Degenerate Parametric Integral Equations System for Laplace Equation and Its Effective Solving

Eugeniusz Zieniuk, Marta Kapturczak, and Andrzej Kużelewski

***Abstract*—In this paper we present application of degenerate kernels strategy to solve parametric integral equations system (PIES) for two-dimensional Laplace equation in order to improve its computing time. The main purpose of this paper is to obtain degenerate kernels for PIES based on non-degenerate kernels and to apply collocation method to solve modified PIES. We verify this method on two examples, which analytical solutions are known. The tests are mainly focused on computing time with respect to accuracy of obtained numerical solutions.**

***Index Terms*—boundary value problems, collocation method, parametric integral equation system, degenerate kernels**

I. Introduction

Differential equations are used as a tool for modeling different physical phenomena. Hence, there are a lot of methods of numerical solving of differential equations. The most popular methods are: finite element method (FEM) [1] and boundary element method (BEM) [5]. Both of these methods possess many advantages (among others: versatility of application, wide software availability), as well as some disadvantages. The main drawback is the necessity of the discretization of the domain (FEM) and the boundary (BEM). Despite the fact, that the complexity of solving BEM was reduced (only boundary was divided on elements), discretization process is still required. Furthermore, in special cases (mass forces, nonlinear problems, and other), there is also necessity of domain discretization in BEM, therefore the complexity of its solving is equal to FEM.

Considering above mentioned problems many researchers constantly looking for new methods of solving the boundary value problems or developing existing methods. We are developing the method called parametric integral equations system method (PIES) [3], [16], [15]. This method is an analytical modification of boundary integral equations (BIE). The fundamental aspect, that distinguish it from classical element methods, is that there is no need of discretization. It has been achieved by inclusion of the shape of boundary (defined using any parametric function) in mathematical formalism of PIES. Until now, we successfully applied the Bézier, Hermite or B-spline [7] curves for modeling the shape of the boundary. The advantage of these curves is the simplicity of their using for modeling and modifying the shape of the boundary. PIES method allow us to separate the approximation of the shape of boundary from approximation of boundary functions. It is very important advantage of PIES method. Therefore, the shape of boundary is effectively modeled by mentioned curves and it is independent of the process of solutions approximation. Then, we started looking for so-effective method for numerical solving of PIES. In previous studies we used and tested the collocation method [9]. We chose this method because of its simplicity connected with requirement of single integration only. However, there are also some disadvantages, such as asymmetric matrix in obtained system of algebraic equations or possible instability of solutions. Moreover, this method requires proper location of collocations points, which affects the accuracy of the results. Despite the fact, that we obtained satisfying results using collocation method for solving PIES, in case of more complex problems (especially 3D), we noticed growing time of computations. Therefore, we decided to develop own strategy of solving PIES and verify its effectiveness on fundamental boundary value problems.

The aim of this work is to develop the method of solving PIES based on degenerate kernels to reduce the time of computations. Firstly, in this paper, we present the strategy of obtaining degenerate kernels in PIES (for two-dimensional Laplace equation) based on non-degenerate (primary) kernels. We test the concept of the method on two examples, which analytical solutions are known. We compare solutions obtained using proposed method of degenerate kernels in PIES with ones from conventional PIES. We check accuracy of the solutions (by relative errors to exact solutions) and effectiveness of the method (by the computing time).

II. PIES and methods of its numerical solution

Parametric integral equations system (PIES) for two-dimensional Laplace equation is presented by the following formula [14]:

$$0.5u_l(\overline{s}) = \sum_{j=1}^{n} \int_{s_{j-1}}^{s_j} \Big\{ \overline{U}^*_{lj}(\overline{s},s)p_j(s) - \overline{P}^*_{lj}(\overline{s},s)u_j(s) \Big\} J_j(s)ds, \tag{1}$$

where $s_{l-1} \le \overline{s} \le s_l, s_{j-1} \le s \le s_j$, $l = 1,2,...,n$, $J_j(s) = \left[\left(\frac{\partial \Gamma_j^{(1)}(s)}{\partial s} \right)^2 + \left(\frac{\partial \Gamma_j^{(2)}(s)}{\partial s} \right)^2 \right]^{0.5}$, n - is the number of segments and $\Gamma(s)$ - any parametric curve modeling the shape of the boundary. In PIES defined in the parametric reference system, s_{l-1} and s_{j-1} correspond to the beginning of l-th and j-th segments, while s_l and s_j to the ends of these segments. Integrands $\overline{U}^*_{lj}(\overline{s},s)$ and $\overline{P}^*_{lj}(\overline{s},s)$ (kernels) in (1) include the shape of the boundary (defined by parametric curves $\Gamma(s)$) in their mathematical formalism, and they are presented as follow:

