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Randomly r-Orthogonal (g, f)-Factorizations in Networks

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***Abstract*—Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. Let $g, f : V(G) \to Z$ be two functions with $f(x) \geq g(x) \geq kr$ for each $x \in V(G)$. Let $H_1, H_2, \cdots, H_k$ be k vertex disjoint mr-subgraphs of G. In this paper, it is verified that every $(mg + (m-1)r, mf - (m-1)r)$-graph contains a (g, f)-factorization randomly r-orthogonal to each H_i $(1 \leq i \leq k)$.**

***Index Terms*—network; graph; (g, f)-factor; r-orthogonal factorization.**

I. Introduction

MANY physical structures can conveniently be modelled by networks. Examples include a communication network with nodes and links modelling cities and communication channels, respectively, or a railroad network with nodes and links representing railroad stations and railways between two stations, respectively. Many problems on network design and optimization, e.g., the file transfer problems on computer networks, building blocks, coding design, scheduling problems and so on, are related to the factors, factorizations and orthogonal factorizations in graphs [1]. The file transfer problem can be modeled as $(0, f)$-factorizations (or f-colorings) in graphs [2]. The designs of Room squares and Latin squares are related to orthogonal factorizations in graphs[1]. It is well known that a network can be represented by a graph. Vertices and edges of the graph correspond to nodes and links between the nodes, respectively. Henceforth we use the term *graph* instead of *network*.

All graphs under consideration are finite, undirected, and simple. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. For every $x \in V(G)$, we use $d_G(x)$ to denote the degree of x in G. Let $g, f : V(G) \to Z$ be two functions with $0 \leq g(x) \leq f(x)$ for any $x \in V(G)$. A (g, f)-graph is a graph G satisfying $g(x) \leq d_G(x) \leq f(x)$ for arbitrary $x \in V(G)$. A spanning subgraph F of a graph G is said to be a (g, f)-factor of G if F itself is a (g, f)-graph. Let m and r be two positive integers. A (g, f)-factorization $\mathcal{F}= \{F_1, F_2, \cdots, F_m\}$ of a graph G is a partition of $E(G)$ into edge-disjoint (g, f)-factors $F_1, F_2, \cdots, F_m$. A subgraph with m edges is said to be an m-subgraph. Let H be an mr-subgraph of a graph G. A (g, f)-factorization $\mathcal{F}= \{F_1, F_2, \cdots, F_m\}$ is said to be r-orthogonal to H if every F_i $(1 \leq i \leq m)$ admits exactly r edges in common with H. A (g, f)-factorization $\mathcal{F}= \{F_1, F_2, \cdots, F_m\}$ of G is called randomly r-orthogonal to H if $A_i \subseteq E(F_i)$ for arbitrary partition $\{A_1, A_2, \cdots, A_m\}$ of $E(H)$ satisfying $|A_i| = r$, $1 \leq i \leq m$. It is obvious that randomly 1-orthogonal is equivalent to 1-orthogonal and 1-orthogonal is also said to be orthogonal.

Alspach et al. [1] put forward the following problem: For a given subgraph H of a graph G, does there exist a factorization $\mathcal{F}$ of a graph G with a given property orthogonal to H?

Zhou et al [3], [4], [5] discussed the existence of graph factors. Research on orthogonal factorizations attracted much attention due to their applications in combinatorial designs, network design, circuit layout and so on [1]. Feng [6] justified that every $(0, mf - m + 1)$-graph has a $(0, f)$-factorization orthogonal to any given m-subgraph. Liu [7] verified that every $(mg + m - 1, mf - m + 1)$-graph has a (g, f)-factorization orthogonal to a matching with m edges. Li and Liu [8] justified that every $(mg+m-1, mf-m+1)$-graph G has a (g, f)-factorization orthogonal to any given m-subgraph. Lam, Liu, Li and Shiu [9] showed that every $(mg+m-1, mf-m+1)$-graph admits a a (g, f)-factorization orthogonal to k vertex-disjoint m-subgraphs $H_1, H_2, \cdots, H_k$. Zhou, Xu and Xu [10] showed the existence of $(0, f)$-factorizations randomly r-orthogonal to any given k vertex disjoint mr-subgraphs in $(0, mf-(m-1)r)$-graphs. Liu and Long [11] proved that every $(mg + m - 1, mf - m + 1)$-graph G has a (g, f)-factorization randomly r-orthogonal to any given mr-subgraph. Many authors studied the orthogonal factorizations in digraphs [12], [13], [14], [15].

