Effects of Subgrades, Shear Deformation, Rotatory Inertia, Contact Area and Inclination on the Deflection of Rectangular Plates

M. C. Agarana, *IAENG Member*, O.O. Ajayi and S. Bishop

Abstract— **Plates are usually subject to moving forces or moving masses, sometimes referred to as moving loads. This paper analyses the effects of foundations, shear deformation, rotatory inertia, angle of inclination and the contact area of the load, on the dynamic behaviour of an elastic rectangular plate under moving load. The method used involved transforming the set of second order partial differential (PDE) equations, governing the moving load problem, to its non-dimensionalised set of first order PDE so as to obtain general solution. Computer programs were then developed and used in conjunction with MATLAB in order to obtain the desired solution. The findings show that the subgrade, rotatory inertia, shear deformation, the angle of inclination and the contact area of the load with the plate, have effects on the dynamic response of the rectangular plate to a moving load. Specifically, the effects of rotatory inertia and shear deformation on the deflection of the Mindlin plate gives a more realistic results for practical application, especially when such plate is considered to rest on a subgrade.**

Index Terms— **Subgrades, Inclination, Rectangular plate, Deflection, Rotatory Inertia, Shear Deformation,**

I. INTRODUCTION

THE moving load problem is a fundamental problem in several fields of Applied Mathematics, Engineering and Applied Physics. The importance of this type of problems also manifested in numerous applications in the area of transportation. Rails and bridges are examples of plates usually designed to support moving loads such as rail vehicles and Cars [1,2,3,4]. Attempts have been made to analyse the dynamic response of a plates under the influence of moving loads, putting into consideration the influence of rotatory inertia and shear deformation by different authors [4,5,6,7]. Such plates are known as Mindlin Plates.[3] One of the works of Gbadeyan and Dada [7] likewise considered the dynamic reaction of versatile rectangular Mindlin plates under uniform somewhat conveyed moving mass [7]. For practical application, it is helpful to consider plates upheld by a versatile foundation[8,9]. For example, an examination including such foundation can be utilized to decide the conduct of scaffolds crossed by rail vehicle. Moreover, auxiliary individuals, especially, plates laying on versatile foundation have wide applications in present day designing practices, for example, railroad spans, roadway asphalts and ceaselessly bolstered pipelines [9,10,11,12]. The objectives of the study were to investigate the dynamic response of Mindlin rectangular elastic inclined plates, resting on subgrades, and being affected by the shear deformation, rotatory inertia, angle of inclination and contact area of the load with the plate, to partially distributed moving load. This was achieved by developing a theory describing the response of the Mindlin rectangular elastic inclined plates, resting on subgrades, to partially distributed moving load. Develop a finite-difference analysis of the problem. Investigate the effects of variation of velocity of moving load on the response of the plate under investigation. Compare the effect of different foundations on the deflection of the Mindlin plate under the influence of a moving load. Investigate the effect of the contact area of a moving load on the response of Mindlin plate resting on a foundation and subjected to partially distributed moving load. In this paper, therefore, the impacts of foundation, rotatory inertia, shear deformation and inclination on the vibration of rectangular plates were investigated. The foundations under consideration were Winkler and Pasternak types.

Manuscript received July 1, 2017; revised July 30, 2017.. This work was supported in full by Covenant University.
M.C. Agarana is with the Department of Mathematics, Covenant University,Nigeria,+2348023214236,michael.agarana@covenantuniversity.edu.ng
O.O. Ajayi is with the Department of Mechanical Engineering, Covenant University, Nigeria.
S. Bishop is with Mathematics Department, Covenant University, Nigeria.

II. PROBLEM DEFINITION

A rectangular plate, with moving load and different boundary conditions is considered. The load is relatively large, that is, its inertia cannot be neglected, and is moving along the mid-space on the surface of the plate, supported by a foundation,.[12]

A. Assumptions

(i) The plate is of constant cross – section,
(ii.) The moving railway vehicle moves with a constant speed.
(iii) The moving load is guided in such a way that it keeps contact with the plate throughout the motion,
(iv). The plate is continuously supported by a subgrade
(v). The moving load is partially distributed.
(vi). The plate is elastic.
(vii).No damping in the system.
(viii).Uniform gravitational field
(ix). Constant mass of the load on the plate.

