Classes of Ordinary Differential Equations Obtained for the Probability Functions of Harris Extended Exponential Distribution

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***Abstract*— In this paper, the differential calculus was used to obtain some classes of ordinary differential equations (ODE) for the probability density function, quantile function, survival function, inverse survival function and hazard function of the Harris extended exponential distribution. The case of reversed hazard function was excluded because of its complexity. The stated necessary conditions required for the existence of the ODEs are consistent with the various parameters that defined the distribution. Solutions of these ODEs by using numerous available methods are new ways of understanding the nature of the probability functions that characterize the distribution. The method can be extended to other probability distributions, functions and can serve an alternative to estimation and approximation.**

***Index Terms*— Harris, exponential, differentiation, quantile function, survival function.**

I. Introduction

THIS is a lifetime probability model proposed by [1]. The distribution is as a result of mixture of Harris [2] and exponential distributions and applications of the distribution as a flexible model was shown by [3]. The distribution is a lifetime with decreasing failure rate [4-5]. Alternatively the distribution can be regarded as an extension or modification of the exponential distribution.

The aim of this paper is to develop ordinary differential equations (ODE) for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF) and hazard function (HF) of Harris extended exponential distribution by the use of differential calculus. Calculus is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. The research is an extension of the ODE to other probability functions other than the PDF. Similar works done where the PDF of probability distributions was expressed as ODE whose solution is the PDF are available. They include: Laplace distribution [6], beta distribution [7], raised cosine distribution [8], Lomax distribution [9], beta prime distribution or inverted beta distribution [10].

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II. Probability Density Function

The probability density function of the Harris extended exponential distribution (HEE) is given as;

$$f(x)=\frac{\lambda\theta^{\frac{1}{k}}\mathrm{e}^{-\lambda x}}{(1-\bar{\theta}\mathrm{e}^{-\lambda kx})^{1+\frac{1}{k}}} \tag{1}$$

Where $\bar{\theta}=1-\theta$, $x,\theta,\bar{\theta},\lambda,k>0$

To obtain the first order ordinary differential equation for the probability density function of the HEE distribution, differentiate equation (1), to obtain;

$$f'(x)=\left\{-\frac{\lambda\mathrm{e}^{-\lambda x}}{\mathrm{e}^{-\lambda x}}-\frac{\bar{\theta}\lambda(k+1)\mathrm{e}^{-\lambda kx}(1-\bar{\theta}\mathrm{e}^{-\lambda kx})^{-(2+\frac{1}{k})}}{(1-\bar{\theta}\mathrm{e}^{-\lambda kx})^{-(1+\frac{1}{k})}}\right\}f(x) \tag{2}$$

$$f'(x)=\left\{-\lambda-\frac{\bar{\theta}\lambda(k+1)\mathrm{e}^{-\lambda kx}}{(1-\bar{\theta}\mathrm{e}^{-\lambda kx})}\right\}f(x) \tag{3}$$

The condition necessary for the existence of equation is $x,\theta,\bar{\theta},\lambda,k>0$.

Differentiate equation (3);

$$f''(x)=\left\{-\lambda-\frac{\bar{\theta}\lambda(k+1)\mathrm{e}^{-\lambda kx}}{(1-\bar{\theta}\mathrm{e}^{-\lambda kx})}\right\}f'(x)+\left\{\frac{(k+1)k\bar{\theta}^2\lambda^2(\mathrm{e}^{-\lambda kx})^2}{(1-\bar{\theta}\mathrm{e}^{-\lambda kx})^2}+\frac{(k+1)k\bar{\theta}\lambda^2\mathrm{e}^{-\lambda kx}}{(1-\bar{\theta}\mathrm{e}^{-\lambda kx})}\right\}f(x) \tag{4}$$

The condition necessary for the existence of equation is $x,\theta,\bar{\theta},\lambda,k>0$.