$$\overline{U}^*_{lj}(\overline{s},s) = \frac{1}{2\pi} ln \frac{1}{\{\eta_1^2 + \eta_2^2\}^{0.5}}, \qquad \overline{P}^*_{lj}(\overline{s},s) = \frac{1}{2\pi} \frac{\eta_1 n_1(s) + \eta_2 n_2(s)}{\eta_1^2 + \eta_2^2}, \tag{2}$$

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where $\eta_1 = \Gamma_1^{(1)}(\overline{s}) - \Gamma_j^{(1)}(\overline{s})$, $\eta_2 = \Gamma_1^{(2)}(\overline{s}) - \Gamma_j^{(2)}(\overline{s})$ and $n_1(s), n_2(s)$ are the direction cosines of the external normal to j-th segment of the boundary.

Solution of PIES (1) is reduced to find unknown functions $u_j(s)$ or $p_j(s)$ on segments, which describe boundary of solving problem. In previous studies, these functions were approximated by the following approximating expressions $\tilde{u}_j(s)$ or $\tilde{p}_j(s)$:

$$\tilde{p}_j(s) = \sum_{k=0}^{M-1} p_j^{(k)} T_j^{(k)}(s), \quad \tilde{u}_j(s) = \sum_{k=0}^{M-1} u_j^{(k)} T_j^{(k)}(s), \quad (3)$$

where $u_j^{(k)}$, $p_j^{(k)}$ are unknown coefficients. One of these coefficients (on j-th segment) is obtained by interpolation of defined boundary condition, then the other one is unknown, M - is the number of coefficients, $T_j^{(k)}(s)$ - base functions (we use Chebyshev and Lagrange polynomials). Hence, to solve the boundary value problem, we need to find unknown coefficients only. For this purpose, in PIES we applied well known collocation method [2], [12] and Galerkin method [8], [13]. Nevertheless, these methods has their advantages and disadvantages. The aim of this paper is to develop new, more effective strategy of solving PIES and to carry out tests of its computing efficiency.

III. Introduction to weighted residual method

At first, the solution obtained using approximation series (3), required to find unknown coefficients $u_j^{(k)}$, $p_j^{(k)}$. For this purpose, we should obtain the system of algebraic equations, which will be the approximation of PIES. We can use many different methods to find the system. Depending on the efficiency of those methods, the solutions could be obtained with different accuracy. We should not expect, that we obtain highly accurate solutions. Therefore, because we just approximate the unknown functions, then we mark them as $\tilde{u}_j(s)$, $\tilde{p}_j(s)$. However, we should apply proper method to obtain the solutions as close to exact solutions as it is possible.

It can be achieved using appropriate methods of finding unknown coefficients in (3) and choosing proper base function in (3) according to the type of boundary value problem. Finally, after substituting $\tilde{u}_j(s)$, $\tilde{p}_j(s)$ (3) into (1), we obtain solutions with some error $R(s) \neq 0.0$ defined as follows:

$$R(s) = 0.5\tilde{u}_l(s) - \sum_{j=1}^{n} \int_{s_{j-1}}^{s_j} \{\overline{U}_{lj}^*(\overline{s}, s)\tilde{p}_j(s) - \overline{P}_{lj}^*(\overline{s}, s)\tilde{u}_j(s)\} J_j(s) ds \neq 0, \quad (4)$$

where $R(s)$ is called residue. To minimize this error, it should be arrange uniformly on boundary and boundary integral should be equal to zero:

$$\int_{\Gamma} R(s) w(s) ds = 0.0, \quad (5)$$

where $w(s)$ is a weight function. Selection of the weight functions gives many versions of weighted residual method. One of the easiest is the pseudospectral method, also known as collocation method [9], previously used by the authors.

In collocation method, the weight function $w(s)$ in (5) is defined as Dirac function $w(s) = \delta(s - s^{(c)})$. After substitution the approximation series (3) and collocation points $s^{(c)}$ into (1) we obtain approximation of PIES. Based on this approximation we can obtain the system of algebraic equations. In result of solving such system we obtain unknown coefficients from (3). We should noted, that collocation points should be properly arrange along the boundary.