In the following, we consider the more general problem: For k given vertex-disjoint mr-subgraphs $H_1, H_2, \cdots, H_k$ of G, does there exist a factorization $\mathcal{F}$ randomly r-orthogonal to every H_i $(i = 1, 2, \cdots, k)$?

We show that this problem above is true by the following theorem which is our main result in this paper.

Theorem 1. Let G be an $(mg + (m-1)r, mf - (m-1)r)$-graph, and let $H_1, H_2, \cdots, H_k$ be k vertex disjoint mr-subgraphs of G, where m, k, r are three positive integers and $g, f : V(G) \to Z$ are two functions with $f(x) \geq g(x) \geq kr$ for any $x \in V(G)$. Then G admits a (g, f)-factorization randomly r-orthogonal to each H_i, $1 \leq i \leq k$.

II. Preliminary lemmas

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let S be a subset of $V(G)$ and A be a subset of $E(G)$. Then we use $G - S$ and $G - A$ to denote the subgraphs induced by $V(G) \setminus S$ and $E(G) \setminus A$, respectively. For any function φ defined on $V(G)$ and $X \subseteq V(G)$, set $\varphi(X) = \sum_{x \in X} \varphi(x)$ and $\varphi(\emptyset) = 0$.

For two disjoint vertex subsets S and T of G, we use $E_G(S, T)$ to denote the set of edges in G with one end in S

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and the other in T, and write $e_G(S,T)$ for $|E_G(S,T)|$. Let S and T be two disjoint vertex subsets of G, and let E_1 and E_2 be two disjoint edge subsets of G. Set

$$U = V(G) \setminus (S \cup T), \quad E(S) = \{xy \in E(G) : x, y \in S\}$$

and

$$E(T) = \{xy \in E(G) : x, y \in T\},$$

Define

$$E_1' = E_1 \cap E(S), \qquad E_1'' = E_1 \cap E_G(S,U),$$

$$E_2' = E_2 \cap E(T), \qquad E_2'' = E_2 \cap E_G(T,U),$$

$$\alpha(S,T;E_1,E_2) = 2|E_1'| + |E_1''|,$$

$$\beta(S,T;E_1,E_2) = 2|E_2'| + |E_2''|$$

and

$$\Delta(S,T;E_1,E_2) = \alpha(S,T;E_1,E_2) + \beta(S,T;E_1,E_2).$$

Under without ambiguity, we use α, β and Δ to denote $\alpha(S,T;E_1,E_2)$, $\beta(S,T;E_1,E_2)$ and $\Delta(S,T;E_1,E_2)$, respectively.

The following lemma is very useful for proving Theorem 1.

Lemma 2.1 (Li and Liu [8]). Let G be a graph, and $g, f : V(G) \to Z$ be two functions defined on $V(G)$ with $0 \le g(x) < f(x) \le d_G(x)$ for any $x \in V(G)$. Let E_1 and E_2 be two disjoint edge subsets of G. Then G contains a (g,f)-factor F with $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \emptyset$ if and only if

$$\delta_G(S,T;g,f) = f(S) + d_{G-S}(T) - g(T) \ge \Delta(S,T;E_1,E_2)$$

for any two disjoint vertex subsets S and T of G.

In the following, we always assume that G is a $(mg+(m-1)r, mf-(m-1)r)$-graph, where m, r are two integers with $m \ge 1$ and $r \ge 1$. Define

$$p(x) = \max\{g(x), d_G(x) - (m-1)f(x) + (m-2)r\}$$

and

$$q(x) = \min\{f(x), d_G(x) - (m-1)g(x) - (m-2)r\}$$

for any $x \in V(G)$. In view of the definitions of $p(x)$ and $q(x)$, it is easy to see that

$$g(x) \le p(x) < q(x) \le f(x)$$

for any $x \in V(G)$. We write

$$\Delta_1(x) = \frac{1}{m} d_G(x) - p(x)$$

and

$$\Delta_2(x) = q(x) - \frac{1}{m} d_G(x)$$

for any $x \in V(G)$.

