On Continuous Dependence of Solution of Quantum Stochastic Differential Equation with Nonlocal Conditions

S. A. Bishop, *IAENG Member* T. A. Anake, M. C. Agarana

Abstract— **In this paper we establish new results on continuous dependence of solution in the sense of Hudson and Parthasarathy formulation of non-classical ordinary differential equation with nonlocal conditions. We employ recent methods to establish the results**.

***Index Terms*— Continuous dependence of solution, Stochastic processes, non-random variables, fixed point.**

I INTRODUCTION

Several results concerning quantum stochastic differential equation (QSDE) of the Hudson and Parthasarathy [3] formulation of quantum stochastic calculus have been established. See [4]. In [1], continuous dependence of solution on parameters was studies. This was accomplished within the framework of the Hudson and Parthasarathy formulations of quantum stochastic differential equation (QSDE) and the Schwabik generalized ordinary differential equations. The Stochastic processes are necessarily functions of bounded variation and satisfy the Lipschitz and Caratheodory conditions. Within the setting of classical ordinary differential equation, some work has been done on continuous dependence of solution on parameters [6]. Of the most recent is the work of [4] where the fixed point approach was adopted.

In this paper, we consider similar results as in [4] using the equivalent form of QSDE formulated by [3]. We also take advantage of the fact that the stochastic processes are differentiable. The result obtained here is a generalization of analogous results due to the reference [4] concerning classical ODE.

We consider the nonlocal QSDE of Hudson and Parthasarathy:

$$dx(t) = E\big(t, x(t)\big)d\Lambda_\pi(t) + F\big(t, x(t)\big)dA_f(t) + G\big(t, x(t)\big)dA_g^+(t) + H\big(t, x(t)\big)dt,$$

$$x_0 = x(0) + \textstyle\sum_{k=1}^n a_k x(\tau_k)\,, a_k > 0, \tau_k \in (0, T) \qquad (1)$$

where x_0 is stochastic process and a_k are positive real integers. The coefficients E, F, G, H $\in L^2_{\text{loc}}(I \times \tilde{\mathcal{A}})_{\text{mvs}}$, where $\tilde{\mathcal{A}}$ is a topological vector space. The integrators Λ_π, A_g^+, A_f in (1) are well defined in [2]. $f, g \in L^\infty_{\gamma,loc}(\mathbb{R}_+), \pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$. Next we introduce the equivalent form of QSDE (1.1) also known as the nonclassical ordinary differential inclusion (NODE) established in [2];

$$\frac{d}{dt}\langle \eta, x(t)\xi\rangle = F(t, x(t))(\eta, \xi)$$

$$\langle \eta, x_0\xi\rangle = \langle \eta, x(0)\xi\rangle + \textstyle\sum_{k=1}^n a_k x(\tau_k)\,(\eta, \xi) \qquad (2)$$

where the map F in (2) is defined in [6], $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ is arbitrary.

The next section will consist of some preliminaries while sections 3 and 4 will contain the major results on existence and continuous dependence of solution respectively. In section 3, we show that a unique solution exists using the contraction method [5], while in section 4, we adopt the method applied in [4] to show the continuous dependence of solution on some stochastic process and the coefficient of the nonlocal problem.

II PRELIMINARIES

We adopt the definitions and notations of the spaces $Ad(\tilde{\mathcal{A}})$, $Ad(\tilde{\mathcal{A}})_{wac}$ $L^p_{loc}(\tilde{\mathcal{A}})$, $L^\infty_{loc}(\mathbb{R}_+), L^2_{loc}(I \times \tilde{\mathcal{A}})$ from the reference [2] and the references therein. The space $C(I, \tilde{\mathcal{A}})$ is the space of stochastic processes that are continuous and the sesquilinear equivalent form will be denoted by $C(I, A(\eta, \xi))$ where $A \subseteq \tilde{\mathcal{A}}$ is dense in $\tilde{\mathcal{A}}$, $I = [0, T]$. The space $C(I, A(\eta, \xi))$ is a Banach space with the associated norm defined by

$$\|x\|_{\eta\xi} = \text{Sup}\{|x(t)(\eta, \xi)| : t \in I\}$$

Let $C\big(I, A(\eta, \xi)\big) =: \mathfrak{B}$. The following assumptions will be used to establish the major results.