In our studies we have tested mainly two arrangements of collocation points: uniform and at points corresponding to the roots of Chebyshev polynomials. The second way was adopted, because it is known as optimal arrangement of nodes in one-dimensional interpolation. Therefore, after process of integration, the system of algebraic equations is obtained. The main advantage of collocation method is, that using Dirac function as weight function, we need single integration only. Unfortunately, we obtain non-symmetric and dense matrix.

The highest average errors are obtained for uniform arrangement of collocation points [4]. Besides the errors, we should also deal with instability of solutions. It can be noticed, that depending on the number of terms in the series (3) we obtain very different error values. Analysing the solutions obtained by the collocation approach with collocation points distributed at points corresponding to the roots of Chebyshev polynomials, we should noticed that they are significantly better than for the uniform distribution.

IV. Concept of the method

The effectiveness of the method can be determine by, among others: 1) accuracy of obtained solutions, 2) stability of the solutions, 3) computing time and 4) computer memory occupation. Direct application of collocation and Galerkin methods applied to solve PIES was previously examined by the authors and described in [3], [4], [16]. We found that the collocation method is very easy to apply, but solutions are less accurate than in Galerkin method. However, solutions obtained by Galerkin method need significantly more computing time (especially in more complex problems). Therefore, we need to develop and test another, more effective method of solving PIES. Firstly, we decided to replace kernels, previously used in PIES, by degenerate kernels to reduce the computing time necessary to solve PIES. Hence, in this paper we apply degenerate kernels in PIES before usage of collocation method. In the literature, we can find different methods of obtaining such kernels, i.e. Taylor [11] or Fourier [6] series expansion. However, in this paper we propose own strategy of replacing kernels in (1) by degenerate kernels.

A. The strategy of interpolation of the kernels using degenerate kernels

In this strategy we obtain degenerated kernels using generalized Lagrange interpolation polynomial of any degree. Generalization of polynomial means its application for interpolation of function of two variables. Lagrange interpolation is very easy, because kernels $\overline{U}_{lj}^*(\overline{s}, s)$ and $\overline{P}_{lj}^*(\overline{s}, s)$ physically are defined in unit square. The square is defined by normalized parameters $\overline{s}, s$, where $0 \leq \overline{s}, s \leq 1$. Interpolation is applied only for kernels, where $l \neq j$, it means except the main diagonal. On the main diagonal (for $l = j$) kernels

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are singular and will not be interpolated. In calculations, for $l = j$ the kernels are not changed.

In the next part of paper, kernels $\overline{U}^*_{lj}(\overline{s}, s)$ and $\overline{P}^*_{lj}(\overline{s}, s)$ (2) are interpolated by generalized Lagrange polynomials and are presented as follow:

$$\begin{aligned} \overline{U}^*_{lj}(\overline{s}, s) &= \sum_{a=0}^{p-1} \sum_{b=0}^{m-1} U^{(ab)}_{lj} L^{(a)}_l(\overline{s}) L^{(b)}_j(s), \\ \overline{P}^*_{lj}(\overline{s}, s) &= \sum_{a=0}^{p-1} \sum_{b=0}^{m-1} P^{(ab)}_{lj} L^{(a)}_l(\overline{s}) L^{(b)}_j(s), \end{aligned} \tag{6}$$

where:

$$\begin{aligned} L^{(a)}_l(\overline{s}) &= \prod_{k=0, k\neq a}^{p-1} \frac{\overline{s} - \overline{s}^{(k)}}{\overline{s}^{(a)} - \overline{s}^{(k)}}, \\ L^{(b)}_j(s) &= \prod_{k=0, k\neq b}^{m-1} \frac{s - s^{(k)}}{s^{(b)} - s^{(k)}}, \end{aligned} \tag{7}$$

and $U^{(ab)}_{lj} = U_{lj}(\overline{s}^{(a)}, s^{(b)})$, $P^{(ab)}_{lj} = P_{lj}(\overline{s}^{(a)}, s^{(b)})$. $U^{(ab)}_{lj}$ and $P^{(ab)}_{lj}$ are the values of kernels, easy to obtain by substituting specified interpolation points $(\overline{s}^{(a)}, s^{(b)})$ defined in unit square $0 \leq \overline{s}, s \leq 1$. These points, to avoid the Runge's phenomenon, are placed corresponding to the roots of Chebyshev polynomials.

