Classes of Ordinary Differential Equations Obtained for the Probability Functions of Half-Normal Distribution

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***Abstract*— In this paper, the differential calculus was used to obtain some classes of ordinary differential equations (ODE) for the probability density function (PDF), quantile function, survival function, inverse survival function, hazard function and reversed hazard function of the half-normal distribution. The stated necessary conditions required for the existence of the ODEs are consistent with the various parameters that defined the distribution. Higher orders ODEs for the PDF yielded different number of ODEs which increases as the order increases.**

***Index Terms*— Differentiation, Ordinary differential equations, half-normal, probability density function, quantile function.**

I. Introduction

HALF-NORMAL distribution is a normal distribution with a mean set at zero and parameterized to the domain of positive real numbers and zero being the lower bound. Pewsey [1] [2] worked on the improved statistical inference for the distribution while [3] proposed unbiased estimators for the parameters of the distribution, which according to them, performs better than the traditional maximum likelihood. Some generalizations are available for the distribution such as: the extended generalized half-normal distribution [4], beta generalized half-normal distribution [5], generalized half-normal distribution [6-7], discrete half-normal distribution [8], an extension of the half-normal distribution called the slashed half-normal distribution [9], Kumaraswamy generalized half-normal distribution [10], beta generalized half-normal geometric distribution [11], gamma half-normal distribution [12] and alpha half-Normal Slash distribution [13]. Also available are epsilon half-normal distribution [14]. The distribution is a sub-models of exponentiated generalized gamma distribution proposed by [15] and generalized half-t distribution by [16]. Also available is the odd log-logistic generalized half-normal lifetime distribution [17].

In addition, the distribution was generalized with the Airy model to obtain the M-Wright distribution [18]. Details of the new method of generating the distribution is given in [19] and its application to quality control were highlighted by [20]. Lang [21] used the distribution to model wage gap between immigrants and natives in Germany. The distribution was among those used by Schoenberg et al. [22] in modeling of the distribution of the sizes of wildfire.

The aim of this research is to develop ordinary differential equations (ODE) for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of half-normal distribution by the use of differential calculus. Calculus is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. The research is an extension of the ODE to other probability functions other than the PDF. Solutions of these ODEs by using numerous available methods are new ways of understanding the nature of the probability functions that characterize the distribution. The method can be extended to other probability distributions, functions and can serve an alternative to approximation. Similar works done where the PDF of probability distributions was expressed as ODE whose solution is the PDF are available. They include: Laplace distribution [23], beta distribution [24], raised cosine distribution [25], Lomax distribution [26], beta prime distribution or inverted beta distribution [27].

II. Probability Density Function

The probability density function of the half-normal distribution is given by;

$$f(x)=\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{2\sigma^2}} \tag{1}$$

To obtain the first order ordinary differential equation for the probability density function of the half-normal distribution, differentiate equation (1), to obtain;

Manuscript received June 30, 2017; revised July 25, 2017. This work was sponsored by Covenant University, Ota, Nigeria.

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$$f'(x) = -\frac{x\sqrt{2}}{\sigma^3\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}} \tag{2}$$

The condition necessary for the existence of the equation is $\sigma > 0$.

Simplify equation (2) using equation (1);

$$f'(x) = -\left(\frac{x}{\sigma^2}\right)\frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}} = -\frac{x}{\sigma^2} f(x) \tag{3}$$

The first order ordinary differential for the probability density function of the half-normal distribution is given as;

$$\sigma^2 f'(x) + x f(x) = 0 \tag{4}$$

$$f(1) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{1}{2\sigma^2}} \tag{5}$$

To obtain the second order ordinary differential equation for the probability density function of the half-normal distribution, differentiate equation (2) to obtain;

$$f''(x) = -\frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{-\frac{x^2}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} + e^{-\frac{x^2}{2\sigma^2}}\right\}$$

(6) $$f''(x) = \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{\frac{x^2}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} - e^{-\frac{x^2}{2\sigma^2}}\right\}$$

(7) The condition necessary for the existence of the equation is $\sigma > 0$.

Two differential equations can be obtained from the simplification of equation (6). They are listed as ODE 1 and 2.