(A_1) The map $F: [0, T] \times A(\eta, \xi) \to A(\eta, \xi)$ is continuous

(A_2) The function $K^F_{\eta\xi} : [0, T] \to \mathbb{R}_+$, is such that for each $t \in [0, T]$, the $\sup_{t\in I} \int_0^t K^F_{\eta\xi}(s)ds \le \ell_{\eta\xi}$ and

$$|F(t, x)(\eta, \xi) - F(t, y)(\eta, \xi)| \le K^F_{\eta\xi}(t)\|x - y\|_{\eta\xi} \qquad (3)$$

(A_3) Let $M_{\eta\xi} > 0$ be a constant such that the

$$\sup_{t\in I} |F(t, 0)(\eta, \xi)| \le M_{\eta\xi} \qquad (4)$$

Proposition 2.1: Assume $\varphi : [0, T] \to A(\eta, \xi)$ is a solution of the problem (2) on [0, T], so for each $t \in [0, T]$, we obtain

$$x(t)(\eta, \xi) =$$
$$a(x_0(\eta, \xi) - \textstyle\sum_{k=1}^n a_k \int_0^{\tau_k} F(s, x(s))(\eta, \xi)ds) +$$
$$+0tF(s, x(s))\eta,\xi ds \qquad (5)$$

where $a = (1 + \sum_{k=1}^n a_k)^{-1}$

Proof: the problem (2) can be written in integral form as

$\langle \eta, x(t)\xi\rangle = \langle \eta, x(0)\xi\rangle + \int_0^t F(s, x(s))(\eta, \xi)\, ds$and

$\langle \eta, x(\tau_k)\xi\rangle = \langle \eta, x(0)\xi\rangle + \int_0^{\tau_k} F(s, x(s))(\eta, \xi)\, ds$

Then

$$\textstyle\sum_{k=1}^n a_k x(\tau_k)\,(\eta, \xi) = \sum_{k=1}^n a_k x(0)\,(\eta, \xi) + + \sum_{k=1}^n a_k \int_0^{\tau_k} F(s, x(s))(\eta, \xi)\, ds$$

This work was supported in full by Covenant University.

S. Bishop is with the Department of Mathematics, Covenant University,Nigeria,+2347081203200,sheila.bishop@covenantuniversity.edu.ng

T. A. Anake is with the Department of Mathematics, Covenant University, Nigeria

M.C. Agarana is with the Department of Mathematics, Covenant University,Nigeria

$\langle \eta, x_0\xi\rangle - \langle \eta, x(0)\xi\rangle =$
$\sum_{k=1}^{n} a_k x_{\eta\xi}(0) + \sum_{k=1}^{n} a_k \int_0^{\tau_k} F(s, x(s))(\eta,\xi)\, ds$
and

$1 + \sum_{k=1}^{n} a_k x_{\eta\xi}(0) =$
$x_{\eta\xi}(0) - \sum_{k=1}^{n} a_k \int_0^{\tau_k} F(s, x(s))(\eta,\xi)\, ds$
Then
$x_{\eta\xi}(0) = \left(x_{\eta\xi}(0) - \sum_{k=1}^{n} a_k \int_0^{\tau_k} F(s, x(s))(\eta,\xi)\, ds\right).$
Note that $\langle \eta, x(0)\xi\rangle = x_{\eta\xi}(0) = x(0)(\eta,\xi)$.
So that (5) follows.
Proposition 2.2: $N : \mathfrak{B} \to \mathfrak{B}$.
Proof: Let $x \in \mathfrak{B}, t_1, t_2 \in I$ such that $|t_1 - t_2| < \delta$. Now define the above mapping by
$N(x(t))(\eta,\xi) =$
$a\left(x_0(\eta,\xi) - \sum_{k=1}^{n} a_k \int_0^{\tau_k} F(s, x(s))(\eta,\xi) ds)\right) +$
$+ \int_0^{t} F(s, x(s))(\eta,\xi)\, ds$, then
$N(x(t_2))(\eta,\xi) - N(x(t_1))(\eta,\xi)$
$= \int_{t_1}^{t_2} F(s, x(s))(\eta,\xi)\, ds$ (6)
By (3) we get
$\|F(t,x) - F(t,0)\|_{\eta\xi} \le \|F(t,x)\|_{\eta\xi} - \|F(t,0)\|_{\eta\xi}$
$\le K_{\eta\xi}^{F}(t)\|x\|_{\eta\xi}$
and
$\|F(t,x)\|_{\eta\xi} \le K_{\eta\xi}^{F}(t)\|x\|_{\eta\xi} + M_{\eta\xi}$ (7)
by substituting (7) in (6), we get
$\left\|N(x(t_2)) - N(x(t_1))\right\|_{\eta\xi} \le$
$\le \int_{t_1}^{t_2} (K_{\eta\xi}^{F}(s)\|x\|_{\eta\xi} + M_{\eta\xi}) ds$
$\le \|x\|_{\eta\xi} \int_{t_1}^{t_2} K_{\eta\xi}^{F}(s) ds + \int_{t_1}^{t_2} M_{\eta\xi}\, ds$
$\le \|x\|_{\eta\xi} \left|h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)\right| + M_{\eta\xi}\delta$
Where $h_{\eta\xi}: [0,T] \to \mathbb{R}$ is a non-decreasing function defined on $[0,T]$. This concludes the proof showing that $N : \mathfrak{B} \to \mathfrak{B}$.