Proposed way of replacing conventional kernels by degenerate kernels is the simplest of other methods, well known in the literature. There is no derivative calculations, contrary to the Taylor series expansion, as well as no integral calculation and solving system of algebraic equations, contrary to least squares method. Our method requires to calculate values of functions in interpolation nodes only and to substitute them into (6). Therefore, an accuracy of interpolation is determined by the number and the kind of nodes arrangement in unit square.

B. Accuracy analysis of the kernels interpolation with different number and kind of nodes arrangement

Kernels $\overline{U}^*_{lj}(\overline{s}, s)$, $\overline{P}^*_{lj}(\overline{s}, s)$ in PIES are functions of two normalized parameters $\overline{s}, s$ defined in unit square $0 \leq \overline{s}, s \leq 1$ and contain relationship between corresponding segments l and j. Accuracy of the solutions obtained by PIES will be conditioned, among others, by the accuracy of kernels interpolation. Therefore, we decide to examine, how the number and kind of nodes arrangement, in square domain ($0 \leq \overline{s}, s \leq 1$), influences the interpolation error. Interpolation accuracy also will be tested in case of different relationship between segments.

Degenerate kernels can be accurately and easily obtained by the interpolation of the kernels $\overline{U}^*_{lj}(\overline{s}, s)$ and $\overline{P}^*_{lj}(\overline{s}, s)$ using generalized (into two variables function) Lagrange polynomials, as presented in (6). The main advantage of such interpolation is that there is only need to calculate values of functions $U^{(ab)}_{lj} = U_{lj}(\overline{s}^{(a)}, s^{(b)})$ and $P^{(ab)}_{lj} = P_{lj}(\overline{s}^{(a)}, s^{(b)})$ (in predefined in square domain nodes $0 \leq \overline{s}, s \leq 1$ of interpolation) and substitute them into generalized interpolation Lagrange polynomials (6). The only problem is that such interpolation should be obtain for different relationship between segments. In other word for different indexes l, j.

We have tested two arrangements of nodes: uniform and at places corresponding to the roots of Chebyshev polynomials of the first kind. Better solutions are obtained for the second way of arrangement. On Fig. 1 we presented graphical visualization of interpolation (5 nodes) of the kernel $\overline{U}^*_{lj}(\overline{s}, s)$ for adjacent segments comparing to exact solution.

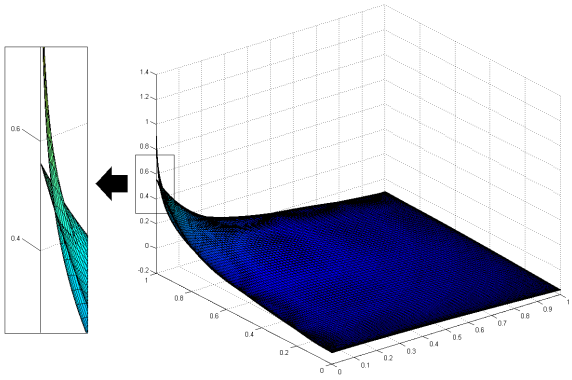

Fig. 1. Interpolation of kernel $\overline{U}^*_{lj}(\overline{s}, s)$ for adjacent segments

Interpolation of kernel $\overline{U}^*_{lj}(\overline{s}, s)$ for such nodes arrangement according to its number are presented in Tab. I, where errors are obtain for 100×100 measuring points using (8).

$$||E(f)||_2 = \left\{ \frac{1}{n} \sum_{i=0}^{n} [f(x_i) - f^*(x_i)]^2 \right\}^{0.5} \cdot 100\%, \tag{8}$$

where $f(x_i)$ is the exact solution in points, $f^*(x_i)$ is the solution obtained by interpolation and n is the number of measuring points. Solutions are obtained for adjacent segments.