ODE 1;

$$f''(x) = -\frac{\sqrt{2}}{\sigma^3\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}}\left\{-\frac{x^2}{\sigma^2} + 1\right\} \tag{8}$$

$$f''(x) = -\frac{f(x)}{\sigma^2}\left\{\frac{-x^2 + \sigma^2}{\sigma^2}\right\} \tag{9}$$

$$\sigma^4 f''(x) + (\sigma^2 - x^2) f(x) = 0 \tag{10}$$

ODE 2;

$$f''(x) = -\left\{-\frac{x^2\sqrt{2}}{\sigma^5\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}} + \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}}\right\} \tag{11}$$

$$f''(x) = -\left\{\frac{x f'(x)}{\sigma^2} + \frac{f(x)}{\sigma^2}\right\} \tag{12}$$

$$\sigma^2 f''(x) + x f'(x) + f(x) = 0 \tag{13}$$

$$f'(1) = -\frac{\sqrt{2}}{\sigma^3\sqrt{\pi}} e^{-\frac{1}{2\sigma^2}} \tag{14}$$

To obtain the third order ordinary differential equation for the probability density function of the half-normal distribution, differentiate equation (6) to obtain;

$$f'''(x) = \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{-\frac{x^3}{\sigma^4} e^{-\frac{x^2}{2\sigma^2}} + \frac{2x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} + \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}\right\} \tag{15}$$

$$f'''(x) = \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{-\frac{x^3}{\sigma^4} e^{-\frac{x^2}{2\sigma^2}} + \frac{3x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}\right\} \tag{16}$$

The condition necessary for the existence of the equation is $\sigma > 0$.

Six differential equations can be obtained from the simplification of equations (15) and (16).

ODE 1;

Simplify equation (15) using equation (1);

$$f'''(x) = \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}}\left\{-\frac{x^3}{\sigma^4} + \frac{2x}{\sigma^2} + \frac{x}{\sigma^2}\right\} \tag{17}$$

$$f'''(x) = \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}}\left\{-\frac{x^3}{\sigma^4} + \frac{3x}{\sigma^2}\right\} = \frac{f(x)}{\sigma^2}\left\{-\frac{x^3}{\sigma^4} + \frac{3x}{\sigma^2}\right\} \tag{18}$$

$$\sigma^6 f'''(x) - (3\sigma^2 x - x^3) f(x) = 0 \tag{19}$$

ODE 2;

Simplify equation (15) using equation (2);

$$f'''(x) = \frac{x\sqrt{2}}{\sigma^3\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}}\left\{-\frac{x^2}{\sigma^4} + \frac{2}{\sigma^2} + \frac{1}{\sigma^2}\right\} \tag{20}$$

$$f'''(x) = -f'(x)\left\{-\frac{x^2}{\sigma^4} + \frac{3}{\sigma^2}\right\} \tag{21}$$

$$\sigma^4 f'''(x) + (3\sigma^2 - x^2) f'(x) = 0 \tag{22}$$

ODE 3;

Simplify equation (15) using equations (1) and (2);

$$f'''(x) = -\frac{x^3\sqrt{2}}{\sigma^7\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}} + \frac{2x\sqrt{2}}{\sigma^5\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}} + \frac{x\sqrt{2}}{\sigma^5\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}} \tag{23}$$

$$\begin{aligned} f'''(x) &= -\frac{x^2 f'(x)}{\sigma^4} + \frac{2x f(x)}{\sigma^4} + \frac{x f(x)}{\sigma^4} \\ &= -\frac{x^2 f'(x)}{\sigma^4} + \frac{3x f(x)}{\sigma^4} \end{aligned} \tag{24}$$

$$\sigma^4 f'''(x) - x^2 f'(x) - 3x f(x) = 0 \tag{25}$$

ODE 4;

Simplify equation (15) using equations (1) and (6);

$$f'''(x) = -\frac{x\sqrt{2}}{\sigma^5\sqrt{\pi}}\left\{\frac{x^2}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} - e^{-\frac{x^2}{2\sigma^2}}\right\} + \frac{2x\sqrt{2}}{\sigma^5\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}} \tag{26}$$

$$f'''(x) = -\frac{x f''(x)}{\sigma^2} + \frac{2x f(x)}{\sigma^4} \tag{27}$$

$$\sigma^4 f'''(x) + \sigma^2 x f''(x) - 2x f(x) = 0 \tag{28}$$

ODE 5;