III. PROBLEM SOLUTION

The set of dynamic equilibrium equations which govern the behaviour of an inclined Mindlin plate supported by subgrade and traversed by a partially distributed moving load may be written as [12]:

$$\frac{M_L B}{A}\left[g\sin\theta+\frac{\partial^2 W}{\partial T^2}+2U\frac{\partial^2 W}{\partial x\partial T}+U^2\frac{\partial^2 W}{\partial x^2}\right]-F_S$$

$$=k^2Gh\left[-\frac{\partial^2 W}{\partial x^2}+\frac{\partial \psi_x}{\partial x}-\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 \psi_y}{\partial y}\right]-KW-M_f\frac{\partial^2 W}{\partial T^2}-G_1\frac{\partial^2 W}{\partial x^2}-G_1\frac{\partial^2 W}{\partial y^2}+\rho h\frac{\partial^2 W}{\partial T^2} \quad (1)$$

$$\frac{B_{\rho L}h_1^3}{12}\left[\frac{\partial^2 \psi_x}{\partial T^2}+2U\frac{\partial^2 \psi_x}{\partial x\partial T}+U^2\frac{\partial^2 \psi_x}{\partial x^2}\right]+\frac{\rho h^3}{12}\frac{\partial^2 \psi_x}{\partial T^2}$$

$$=D\left[\frac{\partial^2 \psi_y}{\partial x^2}+\nu\frac{\partial^2 \psi_y}{\partial x\partial y}\right]+\frac{(1-\nu)}{2}D\left[\frac{\partial^2 \psi_x}{\partial y^2}+\frac{\partial^2 \psi_y}{\partial x\partial y}\right]-k^2Gh\left(\psi_x-\frac{\partial W}{\partial x}\right) \quad (2)$$

$$\frac{B_{\rho L}h_1^3}{12}\left[\frac{\partial^2 \psi_y}{\partial T^2}+2U\frac{\partial^2 \psi_y}{\partial y\partial T}+U^2\frac{\partial^2 \psi_y}{\partial_y^{\,2}}\right]+\frac{\rho h^3}{12}\frac{\partial^2 \psi_y}{\partial T^2}$$

$$=D\left[\frac{\partial^2 \psi_y}{\partial y^2}+\nu\frac{\partial^2 \psi_x}{\partial y\partial x}\right]+\frac{(1-\nu)}{2}D\left[\frac{\partial^2 \psi_x}{\partial x\partial y}+\frac{\partial^2 \psi_y}{\partial x^2}\right]-k^2Gh\left(\psi_y-\frac{\partial W}{\partial y}\right) \quad (3)$$

The notations are as in references [12].
Finite difference technique was used to transform the simplified non-dimensionalised set of first order partial differential equations to a set of algebraic equations.

A. Finite Difference Algorithm for the model

The governing equations were solved using a numerical method based on the finite difference algorithm. These equations were transformed into their equivalent algebraic forms. The finite difference definition of first order partial derivative of a function E(x,y,t) say, with respect to x, y and t respectively are as follows[15,16]:

$$\frac{\partial E}{\partial t}=\frac{1}{4r^*}\left[E_{i+1,j+1}^{k+1}+E_{i+1,j}^{k+1}+E_{i,j+1}^{k+1}+E_{i,j}^{k+1}-E_{i+1,j+1}^{k}-E_{i+1,j}^{k}-E_{i,j+1}^{k}-E_{i,j}^{k}\right] \quad (4)$$

$$\frac{\partial E}{\partial x}=\frac{1}{4h^*}\left[E_{i+1,j+1}^{k+1}+E_{i+1,j}^{k+1}-E_{i,j}^{k+1}-E_{i,j+1}^{k+1}+E_{i+1,j+1}^{k}+E_{i+1,j}^{k}-E_{i,j+1}^{k}-E_{i,j}^{k}\right] \quad (5)$$

$$\frac{\partial E}{\partial y}=\frac{1}{4k^*}\left[E_{i+1,j+1}^{k+1}+E_{i,j+1}^{k+1}-E_{i+1,j}^{k+1}-E_{i,j}^{k+1}+E_{i+1,j+1}^{k}+E_{i,j+1}^{k}-E_{i+1,j}^{k}-E_{i,j}^{k}\right] \quad (6)$$

where E is the function value of the centre of a grid, which is well approximated by the average of its values at the grid nodes. Notations are as in the reference [15, 16,17,18,19].

$$E\left(x+\frac{h^*}{2},y+\frac{k^*}{2},t+\frac{r^*}{2}\right)$$

$$=\frac{1}{8}\left[E_{i+1,j+1}^{k+1}+E_{i+1,j}^{k+1}+E_{i,j+1}^{k+1}+E_{i,j}^{k+1}+E_{i+1,j+1}^{k}+E_{i+1,j}^{k}+E_{i,j+1}^{k}+E_{i,j}^{k}\right] \quad (7)$$

An equivalent set of algebraic equations obtained by using the above finite difference definition was solved after written in matrix form as follows [15,16 17,18,19].

Hi,j+1 S'i,j+1 + Ii+1,j+i S'i+1,j+1= - Gi,jS'I,,j – Ji+1,j S'i+1,j + Ki, j S°i,j + Li,j +i , S°i,j+1 Mi+1 S°i+1,j + Ni+1, j+1 S°i+1,j+1 + P1

(8)
i = 1, 2, 3 ..., N-1; j = 1, 2, 3, ... M -1
where N and M are the numbers of the nodal points along X and Y axes respectively. The matrices were solved with the aid of computer program and MATLAB to arrive at the results.