III Existence of a Unique Solution

Theorem 3.1 Assume that the hypothesis (A_1) − (A_3) hold. If $2\ell_{\eta\xi} < 1$, then Eq. (2) has a unique solution.
Proof: Let $x, \bar{x} \in \mathfrak{B}$, and let the operator N be defined as above for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. We show that N is a contraction and hence has a unique fixed point.
Let $N : \mathfrak{B} \to \mathfrak{B}$, then for each $t \in [0,T]$
$\left\|N(x(t)) - N(\bar{x}(t))\right\|_{\eta\xi} = \left|\int_{t_1}^{t_2} [F(s, x(s))(\eta,\xi) -\right.$
F(s, x(s))η,ξ]ds−ak=1nak0τk[F(s, x(s))η,ξ−F(s, x(s))η,ξ]ds

$$\le \int_{t_1}^{t_2} \left|F(s, x(s))(\eta,\xi) - P(s, \bar{x}(s))(\eta,\xi)\right| ds$$
$$+\ a \sum_{k=1}^{n} a_k \int_0^{\tau_k} \left|F(s, x(s))(\eta,\xi) - F(s, \bar{x}(s))(\eta,\xi)\right| ds$$
$\le \ell_{\eta\xi}\|x - \bar{x}\|_{\eta\xi} + [\sum_{k=1}^{n} a_k]\ell_{\eta\xi}\|x - \bar{x}\|_{\eta\xi}$
Therefore,
$$\left[1 + \sum_{k=1}^{n} a_k\right] \ell_{\eta\xi}\|x - \bar{x}\|_{\eta\xi} \le 2\ell_{\eta\xi}\|x - \bar{x}\|_{\eta\xi}$$
This shows that N is a contraction and hence a unique solution of the problem exists.

IV Continuous Dependence of Solution

*Definition 4.1***:** The solution $x \in \mathfrak{B}$ of Eq. (2) continuously depends on the stochastic process x_0 if for all $\varepsilon > 0, \exists\, \delta > 0$ such that $\|x_0 - \bar{x}_0\|_{\eta\xi} \le \delta \implies \|x - \bar{x}\|_{\eta\xi} \le \varepsilon$.

Theorem 4.1**:** Assume that the conditions (A_1) – (A_3) hold. Then the solution of Eq.(2) continuously depends on x_0 .
Proof: Set (5) to be a solution of Eq. (2) and

$\bar{x}(t)(\eta,\xi) =$
$a(\bar{x}_0(\eta,\xi) - \sum_{k=1}^{n} a_k \int_0^{\tau_k} F(s, \bar{x}(s))(\eta,\xi) ds) +$
+0tF(s, x(s))η,ξds (8)
be a solution of (2) with the nonlocal condition given by

$\bar{x}_0 = \bar{x}(0) + \sum_{k=1}^{n} a_k \bar{x}(\tau_k), a_k > 0, \tau_k \in (0,T)$ (9)
Then
$$|x(t)(\eta,\xi) - \bar{x}(t)(\eta,\xi)| = a[x_0(\eta,\xi) - \bar{x}_0(\eta,\xi)]$$
$$-a \sum_{k=1}^{n} a_k \int_0^{\tau_k} [F(s, x(s))(\eta,\xi) - F(s, \bar{x}(s))(\eta,\xi)]\, ds$$
$$+ \int_0^{t} [F(s, x(s))(\eta,\xi) - F(s, \bar{x}(s))(\eta,\xi)] ds$$
and
$$\|x(t) - \bar{x}(t)\|_{\eta\xi} \le \|x_0 - \bar{x}_0\|_{\eta\xi} + a \sum_{k=1}^{n} a_k \int_0^{\tau_k} |F(s, x(s))(\eta,\xi) - F(s, \bar{x}(s))(\eta,\xi)|\, ds$$
$$+ \int_0^{t} |F(s, x(s))(\eta,\xi) - F(s, \bar{x}(s))(\eta,\xi)| ds$$
$$\le a\delta + 2\ell_{\eta\xi}\|x - \bar{x}\|_{\eta\xi}$$
Hence $\|x(t) - \bar{x}(t)\|_{\eta\xi} \le \frac{a\delta}{\{1 - 2\ell_{\eta\xi}\}} = \varepsilon$ and the proof is completed.
Lastly, we consider the problem (2) with the nonlocal condition given by
$$x_0 = x(0) + \sum_{k=1}^{n} \bar{a}_k x(\tau_k), \tau_k \in (0,T) \qquad (10)$$