Simplify equation (26) using equations (2) and (6);

$$f'''(x) = -\frac{x f''(x)}{\sigma^2} - \frac{2 f'(x)}{\sigma^2} \tag{29}$$

$$\sigma^2 f'''(x) + x f''(x) + 2 f'(x) = 0 \tag{30}$$

ODE 6;

ISBN: 978-988-14048-4-8
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Simplify equation (15) using equations (1), (2) and (6);

$$f'''(x) = -\frac{x\sqrt{2}}{\sigma^5\sqrt{\pi}}\left\{\frac{x^2}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}} - e^{-\frac{x^2}{2\sigma^2}}\right\} + \frac{x\sqrt{2}}{\sigma^5\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}} + \frac{x\sqrt{2}}{\sigma^5\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}} \quad (31)$$

$$f'''(x) = -\frac{xf''(x)}{\sigma^2} - \frac{f'(x)}{\sigma^2} + \frac{xf(x)}{\sigma^4} \quad (32)$$

$$\sigma^4 f'''(x) + \sigma^2 x f''(x) + \sigma^2 f'(x) - xf(x) = 0 \quad (33)$$

To obtain the fourth order ordinary differential equation for the probability density function of the half-normal distribution, differentiate equation (15) to obtain;

$$f^{iv}(x) = \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{\begin{matrix}\frac{x^4}{\sigma^6}e^{-\frac{x^2}{2\sigma^2}} - \frac{3x^2}{\sigma^4}e^{-\frac{x^2}{2\sigma^2}} - \frac{2x^2}{\sigma^4}e^{-\frac{x^2}{2\sigma^2}} \\ + \frac{2}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}} - \frac{x^2}{\sigma^4}e^{-\frac{x^2}{2\sigma^2}} + \frac{1}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}\end{matrix}\right\} \quad (34)$$

$$f^{iv}(x) = \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{\frac{x^4}{\sigma^6}e^{-\frac{x^2}{2\sigma^2}} - \frac{6x^2}{\sigma^4}e^{-\frac{x^2}{2\sigma^2}} + \frac{3}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}\right\} \quad (35)$$

The condition necessary for the existence of the equation is $\sigma > 0$.

Twelve differential equations can be obtained from the simplification of equations (34) and (35).

ODE 1;

Simplify equation (34) using equation (1);

$$f^{iv}(x) = \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}\left\{\begin{matrix}\frac{x^4}{\sigma^6} - \frac{3x^2}{\sigma^4} - \frac{2x^2}{\sigma^4} \\ + \frac{2}{\sigma^2} - \frac{x^2}{\sigma^4} + \frac{1}{\sigma^2}\end{matrix}\right\} \quad (36)$$

$$f^{iv}(x) = \frac{f(x)}{\sigma^2}\left\{\frac{x^4}{\sigma^6} - \frac{6x^2}{\sigma^4} + \frac{3}{\sigma^2}\right\} \quad (37)$$

$$\sigma^8 f^{iv}(x) - (x^4 - 6\sigma^2x^2 + 3\sigma^4) f(x) = 0 \quad (38)$$

ODE 2;

Simplify equation (35) using equation (2);

$$f^{iv}(x) = \frac{x\sqrt{2}}{\sigma^3\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}\left\{\frac{x^3}{\sigma^6} - \frac{6x}{\sigma^4}\right\} - \frac{x\sqrt{2}}{\sigma^3\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}\left\{-\frac{3}{x\sigma^2}\right\} \quad (39)$$

$$f^{iv}(x) = -f'(x)\left\{\frac{x^3}{\sigma^6} - \frac{6x}{\sigma^4}\right\} + f'(x)\left\{-\frac{3}{x\sigma^2}\right\} \quad (40)$$

$$x\sigma^6 f^{iv}(x) + (x^4 - 6\sigma^2x^2 + 3\sigma^4) f'(x) = 0 \quad (41)$$

ODE 3;