IV. RESULTS AND DISCUSSION

The results of a simply supported rectangular plate resting on subgrades and subjected to a moving load are presented and discussed in this section. Figure 1 shows that the Mindlin plate on Winkler foundation deflects more than that on Pasternak foundation.
For a specific value of foundation stiffness K, G and constant area (Ar), deflections of the Mindlin plate were calculated and plotted. It is observed that the response amplitude of the Mindlin plate resting on a Winkler foundation decreases with an increase in the value of the foundation's constant (K), for fixed value of both velocity and constant area of the partial distributed moving load. This was depicted in figures 11, 12 and 13.
The response maximum amplitude of the Mindlin plate supported continuously by an elastic Winkler subgrade decreases with an increase in the contact area(Ar) of the moving partial distributed mass for fixed values of K and velocity (Uw). This was shown in figure 19.
Keeping the values of K and Ar constant, the response maximum amplitudes of the Mindlin plate, resting on a Winkler foundation, increases with an increase in velocity of the moving mass. This can be seen in figure 17.
The response maximum amplitude of Mindlin plate supported by a Pasternak subgrade, decreases as the contact area (Arp) of the moving load deceases for fixed values of velocity foundation constant K and G. Figure 10 shows thi clearly.
For fixed values of K and Arp the response amplitude of a Mindlin plate resting on a Pasternak subgrade increases as K increases. However, the effect of G on the behaviour of the Mindlin plate on the Pasternak foundation is more noticeable than that of K.
The response amplitude of the Mindlin plate not resting on a Pasternak subgrade (i.e. the case K=G=0) is greater than that of the plate continuously supported by the subgrade. As the velocity (Up) of the moving mass increases the response maximum amplitude of the Mindlin plate resting on a Pasternak foundation decreases for fixed values of Arp, K, and G. Figures 8 and 9 depicts this.
For fixed values of K, G, Arp and Up, non-rotatory plate has the highest response maximum amplitude when compared to Mindlin, non-shear and non-Mindlin plates. This can be seen in figure 6
As G increases, the response maximum amplitude for rotatory, Mindlin and non-Mindlin plates, supported by Pasternak foundation, decreases for fixed values of K, Arp and Up. However, the effect of rotatory inertia is minimal when compared with that of shear deformation.
As Arp increases the response maximum amplitude for both Mindlin and rotatory plates resting on Pasternak foundation increase for fixed values of K, G and Up.
Also figures 20, 21 and 22 show that angle of inclination of the plate affects the acceleration of the moving load, thus the deflection of the plate.

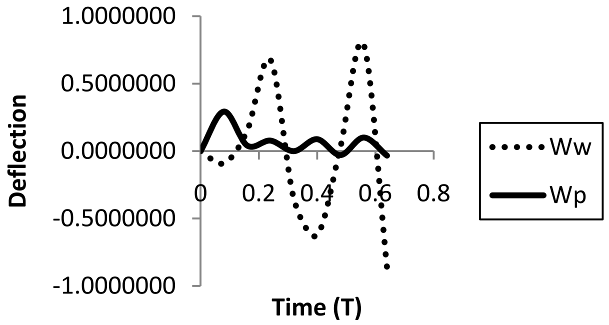

Fig. 1. Deflection of the plate for K=100, G=0.09, Arp=0.5, u=1.5 and various values time

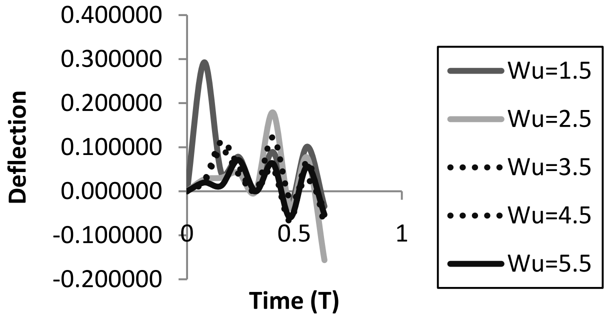

Fig. 2. Deflection of the plate for K=200, G=0.09, Arp=0.5 andvarious values time

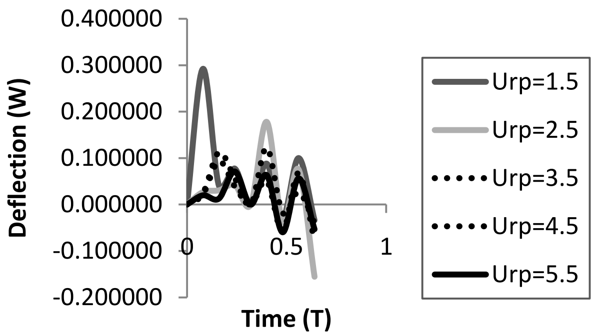

Fig. 3. Deflection of the plate for K=100, G=0.09, Arp=0.5, and various values of time

