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Simulation Study of Failures in Progressively Loaded Multicomponent Systems

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Abstract—**Multicomponent systems are encountered in a variety of applications. Specifically, modern nanodevices are composed of a large number of almost identical parts that function as a unit. If such a system of N components, with components characterized by random load thresholds $\{q_i\}$, $1 \leq i \leq N$, is subjected to a load Q, that irreversibly breaks some weak components, then short sequences of components failures appear. If the load is applied progressively to the system, these initial sequences develop in avalanches of failures, consecutive numbers of functioning components decrease and the system is driven towards an edge of its functionality. This limiting state of the system is characterized by the critical load Q_c and the number $n_c < N$ of still-functioning components, whereas $Q > Q_c$ triggers an ultimate destruction of the system. We employ computer simulations to analyze distributions of Q_c and n_c. We show, that for a class of nanotechnological multicomponent systems, with q governed by a Weibull distribution $p_k(q) = kq^{k-1}\exp(-q^k)$, where k is the shape parameter, the ratio Q_c/n_c is distributed according to a skew-normal distribution, whereas n_c/N is normally distributed with mean $\overline{n}_c$ and variance σ scaled as $(1 - \overline{n}_c/N) \sim 1/k^{7/4}$ and $\sigma/N \sim 1/k$, respectively.**

Index Terms—**avalanches, critical load, evolving failure, multicomponent system, probability distribution.**

I. Introduction

MULTICOMPONENT systems posses a large number of identical components that perform a common task. A possible sequence of failures among these components decreases the device performance and may eventually lead to a catastrophic avalanche of failures. This is because, once the system is subjected to an increasing load it begins to fail immediately when the internal load intensity equals or exceeds the critical value of weakest components and the failure develops in a form of avalanches of simultaneously damaged elements. More specifically, avalanches appear when an increasing load eliminates an element from the working community in such a way that this exclusion alters the internal load pattern sufficiently to trigger the failure of the other elements and, in consequence, provoking a wave of destruction. A common approach to study avalanches of failures is to apply so-called load transfer models. Among them, the Fibre Bundle Models (FBM) and Random Fuse Models [1], [2], [3] are frequently employed in problems related to technological applications.

Our system is a grid of components represented by a collection of components located at nodes of a square lattice, see Fig. 1, and then analyzed within a Fibre Bundle Model framework [4], [5], [6], [7], [8], [9]. We restrict our analysis to the case where each component is characterized by two states: working or failed. We also assume that failed components are not repairable. In our simulations, an ensemble of N components is subjected to a growing load Q, that systematically eliminates week components and involves avalanches of failures. This means that when a component breaks, its load is transferred to the other intact elements and thus the probability of subsequent failures increases. The rule of load transfer is a crucial ingredient of the model. Among many different rules there are two extreme ones: global (equal) load sharing and local load sharing (LLS)[10], [11], [12].

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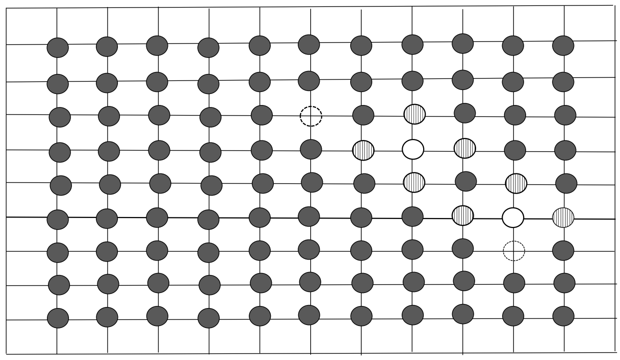

Fig. 1. Schematic view of a multicomponent system. Disks represent components: black disks – intact components, open circles – destroyed components, white discs – just damaged components with their loads transferred to nearest components marked by patterned disks.

Components' imperfections have impact on the behavior of systems under load. Due to these imperfections, components' yields are nonuniform and multiple component-failure modes are represented by the component-load-thresholds. In simulations these load-thresholds are modeled by quenched random variables. The two most frequently employed load-thresholds distributions are uniform and Weibull distributions. The former one is especially well situated in the context of engineering systems [13], [14], [15].