Simplify equation (39) using equations (1) and (2);

$$f^{iv}(x) = \frac{x\sqrt{2}}{\sigma^3\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}\left\{\frac{x^3}{\sigma^6} - \frac{6x}{\sigma^4}\right\} + \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}\left\{\frac{3}{\sigma^2}\right\} \quad (42)$$

$$f^{iv}(x) = -f'(x)\left\{\frac{x^3}{\sigma^6} - \frac{6x}{\sigma^4}\right\} + \frac{f(x)}{\sigma^2}\left\{\frac{3}{\sigma^2}\right\} \quad (43)$$

$$\sigma^6 f^{iv}(x) + (x^3 - 6\sigma^2 x) f'(x) - 3\sigma^4 f(x) = 0 \quad (44)$$

ODE 4;

Simplify equation (34) using equations (1) and (6);

$$f^{iv}(x) = \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{\begin{matrix}\frac{x^4}{\sigma^6}e^{-\frac{x^2}{2\sigma^2}} - \frac{x^2}{\sigma^4}e^{-\frac{x^2}{2\sigma^2}} \\ -\frac{3x^2}{\sigma^4}e^{-\frac{x^2}{2\sigma^2}} + \frac{3}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}} - \frac{2x^2}{\sigma^4}e^{-\frac{x^2}{2\sigma^2}}\end{matrix}\right\} \quad (45)$$

$$f^{iv}(x) = \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{\begin{matrix}\frac{x^2}{\sigma^4}\left(\frac{x^2}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}} - e^{-\frac{x^2}{2\sigma^2}}\right) \\ -\frac{3}{\sigma^2}\left(\frac{x^2}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}} - e^{-\frac{x^2}{2\sigma^2}}\right)\end{matrix}\right\} - \frac{2x^2\sqrt{2}}{\sigma^7\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}} \quad (46)$$

$$f^{iv}(x) = \frac{x^2 f''(x)}{\sigma^4} - \frac{3f''(x)}{\sigma^2} - \frac{2x^2 f(x)}{\sigma^6} \quad (47)$$

$$\sigma^6 f^{iv}(x) - (\sigma^2x^2 - 3\sigma^4) f''(x) + 2x^2 f(x) = 0 \quad (48)$$

ODE 5;

Simplify equation (46) using equations (2) and (6);

$$f^{iv}(x) = \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{\begin{matrix}\frac{x^2}{\sigma^4}\left(\frac{x^2}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}} - e^{-\frac{x^2}{2\sigma^2}}\right) \\ -\frac{3}{\sigma^2}\left(\frac{x^2}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}} - e^{-\frac{x^2}{2\sigma^2}}\right)\end{matrix}\right\} + \frac{2x}{\sigma^4}\left\{-\frac{x\sqrt{2}}{\sigma^3\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}\right\} \quad (49)$$

$$f^{iv}(x) = \frac{x^2 f''(x)}{\sigma^4} - \frac{3f''(x)}{\sigma^2} + \frac{2xf'(x)}{\sigma^4} \quad (50)$$

$$\sigma^4 f^{iv}(x) - (x^2 - 3\sigma^2) f''(x) - 2xf'(x) = 0 \quad (51)$$

ODE 6;

Simplify equation (34) using equations (1) and (15);

$$f^{iv}(x) = -\frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{-\frac{x^4}{\sigma^6}e^{-\frac{x^2}{2\sigma^2}} + \frac{3x^2}{\sigma^4}e^{-\frac{x^2}{2\sigma^2}}\right\} + \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{-\frac{3x^2}{\sigma^4}e^{-\frac{x^2}{2\sigma^2}} + \frac{3}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}\right\} \quad (52)$$

$$f^{iv}(x)=-\frac{x\sqrt{2}}{\sigma^5\sqrt{\pi}}\left\{-\frac{x^3}{\sigma^4}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}+\frac{3x}{\sigma^2}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\right\}+\frac{3\sqrt{2}}{\sigma^4\sqrt{\pi}}\left\{-\frac{x^2}{\sigma^3}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}+\frac{1}{\sigma}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\right\} \tag{53}$$