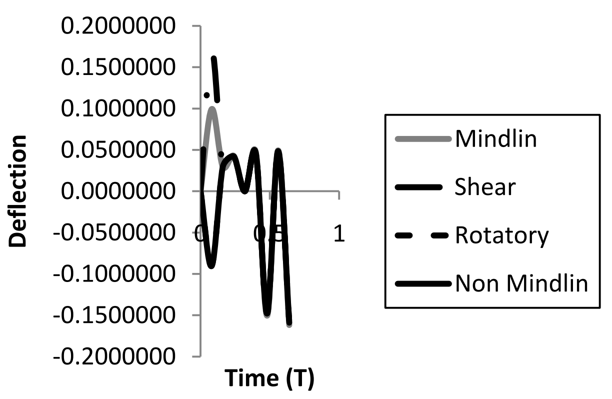

Fig. 4. Deflection of the Mindlin, Non Mindlin, Rotatory and Shear plates for K=100, G=0.18, Arp=0.02, u=1.5 and various values of time

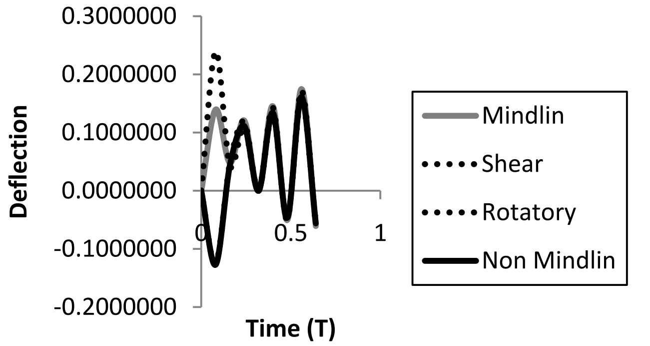

Fig. 5. Deflection of the Mindlin, Non Mindlin, Rotatory and Shear plates for K=100, G=0, Arp=0.02, u=1.5 and various values of time

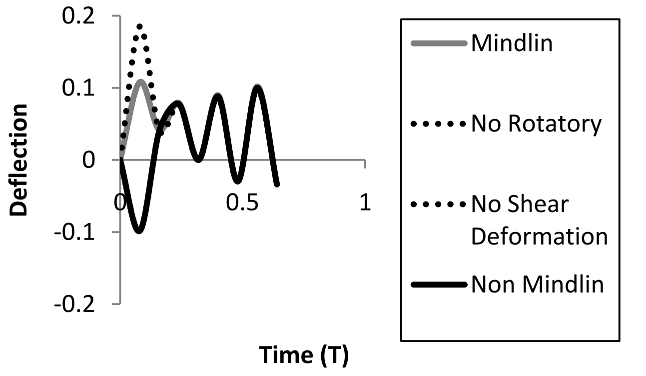

Fig..6. Deflection of the Mindlin, Non Mindlin, Rotatory and Shear plates for K=100, G=0.09, Arp=0.02, u=1.5 and various values of time

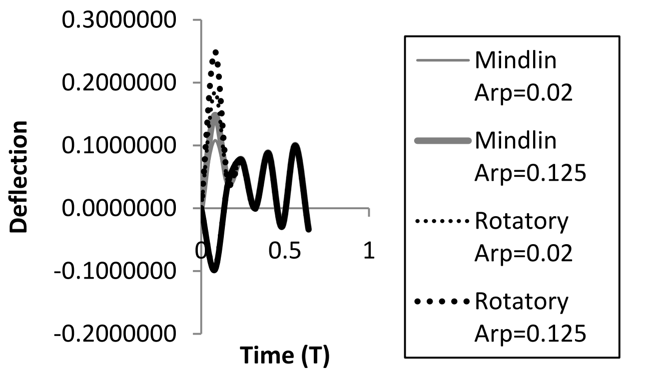

Fig. 7. Deflection of the Mindlin, Non Mindlin, Rotatory, Shear plates at K=100, G=0.09, Up=1.5 and different values of 'Arp' and time

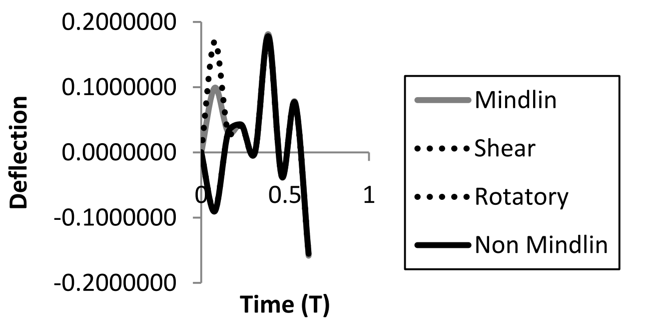

Fig. 8. Deflection of the Mindlin, Non Mindlin, Rotatory and Shear plates for K=100, G=0.09, Arp=0.02, u=2.5 and various values of time