TABLE I
INTERPOLATION ERROR [%] OF KERNEL $\overline{U}^*_{lj}(\overline{s}, s)$

5	8	10	13	15	18	20
1.9027	1.1678	0.9272	0.7057	0.6070	0.4996	0.4456

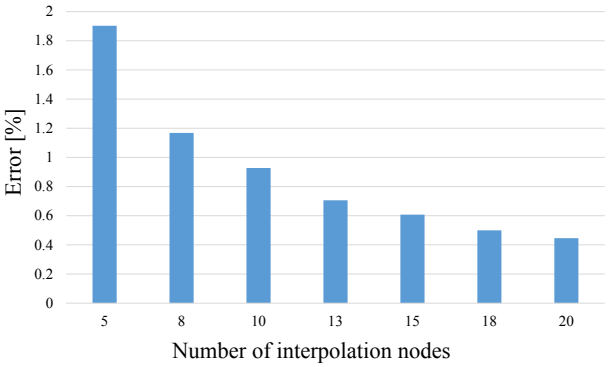

Fig. 2. Interpolation error of kernel $\overline{U}^*_{lj}(\overline{s}, s)$

As we can see in Tab. I (and Fig 2), increasing number of nodes (arranged at places corresponding to the roots of Chebyshev polynomials) results in decreasing interpolation error of kernel $\overline{U}^*_{lj}(\overline{s}, s)$.

Similarly, we can obtain graphical visualization of Lagrange interpolation of kernel $\overline{P}^*_{lj}(\overline{s}, s)$ for different relationship between segments. It is presented in Fig. 3 for adjacent segments using 5 interpolation nodes.

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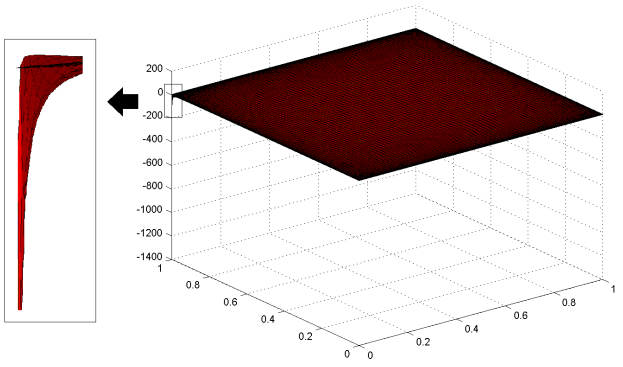
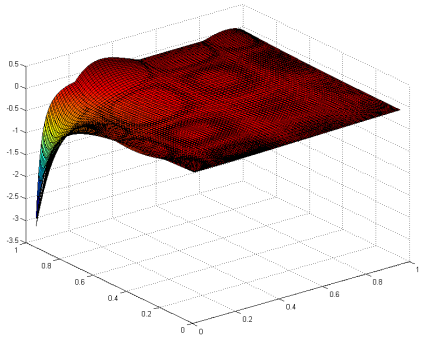
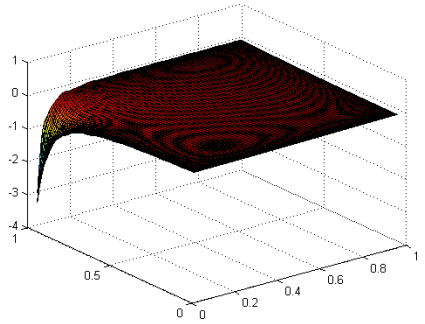


Fig. 3. Interpolation of kernel $\overline{P}^*_{lj}(\overline{s}, s)$ for adjacent segments

In case of solutions located close to points of segments join (corner points), we have rapid change of values of $\overline{P}^*_{lj}(\overline{s}, s)$ (Fig. 3). Therefore, we decided to present in Fig. 4 solutions only in points located at some distance from the corner points for better readability of the chart. We present solutions obtained using 5 (Fig. 4(a)) and 20 (Fig. 4(b)) interpolation nodes.



(a) 5 interpolation nodes



(b) 20 interpolation nodes

Fig. 4. Interpolation of kernel $\overline{P}^*_{lj}(\overline{s}, s)$

Next, we analyse interpolation error of kernel $\overline{P}^*_{lj}(\overline{s}, s)$ for adjacent segments and for different number of nodes arranged at the roots of Chebyshev polynomials. It turns out that, as shown in Tab. II and Fig. 5, we need greater number of interpolation nodes comparing to kernel $\overline{U}^*_{lj}(\overline{s}, s)$ (Tab. I).