II. Computation method

During the loading process, sequences of simultaneous ruptures of several components take place. In order to handle the load partitioning into groups of working components we employ the LLS transfer rule. Within a short interval between consecutive failures the load carried by the destroyed component is transferred only to its closest intact neighbors. The LLS rule is schematically depicted in Fig. (1) where white disks represent destroyed components and their closest neighbors are marked by patterned disks. It is seen that numbers of nearest intact neighbors vary during the loading process. Because of such a limited-range-load-transfer, the distribution of intrinsic load is not homogeneous giving rise to appearance of regions of load accumulation throughout the entire system. The increasing load imposed on the intact components leads to other failures, after which each intact component bears growing load. If the load transfer does not trigger further failures, a stable configuration emerges meaning that this present value of Q is not sufficient to provoke destruction of the entire system, and its value may increase. In the simulations we applied a quasi-static loading procedure, i.e. if the system is in a stable state the external load increases uniformly on all the intact components by an amount δQ sufficient to break the weakest-working component and then the increase of load stops until a new stable state emerges. A series of increases in the value of the external load gives Q_c such that Q_c corresponds to a stable state of the system whereas $Q_{min} = Q_c + \delta Q$ induces an avalanche of failures among all still working components. Application of quasi-static loading allows us to obtain minimal load Q_{min} necessary for destruction of all the components in the system and thus yields Q_c and n_c that characterize the system on the edge of its functionality.

In our simulation, component-load-thresholds $\{q_i\}, i = 1, \ldots, N$ are independent random variables governed by the Weibull distributions [13], [16]. The probability density function of this distribution is given by

$$p_k(q) = kq^{k-1}\exp(-q^k), \tag{1}$$

where the shape parameter $k > 0$ controls the amount of disorder in the system. The Weibull distribution, in its general form, involves a second parameter λ, the so-called scale parameter. Since this λ scales q and $p_k(q)$, respectively as $\tilde{q}/\lambda$ and $\tilde{p}_k(\tilde{q})/\lambda$, we assume $\lambda = 1$ through all our simulations.

An interesting question is how do component-load-thresholds $\{q\}$, distributed according to (1), combine to yield such effective quantities as e.g. critical load Q_c or limiting number of working components n_c. Along with these global characteristics, also some local quantities should be analyzed. Among them, the ratio Q_c/n_c represents a particular interest because it reflects an effective-local-load intensity. In this context, we concentrate on statistical properties of n_c and Q_c/n_c. Distributions and estimators related to Q_c have been reported in [17].

Based on our numerical simulations, we have found that coefficient of skewness of distribution of Q_c/n_c decreases with the system size and takes negative values for systems with $N > 50 \times 50$. It turns out that our skewed data are nicely fitted by a three-parameter skew-normal distribution (SND) [18], [19] defined by

$$SND(x) = \frac{\operatorname{erfc}\left(-\alpha\frac{x-\mu}{\sqrt{2}\sigma}\right)}{\sqrt{2\pi}\sigma}\exp\left[-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2\right] \tag{2}$$

where: μ, σ and α are respectively: location, scale and shape parameters of the SND.

III. Results and discussion

Employing the LLS transfer rule, we simulated the loading process in a two-dimensional square grid of components with a number of components ranging from $N = 50 \times 50$ to $N = 100 \times 100$. We have tuned the amount of component-load-threshold disorder by integer values of k ranging from 2 to 9 and each simulation was repeated at least 10^4 times. Within such a scenario we have collected large data sets involving detailed information about loads (Q) and corresponding numbers of destroyed components (n). Based on these results we have determined statistics relating the both, critical load Q_c and critical number of intact components n_c, along with such empirical estimators as the mean values and the standard deviations.

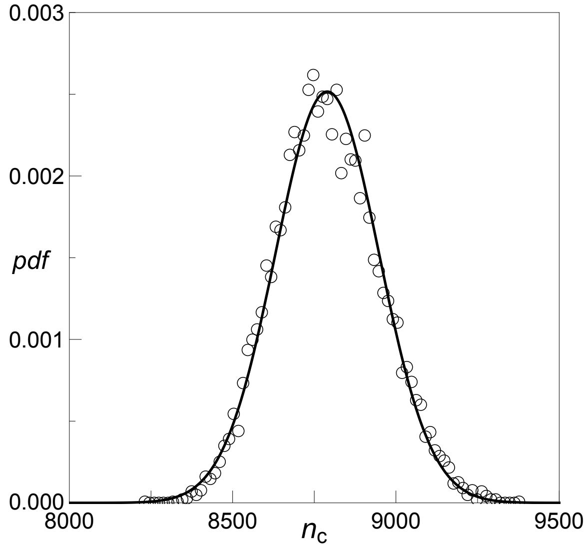

Fig. 2. Empirical probability density function (*pdf*) of n_c for systems with $N = 100^2$ components with component-load-thresholds taken from the Weibull distribution with $k = 2$. The solid lines represent normally distributed n_c with the parameters computed from the simulations. The results are obtained from 10^4 samples.