$$f^{iv}(x)=-\frac{xf'''(x)}{\sigma^2}+\frac{3f(x)}{\sigma^3}\left\{-\frac{x^2}{\sigma^3}+\frac{1}{\sigma}\right\} \tag{54}$$

$$\sigma^6 f^{iv}(x)+x\sigma^4 f'''(x)-3(\sigma^2-x^2)f(x)=0 \tag{55}$$

ODE 7;
Simplify equation (53) using equations (2) and (15);

$$f^{iv}(x)=-\frac{x\sqrt{2}}{\sigma^5\sqrt{\pi}}\left\{-\frac{x^3}{\sigma^4}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}+\frac{3x}{\sigma^2}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\right\}+\left(\frac{3x}{\sigma^4}\right)\left(-\frac{x\sqrt{2}}{\sigma^3\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\right)+\left(-\frac{3}{x\sigma^2}\right)\left(-\frac{x\sqrt{2}}{\sigma^3\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\right) \tag{56}$$

$$f^{iv}(x)=-\frac{xf'''(x)}{\sigma^2}+\frac{3xf'(x)}{\sigma^4}-\frac{3f'(x)}{x\sigma^2} \tag{57}$$

$$x\sigma^4 f^{iv}(x)+x^2\sigma^2 f'''(x)+3(\sigma^2-x^2)f'(x)=0 \tag{58}$$

ODE 8;
Simplify equation (56) using equations (6) and (15);

$$f^{iv}(x)=-\frac{x\sqrt{2}}{\sigma^5\sqrt{\pi}}\left\{-\frac{x^3}{\sigma^4}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}+\frac{3x}{\sigma^2}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\right\}+\left(\frac{3\sqrt{2}}{\sigma^5\sqrt{\pi}}\right)\left(-\frac{x^2}{\sigma^2}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}+\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\right) \tag{59}$$

$$f^{iv}(x)=-\frac{xf'''(x)}{\sigma^2}-\frac{3f''(x)}{\sigma^2} \tag{60}$$

$$\sigma^2 f^{iv}(x)+xf'''(x)+3f''(x)=0 \tag{61}$$

ODE 9;
Simplify equation (34) using equations (1), (2) and (6);

$$f^{iv}(x)=\frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{\begin{array}{l}\frac{x^4}{\sigma^6}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}-\frac{x^2}{\sigma^4}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}-\frac{2x^2}{\sigma^4}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\\+\frac{2}{\sigma^2}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}-\frac{3x^2}{\sigma^4}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}+\frac{1}{\sigma^2}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\end{array}\right\} \tag{62}$$

$$f^{iv}(x)=\frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{\begin{array}{l}-\frac{x^2}{\sigma^4}\left(-\frac{x^2}{\sigma^2}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}+\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\right)\\+\frac{2}{\sigma^2}\left(-\frac{x^2}{\sigma^2}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}+\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\right)\end{array}\right\}-\left(\frac{3x}{\sigma^4}\right)\frac{x\sqrt{2}}{\sigma^3\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}+\left(\frac{1}{\sigma^4}\right)\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{2\sigma^2}} \tag{63}$$

$$f^{iv}(x)=\frac{x^2f''(x)}{\sigma^4}-\frac{2f''(x)}{\sigma^2}+\frac{3xf'(x)}{\sigma^4}+\frac{f(x)}{\sigma^4} \tag{64}$$

$$\sigma^4 f^{iv}(x)-(x^2-2\sigma^2)f''(x)-3xf'(x)-f(x)=0 \tag{65}$$

ODE 10;
Simplify equation (35) using equations (1), (2) and (15);

$$f^{iv}(x)=\frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{\begin{array}{l}\left(\frac{x^4}{\sigma^6}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}-\frac{3x^2}{\sigma^4}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\right)\\-\frac{3x^2}{\sigma^4}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}+\frac{3}{\sigma^2}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\end{array}\right\} \tag{66}$$

$$f^{iv}(x)=\frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{-\frac{x}{\sigma^2}\left(-\frac{x^3}{\sigma^4}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}+\frac{3x}{\sigma^2}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\right)\right\}-\left(\frac{3x}{\sigma^4}\right)\frac{x\sqrt{2}}{\sigma^3\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}+\frac{3\sqrt{2}}{\sigma^5\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{2\sigma^2}} \tag{67}$$