TABLE II
INTERPOLATION ERROR [%] OF KERNEL $\overline{P}^*_{lj}(\overline{s}, s)$

5	8	10	13	15	18	20	50	90	100
1335	1322	1312	1294	1280	1257	1240	886	199	0

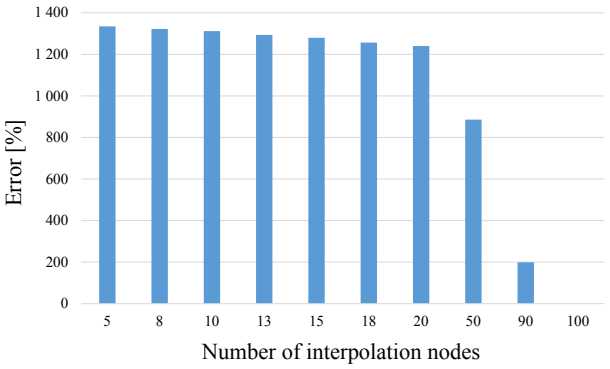


Fig. 5. Interpolation error of kernel $\overline{P}^*_{lj}(\overline{s}, s)$

Furthermore, we test the accuracy of interpolation of the kernel $\overline{P}^*_{lj}(\overline{s}, s)$ in case of not adjacent segments. Obtained solutions using 5 interpolation nodes are presented on Fig. 6.

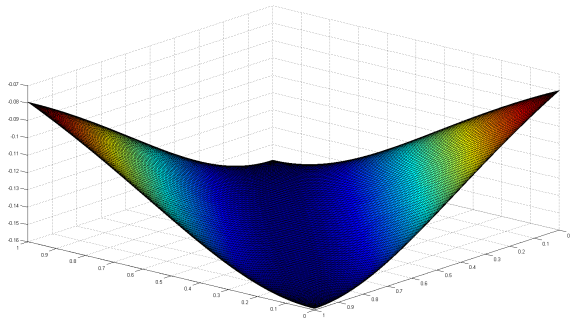


Fig. 6. Interpolation of kernel $\overline{P}^*_{lj}(\overline{s}, s)$ for not adjacent segments

It should be noted, that we can use significantly lower number of nodes comparing to interpolation for adjacent segments.

V. PIES FOR DEGENERATE KERNELS AND IT SOLUTIONS

Substituting kernels (6) into (1) we obtain PIES with degenerate kernels for $l \neq j$ in following form:

$$\frac{1}{2}u_l(\overline{s}) = \sum_{j=1}^{n}\Big\{\sum_{a=0}^{p-1}\sum_{b=0}^{m-1} U_{lj}^{(ab)} L_l^{(a)}(\overline{s}) \int_0^1 L_j^{(b)}(s) p_j(s) - \sum_{a=0}^{p-1}\sum_{b=0}^{m-1} U_{lj}^{(ab)} L_l^{(a)}(\overline{s}) \int_0^1 L_j^{(b)}(s) u_j(s)\Big\} J_j(s) ds, \tag{9}$$

where $j, l = 1, 2, ..., n$ and for $l = j$ with non-degenerate kernels:

$$0.5u_j(\overline{s}) = \int_0^1 \{\overline{U}^*_{jj}(\overline{s}, s) p_j(s) - \overline{P}^*_{jj}(\overline{s}, s) u_j(s)\} J_j(s) ds, \tag{10}$$

It turns out, that in degenerate kernels (for $l \neq j$) we can separate variables and finally we can put Lagrange polynomials $L_l^{(a)}(\overline{s})$ (depend on variable $\overline{s}$ only) outside integral. Now, there are only polynomials $L_j^{(b)}(s)$ (depend on variable s and unknown functions $u_j(s)$ or $p_j(s)$) inside the integrals. Therefore, equations (9) and (10) will be called

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as *degenerate parametric integral equations system (DPIES)* in contrast to conventional PIES with non-degenerate kernels (1).

To solve (9) and (10) we have to find unknown functions $u_j(s)$ or $p_j(s)$. We assumed approximation series $\tilde{u}_j(s)$ or $\tilde{p}_j(s)$ (3), with application of Lagrange polynomials as base functions with unknown functions $u_j^{(k)}$, $p_j^{(k)}$. To find unknown values, we test classical collocation method discussed in section 2.

In collocation method we substitute so-called collocation points $\overline{s}^{(c)}$ (where $s_{l-1} < \overline{s}^{(c)} < s_l$) into (9) and (10). We apply approximation series (3) into (9) and (10) and finally degenerate PIES for $l \neq j$ is presented as follows:

$$0.5\sum_{k=0}^{M} u_l^{(k)} T_l^{(k)}(\overline{s}^{(c)}) = \sum_{j=1}^{n}\left\{\sum_{a=0}^{p-1}\sum_{b=0}^{m-1} U_{lj}^{(ab)} L_l^{(a)}(\overline{s}^{(c)}) \int_0^1 L_j^{(b)}(s)\sum_{k=0}^{M} p_j^{(k)} T_j^{(k)}(s) - \sum_{a=0}^{p-1}\sum_{b=0}^{m-1} P_{lj}^{(ab)} L_l^{(a)}(\overline{s}^{(c)}) \int_0^1 L_j^{(b)}(s)\sum_{k=0}^{M} u_j^{(k)} T_j^{(k)}(s)\right\} J_j(s)ds, \quad (11)$$