The following equations obtained from (3) are required in the simplification of equation (4);

$$\frac{f'(x)}{f(x)}=-\lambda-\frac{\bar{\theta}\lambda(k+1)\mathrm{e}^{-\lambda kx}}{(1-\bar{\theta}\mathrm{e}^{-\lambda kx})} \tag{5}$$

$$\frac{\bar{\theta}\lambda(k+1)\mathrm{e}^{-\lambda kx}}{(1-\bar{\theta}\mathrm{e}^{-\lambda kx})}=-\left(\frac{f'(x)}{f(x)}+\lambda\right) \tag{6}$$

$$\frac{\bar{\theta}^2\lambda^2(k+1)^2(\mathrm{e}^{-\lambda kx})^2}{(1-\bar{\theta}\mathrm{e}^{-\lambda kx})^2}=\left(\frac{f'(x)}{f(x)}+\lambda\right)^2 \tag{7}$$

$$\frac{(k+1)\bar{\theta}^2\lambda^2(\mathrm{e}^{-\lambda kx})^2}{(1-\bar{\theta}\mathrm{e}^{-\lambda kx})^2}=\frac{1}{k+1}\left(\frac{f'(x)}{f(x)}+\lambda\right)^2 \tag{8}$$

$$\frac{k(k+1)(\bar{\theta}\lambda(\mathrm{e}^{-\lambda kx})^2}{(1-\bar{\theta}\mathrm{e}^{-\lambda kx})^2}=\frac{k}{k+1}\left(\frac{f'(x)}{f(x)}+\lambda\right)^2 \tag{9}$$

$$\frac{\bar{\theta}\lambda^2(k+1)\mathrm{e}^{-\lambda kx}}{(1-\bar{\theta}\mathrm{e}^{-\lambda kx})}=-\lambda\left(\frac{f'(x)}{f(x)}+\lambda\right) \tag{10}$$

$$\frac{\bar{\theta}\lambda^2k(k+1)\mathrm{e}^{-\lambda kx}}{(1-\bar{\theta}\mathrm{e}^{-\lambda kx})}=-k\lambda\left(\frac{f'(x)}{f(x)}+\lambda\right) \tag{11}$$

Substitute equations (5), (9) and (11) into equation (4);

$$f''(x)=\frac{f'^2(x)}{f(x)}+\left\{\begin{matrix}\frac{k}{k+1}\left(\frac{f'(x)}{f(x)}+\lambda\right)^2\\ -k\lambda\left(\frac{f'(x)}{f(x)}+\lambda\right)\end{matrix}\right\}f(x) \tag{12}$$

$$f(1)=\frac{\lambda\theta^{\frac{1}{k}}\mathrm{e}^{-\lambda}}{(1-\bar{\theta}\mathrm{e}^{-\lambda k})^{1+\frac{1}{k}}} \tag{13}$$

$$f'(1)=-\lambda\theta^{\frac{1}{k}}\mathrm{e}^{-\lambda}\left\{\frac{\lambda(1-\bar{\theta}\mathrm{e}^{-\lambda k})+\bar{\theta}\lambda(k+1)\mathrm{e}^{-\lambda k}}{(1-\bar{\theta}\mathrm{e}^{-\lambda k})^{2+\frac{1}{k}}}\right\} \tag{14}$$

The second order ordinary differential equation for the probability density function of the HEE distribution can be obtained and evaluated for particular values of k and λ. When $k=1$, equation (12), (13) and (14) become;

$$f''(x)=\frac{f'^2(x)}{f(x)}+\left\{\begin{matrix}\frac{1}{2}\left(\frac{f'(x)}{f(x)}+\lambda\right)^2\\ -\lambda\left(\frac{f'(x)}{f(x)}+\lambda\right)\end{matrix}\right\}f(x) \tag{15}$$

$$f''(x)=\frac{3f'^2(x)}{2f(x)}-\frac{\lambda^2f(x)}{2} \tag{16}$$

$$2f(x)f''(x)-3f'^2(x)+\lambda^2f^2(x)=0 \tag{17}$$

$$f(1)=\frac{\lambda\theta\mathrm{e}^{-\lambda}}{(1-\bar{\theta}\mathrm{e}^{-\lambda})^2} \tag{18}$$

$$f'(1)=-\lambda\theta\mathrm{e}^{-\lambda}\left\{\frac{\lambda(1+\bar{\theta}\mathrm{e}^{-\lambda})}{(1-\bar{\theta}\mathrm{e}^{-\lambda})^3}\right\} \tag{19}$$