ISBN: 978-988-14048-4-8
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

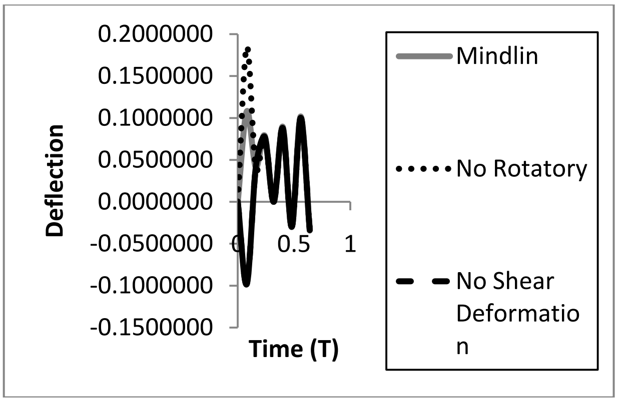

Fig. 9. Deflection of the Mindlin, Non Mindlin, Rotatory, and Shear plates for K=100, G=0.09, Arp=0.02, Up=1.5 and various values of time

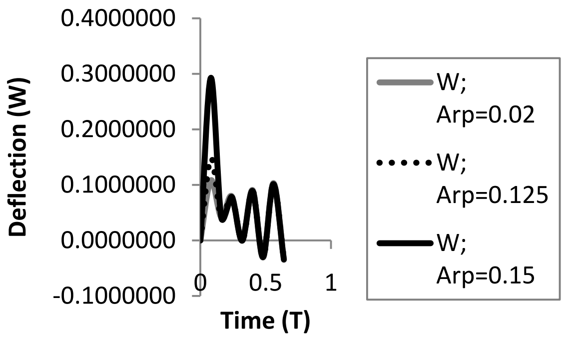

Fig. 10. Deflection of the plate for K=200, G=0.09 and various values of Arp and time

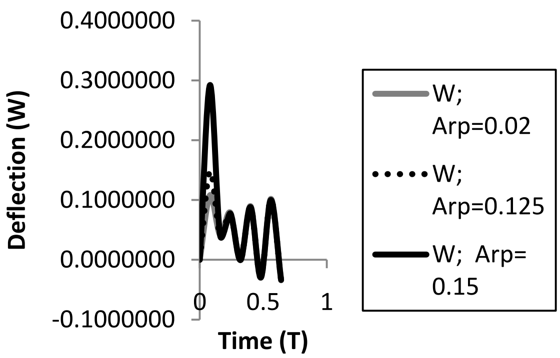

Fig. 11. Deflection of the plate for K=100, G=0.09 and various values of Arp and time

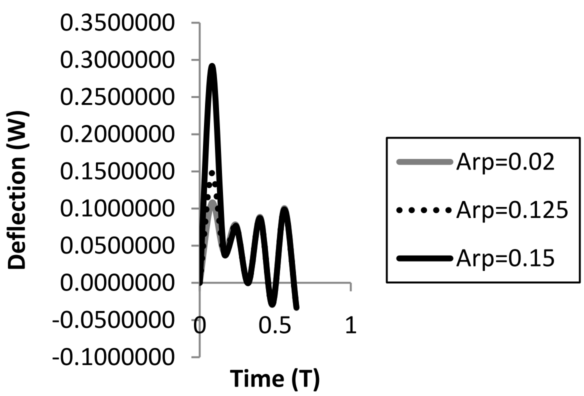

Fig. 12. Deflection of the plate for K=0, G=0.09 and various values of Arp and time

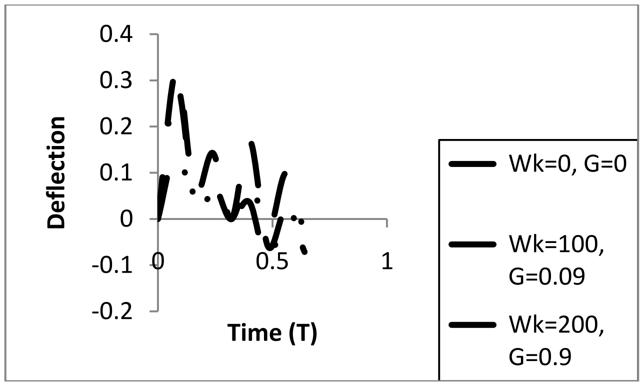

Fig. 13. Deflection of the plate at Arp=0.5 and different values of k, G for various values time

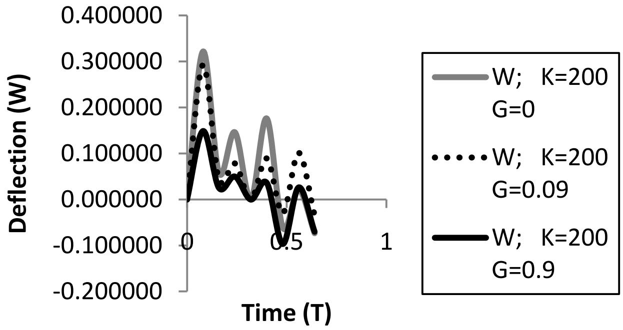

Fig. 14. Deflection of the plate for K=200, Arp=0.5, and various values of G and time t.