Definition 4.2 The solution $x \in \mathfrak{B}$ of Eq.(2) continuously depends on a_k if $\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $\|a_k - \bar{a}_k\| \le \delta \implies \|x - \bar{x}\| \le \varepsilon$.
Theorem 4.2 Assume that the conditions (A_1) − (A_3) hold. Then the solution of Eq. (2) continuously depends on a_k.
Proof. Let
$x(t)(\eta,\xi) =$
$a\left(x_0(\eta,\xi) - \sum_{k=1}^{n} a_k \int_0^{\tau_k} F(s, x(s))(\eta,\xi) ds)\right) +$
$\int_0^{t} F(s, x(s))(\eta,\xi) ds$ be the solution of Eq. (2) and
$\bar{x}(t)(\eta,\xi) =$
$\bar{a}\left(x_0(\eta,\xi) - \sum_{k=1}^{n} \bar{a}_k \int_0^{\tau_k} F(s, \bar{x}(s))(\eta,\xi) ds)\right) +$
$\int_0^{t} F(s, \bar{x}(s))(\eta,\xi) ds$ together with (10) be the solution of Eq.(2). Then,

$x_{\eta\xi}(t) - \bar{x}_{\eta\xi}(t) = [a - \bar{a}]x_0(\eta,\xi) + \int_0^{t} [F(s, x(s))(\eta,\xi) -$
Fs, xsη,ξ]ds
$-a \sum_{k=1}^{n} a_k \int_0^{\tau_k} F(s, x(s))(\eta,\xi) ds +$
$+\bar{a} \sum_{k=1}^{n} \bar{a}_k \int_0^{\tau_k} F(s, \bar{x}(s))(\eta,\xi) ds$ (11)
Now by the definition of a, we get
$$|a - \bar{a}| = \left|\frac{1}{1 + \sum_{k=1}^{n} a_k} - \frac{1}{1 + \sum_{k=1}^{n} \bar{a}_k}\right| (1 + \sum_{k=1}^{n} a_k)\, (1 + \sum_{k=1}^{n} \bar{a}_k)$$
$$= \left|\frac{\sum_{k=1}^{n} (\bar{a}_k - a_k)}{(1 + \sum_{k=1}^{n} a_k)(1 + \sum_{k=1}^{n} \bar{a}_k)}\right| \left(1 + \sum_{k=1}^{n} a_k\right) \times \left(1 + \sum_{k=1}^{n} \bar{a}_k\right)$$
$$\le \left|\sum_{k=1}^{n} (\bar{a}_k - a_k)\right| \le n\delta$$

ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Now from the second term on the right hand side of (11), we get

$$\int_0^t [F(s,x(s))(\eta,\xi) - F(s,\bar{x}(s))(\eta,\xi)]ds \le \ell_{\eta\xi}\|x-\bar{x}\|_{\eta\xi}$$

and from the last two terms on the right hand side of (11),we get

$$\bar{a}\sum_{k=1}^{n}\bar{a}_k\int_0^{\tau_k}F(s,\bar{x}(s))(\eta,\xi)ds - a\sum_{k=1}^{n}a_k\int_0^{\tau_k}F(s,x(s))(\eta,\xi)ds$$

$$= \bar{a}\left(1+\sum_{k=1}^{n}\bar{a}_k\right)\int_0^{\tau_k}F(s,\bar{x}(s))(\eta,\xi)ds - a\left(1 + \sum_{k=1}^{n}a_k\right)\int_0^{\tau_k}F(s,x(s))(\eta,\xi)ds$$

$$-\bar{a}\int_0^{\tau_k}F(s,\bar{x}(s))(\eta,\xi)ds + a\int_0^{\tau_k}F(s,x(s))(\eta,\xi)ds$$