III. Quantile Function

The Quantile function of the Harris extended exponential distribution (HEE) is given as;

$$Q(p)=-\frac{1}{k\lambda}\ln\left(\frac{(1-p)^k}{\theta+\bar{\theta}(1-p)^k}\right) \tag{20}$$

Differentiate equation (20);

$$Q'(p)=-\frac{1}{k\lambda}\left[\frac{\theta+\bar{\theta}(1-p)^k}{(1-p)^k}\left(\begin{matrix}\frac{k\bar{\theta}(1-p)^k(1-p)^{k-1}}{(\theta+\bar{\theta}(1-p)^k)^2}\\ -\frac{k(1-p)^{k-1}}{\theta+\bar{\theta}(1-p)^k}\end{matrix}\right)\right] \tag{21}$$

$$Q'(p)=\frac{1}{\lambda(1-p)}-\frac{\bar{\theta}(1-p)^{k-1}}{\lambda(\theta+\bar{\theta}(1-p)^k)} \tag{22}$$

The condition necessary for the existence of equation is $\theta,\bar{\theta},\lambda,k>0, 0\le p<1$.

The first order ordinary differential equation for the quantile function of the HEE distribution can be obtained and evaluated for particular values of all the parameters especially k, p and λ. This is summarized in the **Table 1.**

Table 1: Classes of differential equations obtained for the quantile function of the HEE distribution for different parameters.

k	λ	θ	$\bar{\theta}$	Ordinary differential equation
1	1	1	1	$(1-p)(2-p)Q'(p)-1=0$
1	1	1	2	$(1-p)(3-2p)Q'(p)-1=0$
1	1	2	1	$(1-p)(3-p)Q'(p)-2=0$
1	2	1	1	$2(1-p)(2-p)Q'(p)-1=0$
1	1	2	2	$(1-p)(2-p)Q'(p)-1=0$
1	2	2	2	$2(1-p)(2-p)Q'(p)-1=0$
1	2	2	1	$(1-p)(3-p)Q'(p)-1=0$
1	2	1	2	$2(1-p)(3-2p)Q'(p)-1=0$
2	1	1	1	$(1-p)(1+(1-p)^2)Q'(p)-1=0$
2	2	1	1	$2(1-p)(1+(1-p)^2)Q'(p)-1=0$
2	2	2	1	$(1-p)(2+(1-p)^2)Q'(p)-1=0$
2	2	2	2	$(1-p)(2+2(1-p)^2)Q'(p)-1=0$
2	1	2	1	$(1-p)(2+(1-p)^2)Q'(p)-2=0$
2	1	1	2	$(1-p)(1+2(1-p)^2)Q'(p)-1=0$
2	1	2	2	$(1-p)(2+2(1-p)^2)Q'(p)-2=0$
2	2	1	2	$2(1-p)(1+2(1-p)^2)Q'(p)-1=0$

In order to obtain a more simplified differential equation that will be the function of the quantile function, differentiate equation (22);

$$Q''(p) = -\frac{1}{\lambda(1-p)^2} - \frac{\bar{\theta}^2 k((1-p)^{k-1})^2}{\lambda(\theta+\bar{\theta}(1-p)^k)^2} + \frac{\bar{\theta}(k-1)(1-p)^{k-2}}{\lambda(\theta+\bar{\theta}(1-p)^k)} \quad (23)$$

The condition necessary for the existence of equation is that $\theta, \bar{\theta}, \lambda, k > 0, 0 \le p < 1$.