Lemma 2.2. For any $x \in V(G)$ and $m \ge 2$, we obtain

$$\Delta_1(x) \ge \begin{cases} \frac{r}{m}, & if\ p(x) > g(x)\ and\ d_G(x) \ge mf(x) \\ & -(m-1)r - r + 1, \\ \frac{(m-1)r}{m}, & otherwise. \end{cases}$$

and

$$\Delta_2(x) \ge \begin{cases} \frac{r}{m}, & if\ q(x) < f(x)\ and\ d_G(x) \le mg(x) \\ & +(m-1)r + r - 1, \\ \frac{(m-1)r}{m}, & otherwise. \end{cases}$$

Proof. If $p(x) = g(x)$, then we have

$$\begin{aligned} \Delta_1(x) &= \frac{1}{m} d_G(x) - p(x) = \frac{1}{m} d_G(x) - g(x) \\ &\ge \frac{1}{m}(mg(x) + (m-1)r) - g(x) \\ &= \frac{(m-1)r}{m}. \end{aligned}$$

Otherwise, it follows from the definition of $p(x)$ that

$$p(x) = d_G(x) - (m-1)f(x) + (m-2)r.$$

If $d_G(x) \le mf(x) - (m-1)r - r$, then we obtain

$$\begin{aligned} \Delta_1(x) &= \frac{1}{m} d_G(x) - p(x) \\ &= \frac{1}{m} d_G(x) - (d_G(x) - (m-1)f(x) \\ &\quad +(m-2)r) \\ &\ge \frac{1-m}{m}(mf(x) - (m-1)r - r) \\ &\quad +(m-1)f(x) - (m-2)r \\ &= r > \frac{(m-1)r}{m}. \end{aligned}$$

Otherwise, we have $mf(x) - (m-1)r \ge d_G(x) \ge mf(x) - (m-1)r - r + 1$. Thus, we have

$$\begin{aligned} \Delta_1(x) &= \frac{1}{m} d_G(x) - p(x) \\ &= \frac{1}{m} d_G(x) - (d_G(x) - (m-1)f(x) \\ &\quad +(m-2)r) \\ &\ge \frac{1-m}{m}(mf(x) - (m-1)r) \\ &\quad +(m-1)f(x) - (m-2)r \\ &= \frac{r}{m}. \end{aligned}$$

So we prove the first inequality. Similarly, we can verify the second inequality. This completes the proof of Lemma 2.2.

Lemma 2.3 (Li and Liu [8]). For arbitrary two disjoint vertex subsets S and T of G, the following equality holds

$$\begin{aligned} \delta_G(S,T;p,q) &= \Delta_1(T) + \Delta_2(S) + \frac{m-1}{m} d_{G-S}(T) \\ &\quad + \frac{1}{m} d_{G-T}(S). \end{aligned}$$

III. The proofs of main results

Let G be a graph, and let g and f be two integer-valued functions defined on $V(G)$ satisfying $f(x) > g(x) \ge kr$ for each $x \in V(G)$. Let $H_1, H_2, \cdots, H_k$ be k vertex-disjoint mr-subgraphs of G. For $i = 1, 2, \cdots, k$, we set

$$A_{i1} = \{xy \in E(H_i) : p(x) \ge g(x)+1\ and\ p(y) \ge g(y)+1\},$$

$$A_{i2} = \{xy \in E(H_i) : p(x) \ge g(x)+1\ or\ p(y) \ge g(y)+1\}$$

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and

$$A_i = \begin{cases} A_{i1}, & if\ A_{i1} \neq \emptyset, \\ A_{i2}, & if\ A_{i1} = \emptyset\ and\ A_{i2} \neq \emptyset, \\ E(H_i), & otherwise. \end{cases}$$

Choose any $Q_i \subseteq A_i$ with $|Q_i| = r$ for $1 \le i \le k$. We write $E_1 = \bigcup_{i=1}^{k} Q_i$ and $E_2 = (\bigcup_{i=1}^{k} E(H_i)) \setminus E_1$. Thus, we obtain $|E_1| = kr$ and $|E_2| = (m-1)kr$. For any two disjoint subsets $S, T \subseteq V(G)$, $E_1', E_1'', E_2', E_2'', \alpha$, β and Δ are defined as in Section 2. It follows from the definitions of α and β that

$$\alpha \le \min\{2kr, r|S|\}$$

and

$$\beta \le \min\{2(m-1)kr, (m-1)r|T|\}.$$

The definitions of $p(x), q(x), \Delta_1(x)$ and $\Delta_2(x)$ are identical to that in Section 2. In order to prove Theorem 1, we first justify the following lemma which plays a crucial role for proving Theorem 1.