A. Distribution of critical number of component

Under the computation method described above we have gathered long records containing critical numbers of components n_c. Based on these records we we have studied the empirical probability density functions. Two of such empirical functions are presented in Fig. (2), for $k = 2$, and in Fig. (9b), for $k = 4$.

Analysis of all our experimental distributions of n_c enable us to fit these distributions by a normal distribution with a mean ($\tilde{\mu}$) and a variance ($\tilde{\sigma}$) that can be approximated by scaling relations: $\tilde{\mu}(N,k) \sim N\mu(k)$ and $\tilde{\sigma}(N,k) \sim N\sigma(k)$. It turns out that the scaled mean can be written as

$$\mu(k) = 1 - \frac{a(N)}{k^{7/4}}, \tag{3}$$

where the coefficient $a(N)$ depends on system size only and $0 < a(N) < 1$ for all $N > 50$. The scaling (3) is presented in Fig. (3) for systems with different number of components. The relative error $(\tilde{\mu}/\mu - 1)$ of this approximation lies in the interval $(-0.002, 0.003)$.

In the same way we have fitted values of $\tilde{\sigma}$ by a function σ defined as:

$$\sigma(k) = \frac{b(N)}{k}, \tag{4}$$

where $0 < b(N) < 1$ for $N >> 1$. The computed standard deviation $\tilde{\sigma}$ and the scaling (4) are displayed in the inset of Fig. (3). The relative error $(\tilde{\sigma}/\sigma - 1)$ of this approximation lies in the interval $(-0.034, 0.025)$ for all simulated systems.

It is worth mentioning that for systems with component-load-threshold uniformly distributed over a segment $[0, 1]$ and LLS transfer rule, the critical number of components is also normally distributed [20].

B. *Distribution of ratio of* Q_c/n_c

Prior to destruction of the system, the applied load attains its maximal value Q_c, i.e. it is the maximal load that can be carried by the system. In the same time the system contains a minimal number of components supporting Q_c. This means that Q_c/n_c represents an average intensity of imposed load. In a case when all intact components equally share a load transferred from destroyed components, the load Q_c is composed from values of load-thresholds of the weakest components. However, within the LLS rule, that we consider in this work, only components that are neighbors of a failure suffer from an extra load. This means that the intensity of imposed load is not uniform and the set of eliminated components does not involve the weakest components only. A closer look at collected sets of Q_c and n_c yields that they are strongly anti correlated, see Fig. (4). The Pearson coefficient (r) computed from their distributions has values $r \in (-0.96, -0.88)$ for all collected data. In Fig. (5) we present an example of an experimental joint probability distribution built by assembling, sample by sample, the critical load Q_c with the number n_c of components working under Q_c.

We start our analysis by comparing values of Q_c/n_c collected from two groups of systems: (i) systems with growing number of components while the strength of disorder is kept constant, i.e. $N \neq const.$, $k = const.$ and (ii) the size of system is fixed whereas the strength of disorder varies. Figures (6) and (7) show empirical probability density functions of Q_c/n_c for systems representing (i) and (ii), respectively. In Fig. (6), that corresponds to the case (i), the maximum of Q_c/n_c is pushed left for a growing number of components. This is because for growing N the number of relatively week components also increases and this gives rise to a growing probability of subsequent failures. This is in contrast to the case (ii), presented in Fig. (7): increasing values of k reflect a decreasing variance of component-load-thresholds and, in consequence, systems with higher values of Q_c/n_c.