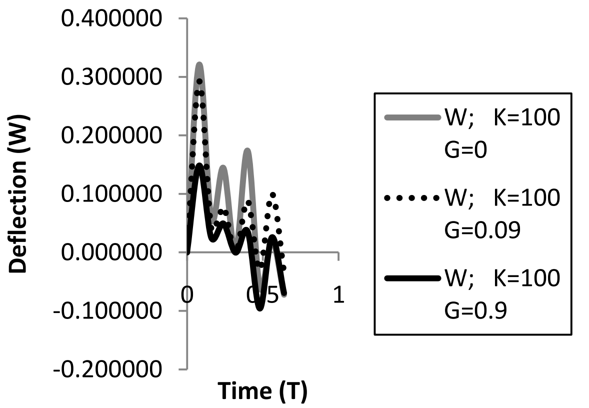

Fig. 15 Deflection of the plate for K=100, Arp=0.5 and various values of G and time t.

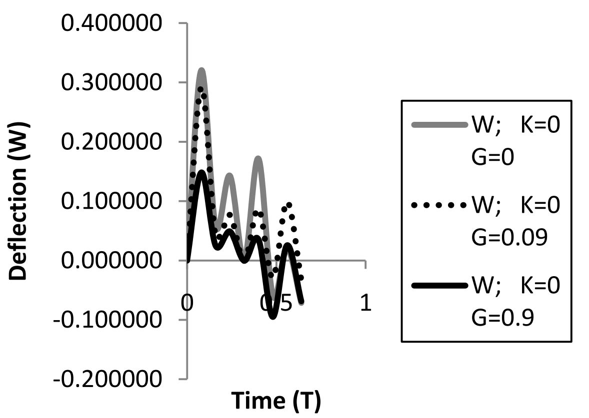

Fig. 16. Deflection of the plate for K=0, Arp=0.5 and various values of G and time t.

ISBN: 978-988-14048-4-8
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

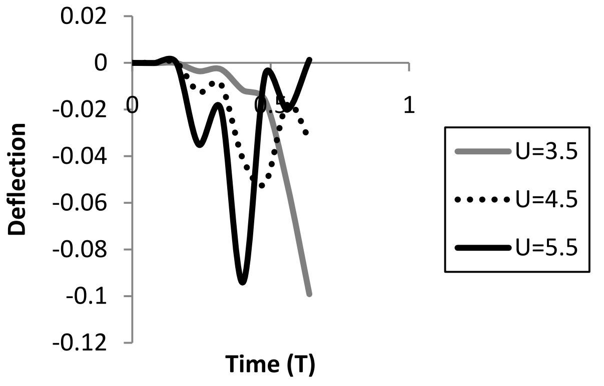

Fig. 17. Deflection of the Mindlin plate for K=200 and different values of velocity and time

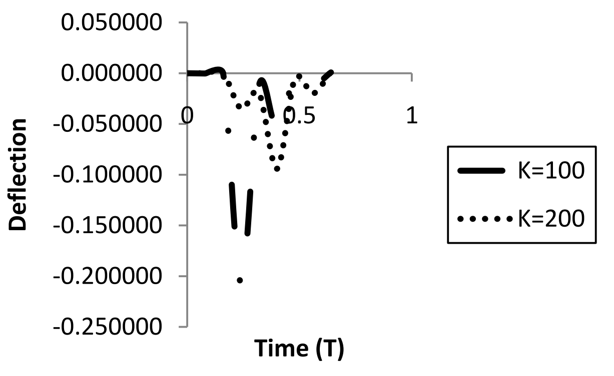

Fig.18. Comparing the effect of K=100 with K=200 on the deflection of Mindlin plate resting on Winkler foundation when velocity U=5.5