Lemma 3.1. Let G be a $(mg + (m-1)r, mf - (m-1)r)$-graph, where $m \ge 2$. Then G admits a (p, q)-factor F_1 with $E_1 \subseteq E(F_1)$ and $E_2 \cap E(F_1) = \emptyset$.

Proof. According to Lemma 2.1, it suffices to show that

$$\delta_G(S, T; p, q) \ge \alpha + \beta$$

for any two disjoint vertex subsets S and T of G.

If $S = \emptyset$, then $\alpha = 0$. In terms of Lemma 2.2, Lemma 2.3 and the condition $g(x) \ge kr$ for any $x \in V(G)$, we obtain

$$\begin{aligned} \delta_G(S, T; p, q) &= \Delta_1(T) + \frac{m-1}{m} d_G(T) \\ &\ge \frac{r|T|}{m} + \frac{m-1}{m}(mg(T) + (m-1)r|T|) \\ &\ge (m-1)r|T| \ge \beta = \alpha + \beta. \end{aligned}$$

If $T = \emptyset$, then $\beta = 0$. In view of Lemma 2.2, Lemma 2.3 and the condition $g(x) \ge kr$ for any $x \in V(G)$, we have

$$\begin{aligned} \delta_G(S, T; p, q) &= \Delta_2(S) + \frac{1}{m} d_G(S) \\ &\ge \frac{r|S|}{m} + \frac{1}{m}(mg(S) + (m-1)r|S|) \\ &\ge r|S| \ge \alpha = \alpha + \beta. \end{aligned}$$

Hence, we may assume that $S \neq \emptyset$ and $T \neq \emptyset$. In the following, let us consider two cases.

Case 1. $|T| \le mk - 1$.

Subcase 1.1. $|S| \le mk - 1$.

According to Lemma 2.2, Lemma 2.3 and the condition $g(x) \ge kr$ for any $x \in V(G)$, we obtain

$$\begin{aligned} \delta_G(S, T; p, q) &\ge \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S) \\ &\ge \frac{m-1}{m}(mg(T) + (m-1)r|T| - |S||T|) \\ &\quad + \frac{1}{m}(mg(S) + (m-1)r|S| - |T||S|) \\ &\ge \frac{m-1}{m}(mkr + (m-1)r - |S|)|T| \\ &\quad + \frac{1}{m}(mkr + (m-1)r - |T|)|S| \\ &\ge \frac{m-1}{m}(mkr + (m-1)r - mk + 1)|T| \\ &\quad + \frac{1}{m}(mkr + (m-1)r - mk + 1)|S| \\ &= \frac{m-1}{m}(mkr - mk + (m-1)r + 1)|T| \\ &\quad + \frac{1}{m}(mkr - mk + (m-1)r + 1)|S| \\ &\ge \frac{m-1}{m}(mr - m + (m-1) + 1)|T| \\ &\quad + \frac{1}{m}(mr - m + (m-1) + 1)|S| \\ &= (m-1)r|T| + r|S| \ge \alpha + \beta. \end{aligned}$$

Subcase 1.2. $|S| \ge mk$.