A careful analysis of data presented in Figs. (6) and (7) reveals that the experimental distributions of Q_c/n_c have statistical properties described by the SND (2). In these plots we have added fitting lines of skew normal probability density functions with parameters computed from the samples. We also present a quantile-quantile (Q-Q) plot of the quantiles related to one of the collected data set against the corresponding quantiles given by the SND. As it is seen in Fig. (8), the points closely follow the straight line which indicates that the set of empirical data comes from the population with underlying skew normal probability distribution. Beside the fact, that we display this Q-Q plot only for an estimate purpose, we have examined our simulated data sets using different goodness of fit tests. We have also estimated values

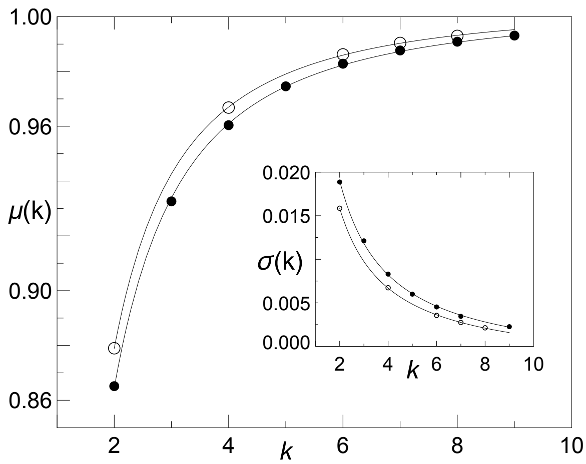

Fig. 3. Mean value $\mu(k)$ of scaled critical number of components $= n_c/N$ as a function of the Weibull shape parameter k. Systems with $N = 100 \times 100$ components - open disks, systems with 60×60 components - filled disks. The solid lines are drawn using (3). The inset presents variance $\sigma(k)$ of n_c/N for the same systems. Solid lines are defined by (4). The results are obtained from 10^4 samples.

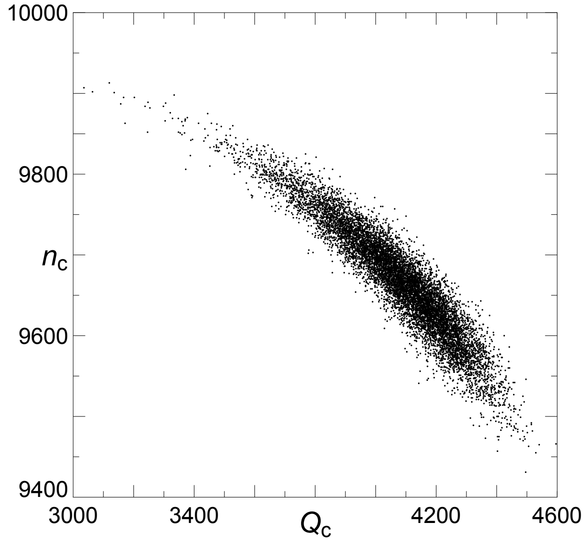

Fig. 4. Critical number of components n_c vs. critical load Q_c for systems with 100^2 components and load-thresholds drawn from the Weibull distribution with $k = 4$. Sample size is 10^4.

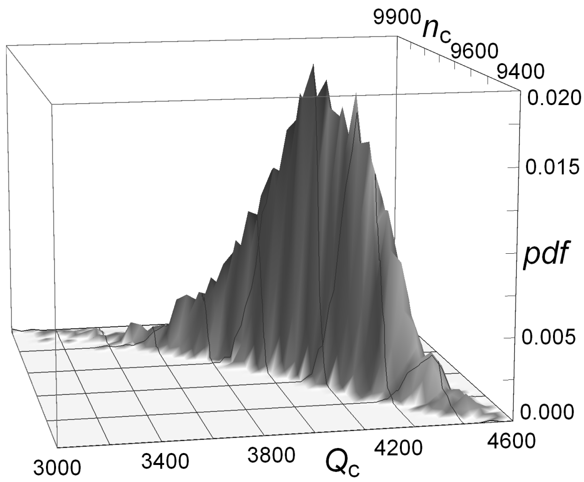

Fig. 5. Empirical joint probability density function of Q_c and n_c for systems with 100^2 components and load-thresholds drawn from the Weibull distribution with $k = 4$. Sample size is 10^4.