$$f^{iv}(x)=-\frac{xf'''(x)}{\sigma^2}+\frac{3xf'(x)}{\sigma^4}+\frac{3f(x)}{\sigma^4} \tag{68}$$

$$\sigma^4 f^{iv}(x)+\sigma^2xf'''(x)-3xf'(x)-3f(x)=0 \tag{69}$$

ODE 11;
Simplify equation (63) using equations (1), (6) and (15);

$$f^{iv}(x)=\frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{\begin{array}{l}-\frac{x}{\sigma^2}\left(-\frac{x^3}{\sigma^4}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}+\frac{3x}{\sigma^2}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\right)\\+\frac{2}{\sigma^2}\left(-\frac{x^2}{\sigma^2}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}+\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\right)\end{array}\right\}-\frac{x^2\sqrt{2}}{\sigma^7\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}+\frac{\sqrt{2}}{\sigma^5\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{2\sigma^2}} \tag{70}$$

$$f^{iv}(x)=-\frac{xf'''(x)}{\sigma^2}-\frac{2f''(x)}{\sigma^2}-\frac{x^2f(x)}{\sigma^6}+\frac{f(x)}{\sigma^4} \tag{71}$$

$$\sigma^6 f^{iv}(x)+\sigma^4xf'''(x)+2\sigma^4f''(x)+(x^2-\sigma^2)f(x)=0 \tag{72}$$

ODE 12;
Simplify equation (70) using equations (2), (6) and (15);

$$f^{iv}(x)=\frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\left\{\begin{array}{l}-\frac{x}{\sigma^2}\left(-\frac{x^3}{\sigma^4}+\frac{3x}{\sigma^2}\right)\\+\frac{2}{\sigma^2}\left(-\frac{x^2}{\sigma^2}+1\right)\end{array}\right\}-\frac{x^2\sqrt{2}}{\sigma^7\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}+\left(-\frac{1}{x}\right)\left(-\frac{x\sqrt{2}}{\sigma^5\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\right) \tag{73}$$

ISBN: 978-988-14048-4-8
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$f^{iv}(x) = -\frac{xf'''(x)}{\sigma^2} - \frac{2f''(x)}{\sigma^2} + \frac{xf'(x)}{\sigma^4} - \frac{f'(x)}{x\sigma^2} \quad (74)$$

$$x\sigma^4 f^{iv}(x) + \sigma^2 x^2 f'''(x) + 2\sigma^2 x f''(x) - (x^2 - \sigma^2) f'(x) = 0 \quad (75)$$

$$f'''(1) = \frac{\sqrt{2}\left(3\sigma^2 - 1\right)}{\sqrt{\pi}\sigma^7} e^{-\frac{1}{2\sigma^2}} \quad (76)$$

III. Quantile Function

The Quantile function of the half-normal distribution is given by;

$$Q(p) = \sigma\sqrt{2} erf^{-1}(p) \quad (77)$$

To obtain the first order ordinary differential equation for the Quantile function of the half-normal distribution, differentiate equation (77), to obtain;

$$Q'(p) = \frac{\sigma\sqrt{2\pi}}{2} e^{[erf^{-1}(p)]^2} \quad (78)$$

The condition necessary for the existence of the equation is $\sigma > 0, 0 \le p < 1$.

Simplify equation (78) using equation (77), however equation (77) becomes;

$$\frac{Q(p)}{\sigma\sqrt{2}} = erf^{-1}(p) \quad (79)$$

$$Q'(p) = \frac{\sigma\sqrt{2\pi}}{2} e^{\frac{Q^2(p)}{2\sigma^2}} \quad (80)$$

$$\ln Q'(p) = \ln\left(\frac{\sigma\sqrt{2\pi}}{2}\right) + \frac{Q^2(p)}{2\sigma^2} \quad (81)$$

$$2\sigma^2 \ln Q'(p) - Q^2(p) - 2\sigma^2 g = 0 \quad (82)$$

Where $g = \ln\left(\frac{\sigma\sqrt{2\pi}}{2}\right)$ (83)

$$Q(0) = 0 \quad (84)$$

To obtain the second order ordinary differential equation for the Quantile function of the half-normal distribution, differentiate equation (78), to obtain;

$$Q''(p) = \frac{\sigma\pi\sqrt{2}}{2} erf^{-1}(p)\left(e^{[erf^{-1}(p)]^2}\right)^2 \quad (85)$$

The condition necessary for the existence of the equation is $\sigma > 0, 0 \le p < 1$.