Note that $d_{G-S}(T) \ge \beta$. It follows from Lemma 2.2, Lemma 2.3, $m \ge 2$ and the condition $g(x) \ge kr$ for any $x \in V(G)$ that

$$\begin{aligned} \delta_G(S, T; p, q) &\ge \Delta_2(S) + \frac{m-1}{m} d_{G-S}(T) \\ &\quad + \frac{1}{m} d_{G-T}(S) \\ &\ge \frac{r|S|}{m} + \frac{(m-1)\beta}{m} + \frac{1}{m}(mg(S) \\ &\quad + (m-1)r|S| - |T||S|) \\ &\ge \frac{r|S|}{m} + \frac{(m-1)\beta}{m} + \frac{1}{m}(mkr \\ &\quad + (m-1)r - |T|)|S| \\ &\ge \frac{r|S|}{m} + \frac{(m-1)\beta}{m} + \frac{1}{m}(mkr \\ &\quad + (m-1)r - mk + 1)|S| \\ &\ge kr + \frac{(m-1)\beta}{m} + \frac{1}{m}(mkr - mk \\ &\quad + (m-1)r + 1)mk \\ &\ge kr + \frac{(m-1)\beta}{m} + (mr - m \\ &\quad + (m-1) + 1)k \\ &= 2kr + \frac{(m-1)\beta}{m} + (m-1)kr \\ &\ge \alpha + \frac{(m-1)\beta}{m} + \frac{2(m-1)kr}{m} \\ &\ge \alpha + \frac{(m-1)\beta}{m} + \frac{\beta}{m} = \alpha + \beta. \end{aligned}$$

Case 2. $|T| \ge mk$.

Note that $d_{G-T}(S) \ge \alpha$ and $d_{G-S}(T) \ge \beta$. Now we choose $v \in T$ with $d_{G[E_2]}(v) = \min\{d_{G[E_2]}(x) : x \in T\}$. Combining this with $|T| \ge mk$ and $|E_2| = (m-1)kr$, we

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have

$$\begin{aligned}|S|+d_{G-S}(T) &= |S|+d_{G-S}(v)+d_{G-S}(T\setminus\{v\})\\ &\geq d_G(v)+\beta-\frac{2(m-1)kr}{mk}\\ &= d_G(v)+\beta-\frac{2(m-1)r}{m} \qquad (1)\end{aligned}$$

Observe that

$$|T|+d_{G-T}(S)\geq d_G(u)+\alpha-r \qquad (2)$$

for some $u\in S$.

Set

$$S_0=\{x\in S: q(x)=f(x) \text{ or } d_G(x)\geq mg(x)+(m-1)r+r\},$$

$$S_1=S\setminus S_0,$$

and

$$T_0=\{x\in T: p(x)=g(x) \text{ or } d_G(x)\leq mf(x)-(m-1)r-r\},$$

$$T_1=T\setminus T_0.$$

Furthermore, we set

$$S_0'=\{x\in S_0: p(x)\geq g(x)+1\},\quad S_1'=S\setminus S_0',$$

and

$$T_0'=\{x\in T_0: p(x)=g(x)\},\quad T_1'=T\setminus T_0'.$$

It is obvious that $p(x)=g(x)$ for each $x\in S_1'\cup T_0'$ and $p(x)\geq g(x)+1$ for each $x\in S_0'\cup T_1'$.

Subcase 2.1. $|S_0'|+|T_0'|\geq 2(k-1)$.

Note that $S\neq\emptyset$ and $T\neq\emptyset$. According to Lemma 2.2, Lemma 2.3, $m\geq 2$, (1), (2) and the condition $g(x)\geq kr$ for any $x\in V(G)$, we obtain