where $j, l = 1, 2, ..., n$ and for $l = j$:

$$0.5\sum_{k=0}^{M} u_j^{(k)} T_j^{(k)}(\overline{s}^{(c)}) = \int_0^1 \left\{\overline{U}_{jj}^{*}(\overline{s}^{(c)}, s)\sum_{k=0}^{M} p_j^{(k)} T_j^{(k)}(s) - \overline{P}_{jj}^{*}(\overline{s}^{(c)}, s)\sum_{k=0}^{M} u_j^{(k)} T_j^{(k)}(s)\right\} J_j(s)ds, \quad (12)$$

We can obtain so-define DPIES in collocation points $\overline{s}^{(c)}$ in explicit form for particular boundary value problem. Values outside the integral are calculated on the base of formulas $U_{lj}^{(ab)} L_l^{(a)}(\overline{s}^{(c)})$ or $P_{lj}^{(ab)} L_l^{(a)}(\overline{s}^{(c)})$. These formulas are the product of the values of Lagrange polynomials in collocation points $\overline{s}^{(c)}$ and values $U_{lj}^{(ab)}$ or $P_{lj}^{(ab)}$ obtained substituting the interpolation nodes into base kernels (2).

In this way we obtain vertical n-dimensional vector for segments $l = 1, 2, 3, ..., n$ (assuming only one collocation point on each segment). While calculated values of integrals (for $j = 1, 2, 3, ..., n$) compose horizontal vector, n-dimensional as well. When we define greater number of collocation points on segment, dimension of the vector is the product of the number of collocation points on segment by the number of segments. We can also set a different number of collocation points on each segment.

Even from theoretical point of view, that strategy has a significant advantage over direct application of collocation method for solving PIES. In conventional PIES we have to calculate integrals for all of matrix elements. In proposed strategy only the elements on diagonal are calculated directly from (12), whilst outside diagonal ($l \neq j$) they are calculated as the product of two vectors, obtained in (11). Such strategy significantly decrease computer memory occupation. The number of calculations is significantly less than in conventional PIES. The proposed concept also significantly reduces the number of integral calculation necessary to obtain coefficients outside the main diagonal. These coefficients are calculated by multiplying previously obtained two vectors. This is less time-consuming operation than integration.

VI. Effectiveness of the Proposed Method - Examples

We considered boundary value problem modeled by Laplace equation in triangular domain. We should define only corner points presented on Fig. 7 to model the linear shape of boundary of the problem. The way of defining the problem is presented in Fig. 7. There is also marked a cross-section inside the domain, where solutions are obtained (10 points).

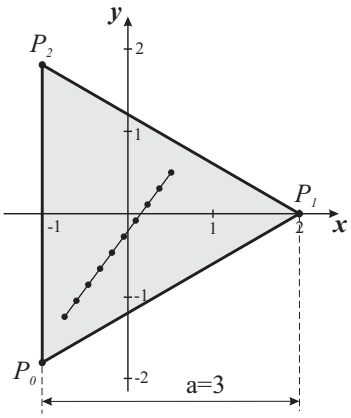

Fig. 7. Way of defining the shape of the boundary

We defined Dirichlet boundary conditions by the following formula [10]:

$$u = 0.5(x^2 + y^2). \quad (13)$$

For so-defined boundary conditions, there are an analytical solutions described by following formula [10]:

$$u_a = \frac{x^3 - 3xy^2}{2a} + \frac{2a^3}{27}. \quad (14)$$

We compare solutions obtained by PIES with degenerate and non-degenerate kernels to analytical ones as is presented in Tab. III. We analyse the impact of the interpolation accuracy of both kernels on solutions accuracy and computing time. Intel Core i5-4590S (4 cores, 4 threads, 3.0 GHz, 6MB cache memory) with 8 GB RAM is used during tests. We use Microsoft Visual Studio Professional 2013 (version: 12.0.21005.1 REL) compiler on Windows 8.1 64-bit system.