The following equations obtained from (22) are needed in the simplification of (23).

$$\frac{\bar{\theta}(1-p)^{k-1}}{\lambda(\theta+\bar{\theta}(1-p)^k)} = \frac{1}{\lambda(1-p)} - Q'(p) \quad (24)$$

$$\frac{\bar{\theta}^2((1-p)^{k-1})^2}{\lambda^2(\theta+\bar{\theta}(1-p)^k)^2} = \left(\frac{1}{\lambda(1-p)} - Q'(p)\right)^2 \quad (25)$$

$$\frac{\bar{\theta}^2 k((1-p)^{k-1})^2}{\lambda(\theta+\bar{\theta}(1-p)^k)^2} = \lambda k\left(\frac{1}{\lambda(1-p)} - Q'(p)\right)^2 \quad (26)$$

$$\frac{\bar{\theta}(k-1)(1-p)^{k-1}}{\lambda(\theta+\bar{\theta}(1-p)^k)} = k-1\left(\frac{1}{\lambda(1-p)} - Q'(p)\right) \quad (27)$$

$$\frac{\bar{\theta}(k-1)(1-p)^{k-2}}{\lambda(\theta+\bar{\theta}(1-p)^k)} = \frac{k-1}{1-p}\left(\frac{1}{\lambda(1-p)} - Q'(p)\right) \quad (28)$$

Substitute equations (26) and (28) into equation (23);

$$Q''(p) = -\frac{1}{\lambda(1-p)^2} - \lambda k\left(\frac{1}{\lambda(1-p)} - Q'(p)\right)^2 + \frac{k-1}{1-p}\left(\frac{1}{\lambda(1-p)} - Q'(p)\right) \quad (29)$$

$$Q(0) = -\frac{1}{k\lambda}\ln\left(\frac{1}{\theta+\bar{\theta}}\right) \quad (30)$$

$$Q'(0) = \frac{1}{\lambda} - \frac{\bar{\theta}}{\lambda(\theta+\bar{\theta})} = \frac{\theta}{\lambda(\theta+\bar{\theta})} \quad (31)$$

The ordinary differential equations can be obtained for the particular values of k and λ.

When k = 1, equations (29)-(31) become;

$$Q''(p) = -\frac{1}{\lambda(1-p)^2} - \lambda\left(\frac{1}{\lambda(1-p)} - Q'(p)\right)^2 \quad (32)$$

$$Q''(p) = -\frac{2}{\lambda(1-p)^2} + \frac{2Q'(p)}{(1-p)} - \lambda Q'^2(p) \quad (33)$$

$$\lambda(1-p)^2 Q''(p) + \lambda^2(1-p)^2 Q'^2(p) - 2\lambda(1-p)Q'(p) + 2 = 0 \quad (34)$$

$$Q(0) = -\frac{1}{\lambda}\ln\left(\frac{1}{\theta+\bar{\theta}}\right) \quad (35)$$

$$Q'(0) = \frac{\theta}{\lambda(\theta+\bar{\theta})} \quad (36)$$

IV. Survival Function

The survival function of the Harris extended exponential distribution (HEE) is given as;

$$S(t) = \left[\frac{\theta e^{-\lambda kt}}{1-\bar{\theta}e^{-\lambda kt}}\right]^{\frac{1}{k}} \quad (37)$$

Differentiate equation (37);

$$S'(t) = \left\{ -\frac{\theta\lambda e^{-\lambda kt}(\theta e^{-\lambda kt})^{\frac{1}{k}-1}}{(\theta e^{-\lambda kt})^{\frac{1}{k}}} - \frac{\bar{\theta}\lambda e^{-\lambda kt}(1-\bar{\theta}e^{-\lambda kt})^{-(\frac{1}{k}+1)}}{(1-\bar{\theta}e^{-\lambda kt})^{-\frac{1}{k}}} \right\} S(t) \quad (38)$$

$$S'(t) = -\left\{\frac{\theta\lambda e^{-\lambda kt}}{\theta e^{-\lambda kt}} + \frac{\bar{\theta}\lambda e^{-\lambda kt}}{(1-\bar{\theta}e^{-\lambda kt})}\right\} S(t) \quad (39)$$

$$S'(t) = -\left\{\lambda + \frac{\bar{\theta}\lambda e^{-\lambda kt}}{(1-\bar{\theta}e^{-\lambda kt})}\right\} S(t) \quad (40)$$

The condition necessary for the existence of equation is that $t, \theta, \bar{\theta}, \lambda, k > 0$.