Substitute equation (77) into equation (85);

$$Q''(p) = \frac{\pi}{2} Q(p)\left(e^{[erf^{-1}(p)]^2}\right)^2 \quad (86)$$

The following equations obtained from the simplification of equation (78) are needed to simplify equation (86);

$$Q'^2(p) = \frac{\pi\sigma^2}{2}\left(e^{[erf^{-1}(p)]^2}\right)^2 \quad (87)$$

$$\frac{Q'^2(p)}{\sigma^2} = \frac{\pi}{2}\left(e^{[erf^{-1}(p)]^2}\right)^2 \quad (88)$$

Substitute equation (88) into equation (86);

$$Q''(p) = Q(p)\frac{Q'^2(p)}{\sigma^2} \quad (89)$$

$$\sigma^2 Q''(p) - Q(p)Q'^2(p) = 0 \quad (90)$$

$$Q'(0) = \frac{\sigma\sqrt{2\pi}}{2} \quad (91)$$

IV. Survival Function

The survival function of the half-normal distribution is given by;

$$S(t) = 1 - erf\left(\frac{t}{\sigma\sqrt{2}}\right) \quad (92)$$

To obtain the first order ordinary differential equation for the Survival function of the half-normal distribution, differentiate equation (92), to obtain;

$$S'(t) = -\frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{t^2}{2\sigma^2}} = -f(t) \quad (93)$$

The condition necessary for the existence of the equation is $\sigma > 0$.

$$S'(t) + f(t) = 0 \quad (94)$$

$$S(0) = 1 \quad (95)$$

The second and third order ordinary differential equations for the Survival function of the half-normal distribution can also be obtained using equation (93);

$$\sigma^2 S''(t) + tS'(t) = 0 \quad (96)$$

$$S'(0) = -\frac{\sqrt{2}}{\sigma\sqrt{\pi}} \quad (97)$$

$$\sigma^4 S'''(t) + (\sigma^2 - t^2)S'(t) = 0 \quad (98)$$

$$\sigma^2 S'''(t) + \sigma t S''(t) + S(t) = 0 \quad (99)$$

V. Inverse Survival Function

The inverse survival function of the half-normal distribution is given by;

$$Q(p) = \sigma\sqrt{2} erf^{-1}(1-p) \quad (100)$$

To obtain the first order ordinary differential equation for the Quantile function of the half-normal distribution, differentiate equation (100), to obtain;

$$Q'(p) = -\frac{\sigma\sqrt{2\pi}}{2} e^{[erf^{-1}(1-p)]^2} \quad (101)$$

The condition necessary for the existence of the equation is $\sigma > 0, 0 \le p < 1$.

Simplify equation (101) using equation (100), however equation (100) becomes;

$$\frac{Q(p)}{\sigma\sqrt{2}} = erf^{-1}(1-p) \quad (102)$$

$$Q'(p) = -\frac{\sigma\sqrt{2\pi}}{2} e^{\frac{Q^2(p)}{2\sigma^2}} \quad (103)$$

ISBN: 978-988-14048-4-8
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$\ln Q'(p) = -\ln\left(\frac{\sigma\sqrt{2\pi}}{2}\right) - \frac{Q^2(p)}{2\sigma^2} \quad (104)$$

$$2\sigma^2 \ln Q'(p) + Q^2(p) + 2\sigma^2 g = 0 \quad (105)$$

Where $g = \ln\left(\frac{\sigma\sqrt{2\pi}}{2}\right)$ (106)

$$Q(0) = 0 \quad (107)$$

VI. Hazard Function

The hazard function of the half-normal distribution is given by; $h(t) = \left(\frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{t^2}{2\sigma^2}}\right)\left(1 - erf\left(\frac{t}{\sigma\sqrt{2}}\right)\right)^{-1}$