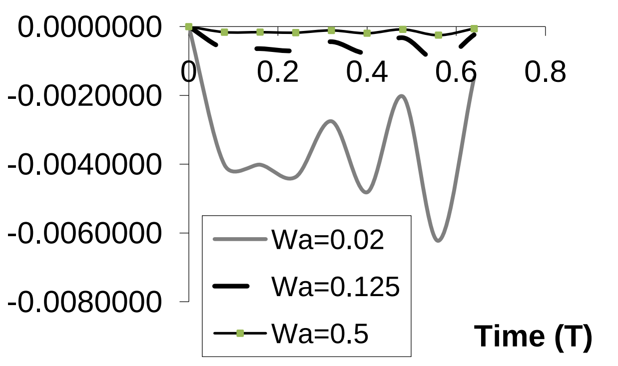

Fig.19. Deflection of the plate for K=100 and various values of Ar and time

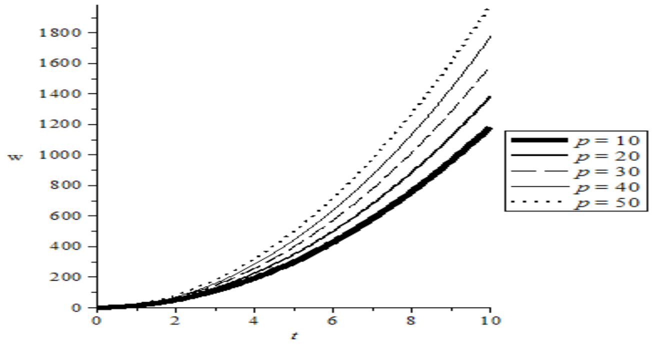

Fig.20 Deflection of plate at various values of applied load and different time.

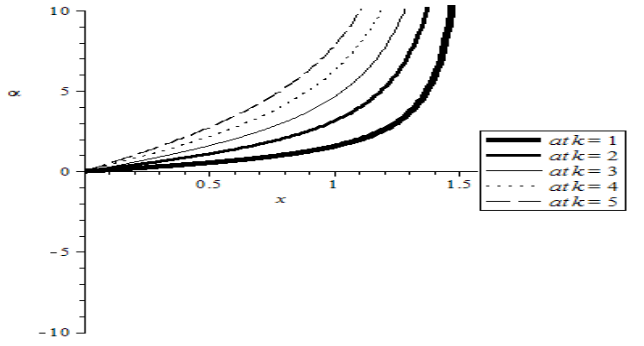

Fig.21. Acceleration of deflection of inclined plate at various values of . k

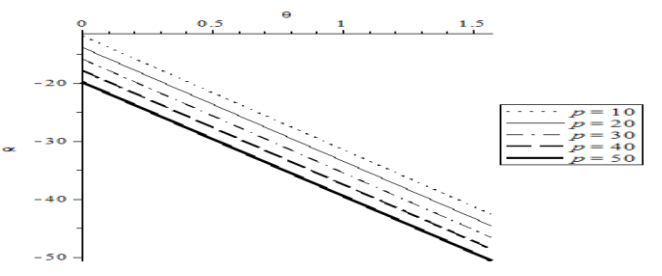

Fig.22. Acceleration of deflection of the plate at different values of applied load and different angles of inclination.

V. CONCLUSION

The structure of interest was a Mindlin rectangular elastic inclined plate on either Winkler or Pasternak elastic foundation, respectively, under the influence of a uniform partially distributed moving load. The problem was to determine the dynamic response of the whole system. Finite Difference technique was adopted in solving the resulting first order coupled partial differential equations obtained from the governing equations, for the simply supported Mindlin plate. The study has contributed to scientific knowledge by showing that elastic subgrade, on which the Mindlin plate rests has a significant effect on the dynamic response of the plate to partially distributed load. The effect is rotatory inertia and shear deformation on the dynamic response of the Mindlin plate to the moving load gives a more realistic results for practical application , especially when such plate is considered to rest on a foundation.

The study has contributed to scientific knowledge by showing that the elastic subgrade, on which the Mindling plate rests has a significant effect on the dynamic response of the plate, to a partially distributed load. In conclusion, the effect of rotatory inertia, shear deformation, subgrades, contact area and inclination on the dynamic response of the Mindlin plate to the moving load gives more realistic results for practical application, especially when such plate is considered to rest on a foundation. The main contribution of the research are :

The development of theory for the dynamic response of Mindlin Elastic plate resting on a Winkler foundation under partially distributed moving load.

The development of theory for the dynamic response of Mindlin Elastic plate resting on a Pasternak foundation under partially distributed moving load

ISBN: 978-988-14048-4-8
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

The use of Finite difference method for solving the governing equations for the above moving load problems.

REFERENCES

[1] Agarana, M. C. and Gbadeyan, J.A. and Emetere, Moses (2016) *On Response of Elastic Isotropic Damped Shear Highway Bridge Supported by Sub-grade to Uniform Partially Distributed MovingVehicle.* International Journal of Applied Engineering Research, 11 (1). pp. 244-258. ISSN 0973-4562

[2] Agarana, M. C. and Gbadeyan, J.A. and Agboola, O.O. and Anake, T. A. and Adeleke, O. J. (2015) *Dynamic Response of an Inclined Railway Bridge Supported by Winkler Foundation Under a Moving Railway Vehicle.* Australian Journal of Basic and Applied Sciences,, 9 (11). pp. 355-361. ISSN 1991-8178 .