TABLE III
Solutions in domain (average relative error and time)

	PIES	DPIES (U-5, P-5)	DPIES (U-5, P-15)
average relative error [%]	0.01	5.85	0.12
computing time [ms]	1.625	1.5	1.654

As can be seen in Tab. III, application of 5 interpolation nodes (in both kernels) causes a slight reduction of computing time, however the accuracy of the solutions decreases a lot. Therefore, we decide to increase accuracy of interpolation of kernel P (15 interpolation nodes). It results in a significant improve of accuracy, but increase computing time. Because presented example is elementary we are not able to draw unambiguous conclusion. Hence, we decide to consider an example with more complex shape of boundary with known analytical solution. The shape will be defined with different number of boundary points.

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We consider the problem of elliptical domain described by formula: $x^2/a^2 + y^2/b^2 = 1$ where the following Dirichlet boundary conditions are defined: $u = 0.5(x^2+y^2)$ [10]. The shape of boundary is modeled using 4 (Fig. 8a) or 8 (Fig. 8b) boundary points. The number of segments have an impact on the accuracy of modeled shape of boundary, therefore finally it have an impact on the accuracy of the solutions.

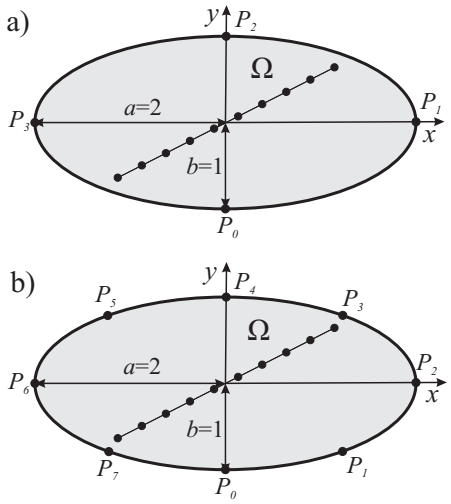

Fig. 8. Defining the shape of the boundary using a) 4, b) 8 boundary points

Analytical solution for so-defined problem is presented in the following form [10]:

$$u_a = \frac{x^2+y^2}{2} - \frac{a^2b^2(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1)}{a^2+b^2}. \tag{15}$$

Solutions are obtained in cross-section presented in Fig. 8. In Tab. IV we present analytical solutions comparing to ones obtained using degenerate PIES and PIES with non-degenerate kernels. Similarly to the previous example, in case of small number of segments (8a), computing time with a similar accuracy of the results is almost the same.

TABLE IV
SOLUTIONS IN DOMAIN (AVERAGE RELATIVE ERROR AND TIME)

	Fig. 8a			Fig. 8b		
	PIES	DPIES U - 5 P - 5	DPIES U - 5 P - 15	PIES	DPIES U - 5 P - 5	DPIES U - 5 P - 15
error [%]	1.34	1.13	1.14	0.09	0.07	0.07
time [ms]	2.282	2.031	2.267	5.734	4.312	5.721

It changes in case of greater number of segments (8b), what additionally improve the accuracy of modeling, and hence solutions accuracy also. Then the computing time of solutions, with similar accuracy, in PIES method is 5.7 ms, when in DPIES is 4.3 ms. We can also note, that sufficient accuracy was obtained with just 5 interpolation nodes (for both kernels).

In conclusion there are many factors that affect the computation time and the accuracy of the solutions. But presented examples prove, that the proposed strategy gives hope for the effective solving of the boundary value problems, however it requires more detailed studies.

VII. CONCLUSIONS

In this paper, we replaced kernels in conventional PIES by degenerate kernels and obtained DPIES. To obtain degenerate kernels we used generalized Lagrange interpolation for two variables. The accuracy of interpolation of the kernels is determined by the number of nodes and way of its arrangement in unit square(the domain of kernel definition). It was noted, that in order to obtain satisfactory solutions, we need more interpolation points in case of kernel P than in kernel U. We should also use more interpolation nodes for adjacent segments. The collocation method, previously used for solving conventional PIES with non-degenerate kernels, is used in DPIES, as well. The effectiveness of such strategy was tested on examples of problems described by Laplace equation. We solved two examples and we compared solutions obtained by DPIES, conventional PIES with non-degenerate kernels and analytical ones. Based on obtained solutions, we can summarized, that application of degenerate kernels in PIES reduces the computing time without significant loss of solutions accuracy. However, in order to confirm the reliability of final conclusions, we should conduct tests on more complex examples. It will be the next step of our studies.

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