Equation (37) can also be written as;

$$S^k(t) = \frac{\theta e^{-\lambda kt}}{1-\bar{\theta}e^{-\lambda kt}} \quad (41)$$

$$\frac{S^k(t)}{\theta} = \frac{e^{-\lambda kt}}{1-\bar{\theta}e^{-\lambda kt}} \quad (42)$$

$$\frac{\bar{\theta}S^k(t)}{\theta} = \frac{\bar{\theta}e^{-\lambda kt}}{1-\bar{\theta}e^{-\lambda kt}} \quad (43)$$

Substitute equation (43) into equation (40);

$$S'(t) = -\left\{\lambda + \frac{\bar{\theta}S^k(t)}{\theta}\right\} S(t) \quad (44)$$

The first order ordinary differential equation for the survival function of the HEE distribution is given by;

$$\theta S'(t) + \theta\lambda S(t) + \bar{\theta}S^{k+1}(t) = 0 \quad (45)$$

$$S(1) = \left[\frac{\theta e^{-\lambda k}}{1-\bar{\theta}e^{-\lambda k}}\right]^{\frac{1}{k}} \quad (46)$$

The ordinary differential equations can be obtained for the particular values of k and λ.

When k = 1, equations (45) and (46) become;

$$\theta S'(t) + \theta\lambda S(t) + \bar{\theta}S^2(t) = 0 \quad (47)$$

$$S(1) = \frac{\theta e^{-\lambda}}{1-\bar{\theta}e^{-\lambda}} = \frac{\theta}{e^{\lambda}-\bar{\theta}} \quad (48)$$

In order to obtain the second order differential equation, differentiate equation (40);

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$$S''(t)=\left\{-\lambda-\frac{\bar{\theta}\lambda\,\mathrm{e}^{-\lambda kt}}{(1-\bar{\theta}\,\mathrm{e}^{-\lambda kt})}\right\}S'(t)+\left\{\frac{k(\bar{\theta}\lambda\,\mathrm{e}^{-\lambda kt})^2}{(1-\bar{\theta}\,\mathrm{e}^{-\lambda kt})^2}+\frac{\bar{\theta}k\lambda^2\,\mathrm{e}^{-\lambda kt}}{(1-\bar{\theta}\,\mathrm{e}^{-\lambda kt})}\right\}S(t) \tag{49}$$

The condition necessary for the existence of equation is that $t,\theta,\bar{\theta},\lambda,k>0$.

The following equations obtained from equation (40) are needed in the simplification of equation (49);

$$\frac{S'(t)}{S(t)}=-\lambda-\frac{\bar{\theta}\lambda\,\mathrm{e}^{-\lambda kt}}{(1-\bar{\theta}\,\mathrm{e}^{-\lambda kt})}$$

(50) $$\frac{\bar{\theta}\lambda\,\mathrm{e}^{-\lambda kt}}{(1-\bar{\theta}\,\mathrm{e}^{-\lambda kt})}=-\left(\frac{S'(t)}{S(t)}+\lambda\right)$$

(51) $$\left(\frac{\bar{\theta}\lambda\,\mathrm{e}^{-\lambda kt}}{1-\bar{\theta}\,\mathrm{e}^{-\lambda kt}}\right)^2=\left(\frac{S'(t)}{S(t)}+\lambda\right)^2$$