$$\begin{aligned}\delta_G(S,T;p,q) &\geq \Delta_1(T)+\Delta_2(S)+\frac{m-1}{m}d_{G-S}(T)\\ &\quad+\frac{1}{m}d_{G-T}(S)\\ &\geq \frac{r|T_1'|+(m-1)r|T_0'|}{m}\\ &\quad+\frac{r|S_1'|+(m-1)r|S_0'|}{m}\\ &\quad+\frac{m-1}{m}d_{G-S}(T)+\frac{1}{m}d_{G-T}(S)\\ &= \frac{r|T|+(m-2)r|T_0'|}{m}\\ &\quad+\frac{r|S|+(m-2)r|S_0'|}{m}\\ &\quad+\frac{m-1}{m}d_{G-S}(T)+\frac{1}{m}d_{G-T}(S)\\ &= \frac{(r-1)|S|+(r-1)|T|}{m}\\ &\quad+\frac{(m-2)r(|S_0'|+|T_0'|)}{m}\\ &\quad+\frac{|T|+d_{G-T}(S)}{m}\\ &\quad+\frac{|S|+d_{G-S}(T)}{m}+\frac{m-2}{m}d_{G-S}(T)\\ &\geq \frac{2(r-1)+2(m-2)(k-1)r}{m}\\ &\quad+\frac{d_G(u)+\alpha-r}{m}\\ &\quad+\frac{d_G(v)+\beta-\frac{2(m-1)r}{m}}{m}+\frac{(m-2)\beta}{m}\\ &> \frac{2(m-2)(k-1)r+mg(u)}{m}\\ &\quad+\frac{(m-1)r+\alpha-r+mg(v)}{m}\\ &\quad+\frac{(m-1)r+(m-1)\beta-2}{m}\\ &\geq \frac{2(m-2)kr+2mkr+\alpha}{m}\\ &\quad+\frac{(m-1)\beta+r-2}{m}\\ &= \frac{2(m-1)kr+2(m-1)kr+\alpha}{m}\\ &\quad+\frac{(m-1)\beta+r-2}{m}\\ &\geq \frac{(m-1)\alpha+\beta+\alpha+(m-1)\beta-1}{m}\\ &> \alpha+\beta-1.\end{aligned}$$

Since $\delta_G(S,T;p,q)$ is an integer, we obtain

$$\delta_G(S,T;p,q)\geq\alpha+\beta.$$

Subcase 2.2. $|S_0'|+|T_0'|<2(k-1)$.

We write $n=|V(E_1)\cap T_1'|$. Then we obtain

$$|V(E_1)\cap(S_0'\cup T_1')|\leq|S_0'|+n. \qquad (3)$$

Let $Q_i\subseteq E_1\cap E(H_i)$, where $V(Q_i)\subseteq S\cup T$, $|Q_i|=r$ and $|V(Q_i)|=q_i$ ($2\leq q_i\leq 2r$). According to the choice of E_1, if $V(Q_i)\cap(S_0'\cup T_1')=\emptyset$, then $V(H_i)\cap T_1'=\emptyset$; if $|V(Q_i)\cap(S_0'\cup T_1')|=1$, then $|V(Q)\cap T_1'|\leq 1$ for any $Q\subseteq E(H_i)$ with $|Q|=r$ and $|V(Q)|=q_i$; if $|V(Q_i)\cap(S_0'\cup T_1')|=2$, then $|V(Q)\cap T_1'|\leq 2$ for any $Q\subseteq E(H_i)$ with $|Q|=r$ and $|V(Q)|=q_i$; ......; if $|V(Q_i)\cap(S_0'\cup T_1')|=q_i$, then $|V(Q)\cap T_1'|\leq q_i$ for any $Q\subseteq E(H_i)$ with $|Q|=r$ and $|V(Q)|=q_i$. We write $n'=|V(E_1)\cap U|$, where $U=V(G)\setminus(S\cup T)$. Combining these with (3) and the definition of $\beta(S,T)$, we have

$$\begin{aligned}\beta(S,T_1') &\leq |V(E_1)\cap(S_0'\cup T_1')|(m-1)r+n'(m-1)r\\ &\leq (|S_0'|+n+n')(m-1)r.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\beta(S,T) &= \beta(S,T_0')+\beta(S,T_1')\leq|T_0'|(m-1)r\\ &\quad+(|S_0'|+n+n')(m-1)r\\ &= (|S_0'|+|T_0'|+n+n')(m-1)r, \qquad (4)\end{aligned}$$

and

$$\alpha(S,T)\leq 2kr-(n+n')r. \qquad (5)$$

Note that $S\neq\emptyset$ and $T\neq\emptyset$. In terms of Lemma 2.2, Lemma 2.3, $m\geq 2$, (1), (2), (4), (5) and the condition $g(x)\geq$