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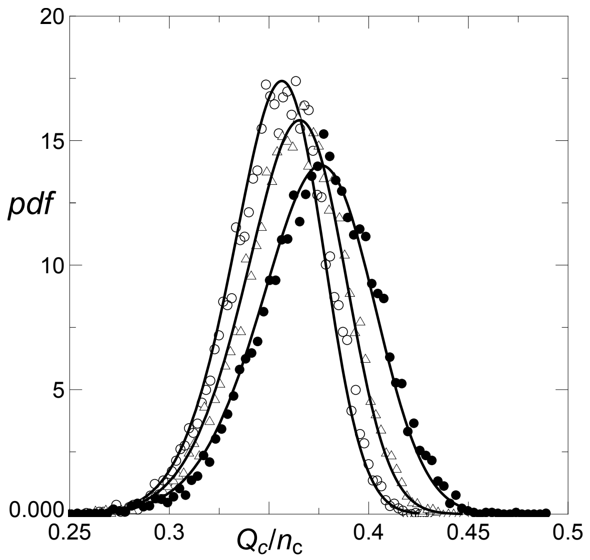

Fig. 6. Empirical probability density functions (*pdf*) of Q_c/n_c for systems with $N = 100^2$ (open circles), $N = 80^2$ (triangles) and $N = 60^2$ (filled circles) components. Component-load-thresholds are governed by the Weibull distribution with $k = 2$ for all presented systems. The solid lines represent skew-normally distributed Q_c/n_c with the parameters computed from the simulations. The results are obtained from at least 10^4 samples for each value of N.

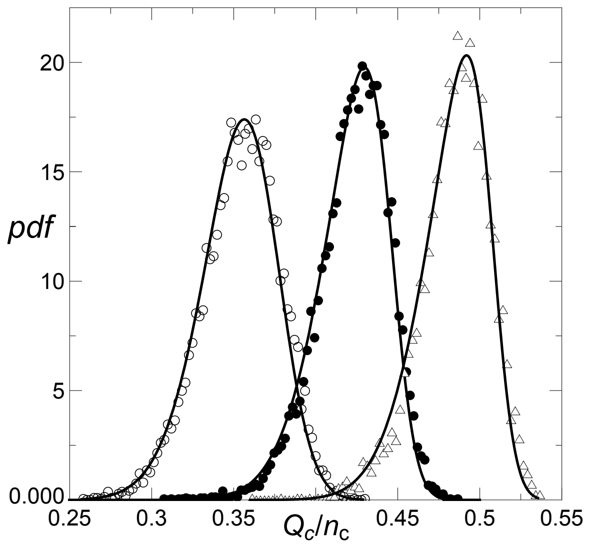

Fig. 7. Empirical probability density functions (*pdf*) of Q_c/n_c for systems with 100^2 components and load-thresholds drawn from the Weibull distribution: $k = 2$ (open circles), $k = 4$ (filled circles) and $k = 6$ (triangles). The solid lines represent skew-normally distributed Q_c/n_c with the parameters computed from the simulations. The results are obtained from 10^4 samples for each value of k.

of the location, scale and shape parameters of the SND by employing the maximum likelihood procedure.

We finalize our analysis of quantities, that represent systems on their edge of functionality, with an example of the system whose 100×100 component-load-thresholds are characterized by the Weibull shape parameter $k = 4$. This system is sufficiently large, with still moderate disorder, to be representative for multicomponent systems studied in this work. In Fig. (9) we present experimental distributions of quantities collected during simulations carried out with this system, i.e. distributions of: $n_c, Q_c, Q_c/n_c$ as well as Q_c

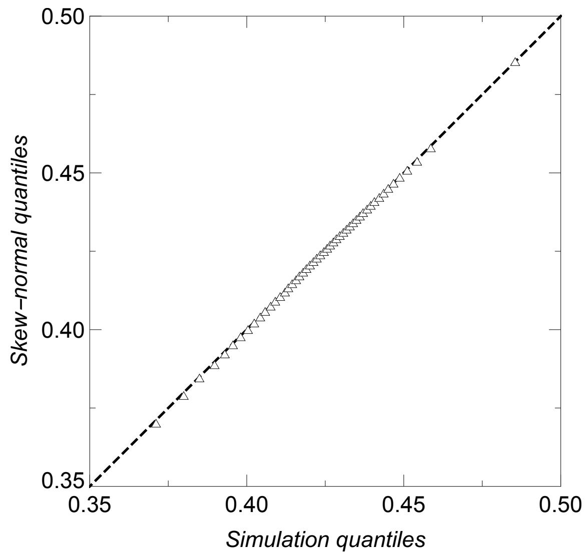

Fig. 8. Quantile-Quantile plot of the quantiles of the set of computed Q_c/n_c vs. the quantiles of the skew normal probability distribution for systems with 100^2 components and load-thresholds drawn from the Weibull distribution with $k = 4$. Sample size is 10^4.

vs. n_c. We have already mentioned that the experimental distribution of Q_c, presented in Fig. (9a) can be fitted correctly by the Weibull distribution [17]. Also Figs. (4) and (5) are related to this system, namely they show the experimental joint probability distribution of Q_c and n_c.