[3] Agarana M.C., and Ede A., "Free Vibration Analysis of Elastic Orthotropic Rectangular Inclined Damped Highway Supported by Pasternak Foundation under Moving Aerodynamic Automobile," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2016, 29 June - 1 July, 2016, London, U.K., pp978-981

[4] Agarana, M. C. and Agboola, O.O. (2015) *Dynamic Analysis of Damped Driven Pendulum using Laplace Transform Method.* International Journal of Mathe , 26 (3). ISSN 0974-570X (online) 0974-5718(Print)

[5] Agarana, M. C. and Gbadeyan, J.A. (2015) *Dynamic Response of Mindlin Elastic Plate Supported by Pasternak Foundation under Uniform Partially Distributed Moving Load.* Australian Journal of Applied Mathematics. pp. 17-26. ISSN 203-9478 (Print) ISSN: 2203-9486 (Online)

[6] Gbadeyan, J.A. and Agarana, M. C. (2008) *Dynamic Analysis of Railway Bridges Supported by Winkler Foundation under Uniform Partially Distributed Moving Railway Vehicle.* Dynamic Analysis of Railway Bridges Supported by Winkler Foundation under Uniform Partially Distributed Moving Railway Vehicle. pp. 1-12

[7] Gbadeyan, J.A. and Dada, M.S. (2006). Dynamic Response of a Mindlin Elastic Rectangular Plate Under a Distributed Moving Mass. International Journal of Mechanical Sciences 48, 323-340.197

[8] Ajayi, O. O. and Agarana, M. C. and Adeniyi, M.O and Ogbonnaya, Mercy (2017) *Modelling and Control of Flow Induced Vibration of Top–Tensioned Marine Risers Using Analytical Methods.* Procedia Manufacturing, 7. pp. 609-615. ISSN 2351-9789

[9]] Ajayi, O. O. and Agarana, M. C. and Animasaun, T.O (2017) *Vibration Analysis of the Low Speed Shaft and Hub of a Wind Turbine Using Sub Structuring Techniques.* Procedia Manufacturing, 7. pp. 602-608. ISSN 2351-9789

[10] Agarana, M. C. and Iyase, S.A . (2015) *Analysis of Hermite's equation governing the motion of damped pendulum with small displacement.* International Journal of Physical Sciences, 10 (12). pp. 364-370. ISSN 1992 - 1950

[11] Zhang, T. and Zheng, G. (2010). Vibration Analysis of an Elastic Beam Subjected to a Moving Beam with Flexible Connections. America Association of Civil Engineers, 136(1), 120-130.

[12] Idowu, I.A., Gbolagade, A.W. and Olayiwola, M.O. (2009). The Response of loaded beam subjected to moving masses and external forces. Report and Opinion, 68-78.

[13] Gbadeyan, J.A. and Dada, M.S. (2007). The effects of linearly Varying Distributed Moving Loads on Beams. Journal of Engineering and Applied Sciences, 2(6), Pp 1006-1011

[14] Friswell, M.I., Adhikan, S. and Lei, Y. (2007). Vibration Analysis of Beams with non-local foundations using the finite element method.International Journals of Numerical Methods in Engineering, 71, 1365-1386.

[15] M. C. Agarana, and J. A. Gbadeyan, "Finite Difference Dynamic Analysis of Railway Bridges Supported by Pasternak Foundation under Uniform Partially Distributed Moving Railway Vehicle," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2015, 21-23 October, 2015, San Francisco, USA, pp996-1000

[16] Agarana M.C., Gbadeyan J.A., and Ajayi O.O., "Dynamic Response of Inclined Isotropic Elastic Damped Rectangular Mindlin Plate resting on Pasternak Foundation under a Moving Load," Lecture Notes in Engineering and Computer Science: Proceedings of The International MultiConference of Engineers and Computer Scientists 2016, 16-18 March, 2016, Hong Kong, pp713-718

[17] Agarana, M. C. and Gbadeyan, J.A. (2016) *A Comparison of Dynamic Behaviours of Mindlin, Shear, Rotatory and Kirchoff Plates Supported by Subgrade under Moving Load.* In: International Conference on "Engineering & Technology, Computer, Basic & Applied Science , January 22-23, 2016. , Dubai, UAE.

[18] Agarana M.C., and Ede A.N., "Application of Differential Transform Method to Vibration Analysis of Damped Railway Bridge on Pasternak Foundation under Moving Train," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2016, 29 June - 1 July, 2016, London, U.K., pp1177-1179

[19] Agarana M.C., Gbadeyan J.A., and Ajayi O.O., "Dynamic Response of Inclined Isotropic Elastic Damped Rectangular Mindlin Plate resting on Pasternak Foundation under a Moving Load," Lecture Notes in Engineering and Computer Science: Proceedings of The International MultiConference of Engineers and Computer Scientists 2016, 16-18 March, 2016, Hong Kong, pp713-718

ISBN: 978-988-14048-4-8
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)