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kr for any $x \in V(G)$, we obtain

$$\begin{aligned}
\delta_G(S,T;p,q) &\geq \Delta_1(T)+\Delta_2(S)+\frac{m-1}{m}d_{G-S}(T)\\
&\quad +\frac{1}{m}d_{G-T}(S)\\
&\geq \frac{r|T_1'|+(m-1)r|T_0'|}{m}\\
&\quad +\frac{r|S_1'|+(m-1)r|S_0'|}{m}\\
&\quad +\frac{m-1}{m}d_{G-S}(T)+\frac{1}{m}d_{G-T}(S)\\
&= \frac{r|T|+(m-2)r|T_0'|}{m}\\
&\quad +\frac{r|S|+(m-2)r|S_0'|}{m}\\
&\quad +\frac{m-1}{m}d_{G-S}(T)+\frac{1}{m}d_{G-T}(S)\\
&\geq \frac{(m-2)r(|S_0'|+|T_0'|)}{m}+\frac{|T|+d_{G-T}(S)}{m}\\
&\quad +\frac{|S|+d_{G-S}(T)}{m}+\frac{m-2}{m}d_{G-S}(T)\\
&\geq \frac{(m-2)r(|S_0'|+|T_0'|)}{m}+\frac{d_G(u)+\alpha-r}{m}\\
&\quad +\frac{d_G(v)+\beta-\frac{2(m-1)r}{m}}{m}+\frac{(m-2)\beta}{m}\\
&\geq \frac{(m-2)r(|S_0'|+|T_0'|)}{m}\\
&\quad +\frac{mg(u)+(m-1)r+\alpha-r}{m}\\
&\quad +\frac{mg(v)+(m-1)r+(m-1)\beta}{m}\\
&\quad -\frac{2(m-1)r}{m^2}\\
&\geq \frac{(m-2)r(|S_0'|+|T_0'|)+2mkr}{m}\\
&\quad +\frac{2(m-1)r+\alpha+(m-1)\beta-r}{m}\\
&\quad -\frac{2(m-1)r}{m^2}\\
&> \frac{(m-2)r(|S_0'|+|T_0'|)+2mkr+\alpha}{m}\\
&\quad +\frac{(m-1)\beta-2r}{m}\\
&= \frac{(m-2)r(|S_0'|+|T_0'|)+2(k-1)r}{m}\\
&\quad +\frac{2(m-1)kr+\alpha+(m-1)\beta}{m}\\
&> \frac{(m-1)r(|S_0'|+|T_0'|)}{m}\\
&\quad +\frac{(m-1)(\alpha+(n+n')r)+\alpha}{m}\\
&\quad +\frac{(m-1)\beta}{m}\\
&= \frac{(m-1)r(|S_0'|+|T_0'|+n+n')}{m}\\
&\quad +\frac{m\alpha+(m-1)\beta}{m}\\
&\geq \frac{m\alpha+m\beta}{m}=\alpha+\beta.
\end{aligned}$$

This completes the proof of Lemma 3.1.

Proof of Theorem 1. We apply induction on m. Theorem 1 holds for $m=1$. We assume that Theorem 1 is true for $m-1$, where $m \geq 2$. In view of Lemma 3.1, G admits a (p,q)-factor F_1 with $E_1 \subseteq E(F_1)$ and $E_2 \cap E(F_1)=\emptyset$. It is obvious that F_1 is also a (g,f)-factor of G. We write $G'=G-E(F_1)$. In terms of the definitions of $p(x)$ and $q(x)$, we have

$$\begin{aligned}
d_{G'}(x) &= d_G(x)-d_{F_1}(x)\geq d_G(x)-q(x)\\
&\geq d_G(x)-(d_G(x)-(m-1)g(x)-(m-2)r)\\
&= (m-1)g(x)+(m-2)r.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
d_{G'}(x) &= d_G(x)-d_{F_1}(x)\leq d_G(x)-p(x)\\
&\leq (m-1)f(x)-(m-2)r.
\end{aligned}$$

Therefore, G' is an $((m-1)g+(m-2)r,(m-1)f-(m-2)r)$-graph. Set $H_i'=H_i-E_1$, $i=1,2,\cdots,k$. According to the induction hypothesis, G' has a (g,f)-factorization $\mathcal{F}'=\{F_2,\cdots,F_m\}$ randomly r-orthogonal to each H_i', $1 \leq i \leq k$. Thus, G has a (g,f)-factorization $\mathcal{F}=\{F_1,F_2,\cdots,F_m\}$ randomly r-orthogonal to each H_i, $i=1,2,\cdots,k$. This completes the proof of Theorem 1.

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