(52) $$\frac{k(\bar{\theta}\lambda\,\mathrm{e}^{-\lambda kt})^2}{(1-\bar{\theta}\,\mathrm{e}^{-\lambda kt})^2}=k\left(\frac{S'(t)}{S(t)}+\lambda\right)^2$$

(53) $$\frac{k\bar{\theta}\lambda\,\mathrm{e}^{-\lambda kt}}{(1-\bar{\theta}\,\mathrm{e}^{-\lambda kt})}=-k\left(\frac{S'(t)}{S(t)}+\lambda\right)$$

(54) $$\frac{k\bar{\theta}\lambda^2\,\mathrm{e}^{-\lambda kt}}{(1-\bar{\theta}\,\mathrm{e}^{-\lambda kt})}=-k\lambda\left(\frac{S'(t)}{S(t)}+\lambda\right)$$

(55) Substitute equations (50), (53) and (55) into equation (49);

$$S''(t)=\frac{S'^2(t)}{S(t)}+\left\{\begin{array}{l}k\left(\dfrac{S'(t)}{S(t)}+\lambda\right)^2\\ -k\lambda\left(\dfrac{S'(t)}{S(t)}+\lambda\right)\end{array}\right\}S(t) \tag{56}$$

Simplify equation (56);

$$S''(t)=\frac{(k+1)S'^2(t)}{S(t)}+k\lambda S'(t) \tag{57}$$

The second order ordinary differential equation for the survival function of the HEE distribution is given by;

$$S(t)S''(t)-(k+1)S'^2(t)-k\lambda S(t)S'(t)=0 \tag{58}$$

$$S'(1)=-\left\{\lambda+\frac{\bar{\theta}\lambda\,\mathrm{e}^{-\lambda k}}{(1-\bar{\theta}\,\mathrm{e}^{-\lambda k})}\right\}S(1)=-\left(\lambda+\frac{\bar{\theta}\lambda\,\mathrm{e}^{-\lambda k}}{(1-\bar{\theta}\,\mathrm{e}^{-\lambda k})}\right)\left(\frac{\theta\,\mathrm{e}^{-\lambda k}}{1-\bar{\theta}\,\mathrm{e}^{-\lambda k}}\right)^{\frac{1}{k}} \tag{59}$$

One case of equation (58) is considered, that is when k = 1, equations (58), (46) and (59) become;

$$S(t)S''(t)-2S'^2(t)-\lambda S(t)S'(t)=0 \tag{60}$$

$$S(1)=\left(\frac{\theta\,\mathrm{e}^{-\lambda}}{1-\bar{\theta}\,\mathrm{e}^{-\lambda}}\right) \tag{61}$$

$$S'(1)=-\left(\lambda+\frac{\bar{\theta}\lambda\,\mathrm{e}^{-\lambda}}{(1-\bar{\theta}\,\mathrm{e}^{-\lambda})}\right)\left(\frac{\theta\,\mathrm{e}^{-\lambda}}{1-\bar{\theta}\,\mathrm{e}^{-\lambda}}\right)=-\frac{(\lambda-\bar{\theta}\,\mathrm{e}^{-\lambda}(\lambda-1))\bar{\theta}\,\mathrm{e}^{-\lambda}}{(1-\bar{\theta}\,\mathrm{e}^{-\lambda})^2} \tag{62}$$

V. Inverse Survival Function

The inverse survival function of the Harris extended exponential distribution (HEE) can be obtained from the survival function;

$$\left[\frac{\theta\,\mathrm{e}^{-\lambda kQ(p)}}{1-\bar{\theta}\,\mathrm{e}^{-\lambda kQ(p)}}\right]^{\frac{1}{k}}=p \tag{63}$$

$$\frac{\theta}{\mathrm{e}^{\lambda kQ(p)}-\bar{\theta}}=p^k \tag{64}$$

$$\mathrm{e}^{\lambda kQ(p)}=\frac{\theta+\bar{\theta}p^k}{p^k} \tag{65}$$

Taking logarithm;

$$\lambda kQ(p)=\ln(\theta+\bar{\theta}p^k)-\ln p^k \tag{66}$$

Differentiating;

$$\lambda kQ'(p)=\frac{k\bar{\theta}p^{k-1}}{\theta+\bar{\theta}p^k}-\frac{kp^{k-1}}{p^k} \tag{67}$$

$$\lambda Q'(p)=\frac{\bar{\theta}p^{k-1}}{\theta+\bar{\theta}p^k}-\frac{1}{p} \tag{68}$$

The condition necessary for the existence of equation (68) is that $\theta,\bar{\theta},\lambda,k>0, 0\le p<1$.