(108) To obtain the first order ordinary differential equation for the Hazard function of the half-normal distribution, differentiate equation (108), to obtain;

$$h'(t) = \left\{ -\frac{\frac{t\sqrt{2}}{\sigma^3\sqrt{\pi}} e^{-\frac{t^2}{2\sigma^2}}}{\frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{t^2}{2\sigma^2}}} - \frac{\left(\frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{t^2}{2\sigma^2}}\right)\left(1 - erf\left(\frac{t}{\sigma\sqrt{2}}\right)\right)^{-2}}{\left(1 - erf\left(\frac{t}{\sigma\sqrt{2}}\right)\right)^{-1}} \right\} h(t) \quad (109)$$

$$h'(t) = -\left\{ \frac{t}{\sigma^2} + \frac{\left(\frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{t^2}{2\sigma^2}}\right)}{\left(1 - erf\left(\frac{t}{\sigma\sqrt{2}}\right)\right)} \right\} h(t) \quad (110)$$

The condition necessary for the existence of the equation is $\sigma > 0$.

$$h'(t) = -\left\{\frac{t}{\sigma^2} + h(t)\right\} h(t) \quad (111)$$

$$\sigma^2 h'(t) = -(t + \sigma^2 h(t))h(t) \quad (112)$$

$$\sigma^2 h'(t) + \sigma^2 h^2(t) + th(t) = 0 \quad (113)$$

$$h(0) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \quad (114)$$

Higher order ordinary differential equations can also be obtained such as;

$$\sigma^2 h''(t) + (t + 2\sigma^2 h(t))h'(t) + h(t) = 0 \quad (115)$$

$$\sigma^2 h'''(t) + (t + 2\sigma^2 h(t))h''(t) + 2\sigma^2 h'^2(t) + h'(t) + h(t) = 0 \quad (116)$$

VII. Reversed Hazard Function

The reversed hazard function of the half-normal distribution is given by;

$$j(t) = \frac{\frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{t^2}{2\sigma^2}}}{erf\left(\frac{t}{\sigma\sqrt{2}}\right)} = \left(\frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{t^2}{2\sigma^2}}\right)\left(erf\left(\frac{t}{\sigma\sqrt{2}}\right)\right)^{-1} \quad (117)$$

To obtain the first order ordinary differential equation for the reversed hazard function of the half-normal distribution, differentiate equation (117), to obtain;

$$j'(t) = -\left\{ \frac{t}{\sigma^2} + \frac{\left(\frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{t^2}{2\sigma^2}}\right)}{\left(erf\left(\frac{t}{\sigma\sqrt{2}}\right)\right)} \right\} j(t) \quad (118)$$

The condition necessary for the existence of the equation is $\sigma > 0$.

$$j'(t) = -\left\{\frac{t}{\sigma^2} + j(t)\right\} j(t) \quad (119)$$

$$\sigma^2 j'(t) + \sigma^2 j^2(t) + tj(t) = 0 \quad (120)$$

$$j(1) = \frac{\frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{1}{2\sigma^2}}}{erf\left(\frac{1}{\sigma\sqrt{2}}\right)} \quad (121)$$

The ODEs of all the probability functions considered can be obtained for the particular values of the distribution. Several analytic, semi-analytic and numerical methods can be applied to obtain the solutions of the respective differential equations [28-43]. Also comparison with two or more solution methods is useful in understanding the link between ODEs and the probability distributions.

VIII. Concluding Remarks

In this work, differentiation was used to obtain some classes of ordinary differential equations for the probability density function (PDF), quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of the half-normal distribution. The number of ODEs increases with the order of the differential equation for the case of the PDF. Generally, the work was simplified by the application of simple algebraic procedures and the ODE of higher orders is easily obtained from the lower ones. In all, the parameters that define the distribution determine the nature of the respective ODEs and the range determines the existence of the ODEs.

Acknowledgment

The authors are unanimous in appreciation of financial sponsorship from Covenant University. The constructive suggestions of the reviewers are greatly appreciated.

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