C. Conclusion

We carried out simulations of progressively loaded multicomponent systems. We considered components placed in nodes of a square lattice and characterized by quench random components-load-thresholds drawn from the Weibull probability distribution. Based on the presented results of simulation study we conclude that the experimental distributions of the critical number of components n_c as well as the local-load intensity Q_c/n_c can be effectively estimated. By fitting discrete distributions, we have found that on the edge of the system functionality: (i) the ratio Q_c/n_c is skew-normally distributed, (ii) the number of intact components is normally distributed and (iii) for $N >> 1$, the mean and variance of normally distributed n_c/N scale like $(1 - \mu/N) \sim 1/k^{7/4}$ and $\sigma \sim 1/k$, respectively.

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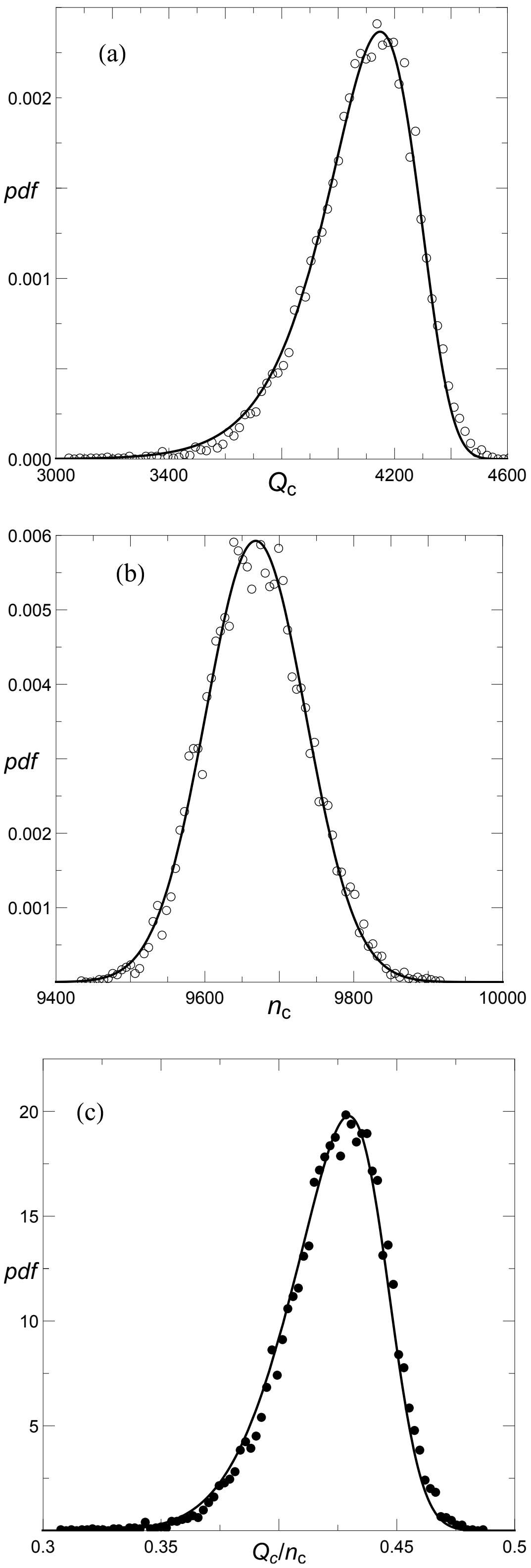

Fig. 9. Empirical probability density functions (*pdf*): (a) critical load Q_c, (b) critical number of components n_c and (c) ratio Q_c/n_c. System involves $N = 100^2$ components and component-load-thresholds are taken from the Weibull distribution with $k = 4$. The solid lines represent fitted distributions: (a) Weibull, (b) normal (c) skew-normal, with the parameters computed from the simulations. The results are obtained from 10^4 samples.

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