle ds\right|$$
$$\leq \int_0^t|\langle\eta,V(s)\xi\rangle - \langle\eta,\overline{V}(s)\xi\rangle|ds$$
$$\leq \int_0^t \rho\left((s,\Phi(s))(\eta,\xi) - \left(s,\overline{\Phi}(s)\right)(\eta,\xi)\right)ds$$
$$\leq W(\int_0^t K^P_{\eta\xi}(s)\|\Phi(s)-\overline{\Phi}(s)\|_{\eta\xi}ds)$$

Since the integral $\int_0^t K^P_{\eta\xi}(s)\,ds$ exists on [0, T], it is also essentially bounded on the given interval. Hence, there exists a constant $C_{\eta\xi}$ such that $\operatorname{ess\,sup} K^P_{\eta\xi}(s) = C_{\eta\xi}, s \in [0,T]$. Thus

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$$\left\|\Phi(t)-\overline{\Phi}(t)\right\|_{\eta\xi} \leq W\left(\mathrm{C}_{\eta\xi}, t\int_0^t \left\|\Phi(s)-\overline{\Phi}(s)\right\|_{\eta\xi} ds\right).$$

By the Gronwall's inequality, we conclude that $\Phi(t)=\overline{\Phi}(t), t\in[0,T]$. Hence the solution is unique.

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CONFLICT OF INTEREST

The authors declare that no conflict of interest with respect to the publication of this paper.

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