$$= \bar{a}(\bar{a}^{-1})\int_0^{\tau_k}F(s,\bar{x}(s))(\eta,\xi)ds - a(a^{-1})\int_0^{\tau_k}F(s,x(s))(\eta,\xi)ds$$

$$-\bar{a}\int_0^{\tau_k}F(s,\bar{x}(s))(\eta,\xi)ds + a\int_0^{\tau_k}F(s,x(s))(\eta,\xi)ds$$

$$= \int_0^{\tau_k}F(s,\bar{x}(s))(\eta,\xi)ds - \int_0^{\tau_k}F(s,x(s))(\eta,\xi)ds$$

$$-\bar{a}\int_0^{\tau_k}F(s,\bar{x}(s))(\eta,\xi)ds + a\int_0^{\tau_k}F(s,x(s))(\eta,\xi)ds$$

$$a\int_0^{\tau_k}F(s,\bar{x}(s))(\eta,\xi)ds - a\int_0^{\tau_k}F(s,x(s))(\eta,\xi)ds$$

$$\le \int_0^{\tau_k}[F(s,\bar{x}(s))(\eta,\xi) - F(s,x(s))(\eta,\xi)]ds$$

$$+(a-\bar{a})\int_0^{\tau_k}F(s,\bar{x}(s))(\eta,\xi)ds + a\int_0^{\tau_k}[F(s,x(s))(\eta,\xi) - F(s,\bar{x}(s))(\eta,\xi)]ds$$

$$= \|x(t)-\bar{x}(t)\|_{\eta\xi}\int_0^{\tau_k}K_{\eta\xi}^F(s)ds + \delta\,\|x(t)\|_{\eta\xi} + a\|x(t)-\bar{x}(t)\|_{\eta\xi}\int_0^{\tau_k}K_{\eta\xi}^F(s)ds$$

$$\le \|x(t)-\bar{x}(t)\|_{\eta\xi}\int_0^{\tau_k}K_{\eta\xi}^F(s)ds[1+a] + \delta\|x(t)\|_{\eta\xi}$$

$$\le \ell_{\eta\xi}\|x-\bar{x}\|_{\eta\xi}(1+a) + \delta\|x(t)\|_{\eta\xi}$$

Now adding up everything together yields

$$\|x(t)-\bar{x}(t)\|_{\eta\xi} \le n\delta\|x_0\|_{\eta\xi} + \delta\|x(t)\|_{\eta\xi} + 2\ell_{\eta\xi}\|x-\bar{x}\|_{\eta\xi} + \ell_{\eta\xi}\|x-\bar{x}\|_{\eta\xi}$$

$$\le 3\ell_{\eta\xi}\|x-\bar{x}\|_{\eta\xi} + \delta\|x(t)\|_{\eta\xi} + n\delta\|x_0\|_{\eta\xi}$$

$$\le \frac{\delta(n\|x_0\|_{\eta\xi} + \|x(t)\|_{\eta\xi})}{(1-3\ell_{\eta\xi})} = \varepsilon$$

And this completes the proof.

Acknowledgement

The authors appreciate the management of Covenant University for their financial support and the constructive suggestions of the reviewers.

References

[1] S. A. Bishop, "Continuous Dependence on Parameter and the Kurzweil Equation associated with Nonclassical Stochastic Differential Equation," ICASTOR J. of Mathematical Sciences, Vol. 5, No. 2, 255-265, 2011.

[2] G.O.S. Ekhaguere, "Lipschitzian quantum stochastic differential inclusions," Int. J. Theor. Phys., Vol. 31 pp. 2003-2034, 1992, DOI: 10.1007/BF0067196

[3] R. L. Hudson and K. R. Parthasarathy, "Quantum Ito's formula and stochastic evolutions," Comm. Math. Phys., Vol. 93, pp. 301-324, 1984, DOI: 10.1007/BF01258530

[4] A. M. A. El-Sayed, R. O. Abd-El-Rahman R.O.,and M. El-Gendy, "Continuous Dependence of the Solution of A Stochastic Differential Equation With Nonlocal Conditions," Journal of Statistics and Mathematical Sciences. 2016

[5] S. Itoh, "Random fixed point theorems with an application to random differential equations in banach spaces," J Math Anal. Appl. Vol. 6, pp. 261-273, 1979.

[6] S. Schwabik, "Generalized Ordinary Differential Equations," World Scientific, 1992