The first order ordinary differential equation for the inverse survival function of the HEE distribution can be obtained and evaluated for particular values of all the parameters especially k, p and λ. This is summarized in the **Table 2**.

Table 2: Classes of differential equations obtained for the inverse survival function of the HEE distribution for different parameters.

k	λ	θ	$\bar{\theta}$	ordinary differential equation
1	1	1	1	$p(p+1)Q'(p)+1=0$
1	1	1	2	$p(2p+1)Q'(p)+1=0$
1	1	2	1	$p(p+2)Q'(p)+2=0$
1	1	2	2	$p(p+1)Q'(p)+1=0$
1	2	1	1	$2p(p+1)Q'(p)+1=0$
1	2	1	2	$2p(2p+1)Q'(p)+1=0$
1	2	2	1	$p(p+2)Q'(p)+1=0$
1	2	2	2	$p(p+1)Q'(p)+1=0$
2	1	1	1	$p(p^2+1)Q'(p)+1=0$
2	1	1	2	$p(2p^2+1)Q'(p)+1=0$

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Table 2 continued

k	λ	θ	$\bar{\theta}$	ordinary differential equation
2	1	2	1	$p(p^2+2)Q'(p)+2=0$
2	1	2	2	$p(p^2+1)Q'(p)+1=0$
2	2	1	1	$2p(p^2+1)Q'(p)+1=0$
2	2	1	2	$2p(2p^2+1)Q'(p)+1=0$
2	2	2	1	$p(p^2+2)Q'(p)+2=0$
2	2	2	2	$p(p^2+1)Q'(p)+1=0$

VI. Hazard Function

The hazard function of the Harris extended exponential distribution (HEE) is given as;

$$h(t)=\frac{\lambda}{1-\bar{\theta}\mathrm{e}^{-\lambda kt}} \tag{69}$$

Differentiate equation (69);

$$h'(t)=-\frac{\bar{\theta}\lambda^2 k\,\mathrm{e}^{-\lambda kt}}{(1-\bar{\theta}\mathrm{e}^{-\lambda kt})^2} \tag{70}$$

The condition necessary for the existence of equation (70) is $t,\theta,\bar{\theta},\lambda,k>0$.

Use equation (69) in (70);

$$h'(t)=-\bar{\theta}k\,\mathrm{e}^{-\lambda kt}\,h^2(t) \tag{71}$$

Equation (69) can also be written as;

$$(1-\bar{\theta}\mathrm{e}^{-\lambda kt})h(t)=\lambda \tag{72}$$

$$\bar{\theta}\mathrm{e}^{-\lambda kt}=\frac{h(t)-\lambda}{h(t)} \tag{73}$$

Substitute equation (73) into equation (71);

$$h'(t)=-k(h(t)-\lambda)h(t) \tag{74}$$

The first order ordinary differential equation for the hazard function of the HEE distribution is given by;

$$h'(t)+kh^2(t)-k\lambda h(t)=0 \tag{75}$$

$$h(1)=\frac{\lambda}{1-\bar{\theta}\mathrm{e}^{-\lambda k}} \tag{76}$$

One case of equation (75) is considered, that is when k = 1, equations (75) and (76) become;

$$h'(t)+h^2(t)-\lambda h(t)=0 \tag{77}$$

$$h(1)=\frac{\lambda}{1-\bar{\theta}\mathrm{e}^{-\lambda}} \tag{78}$$

The ODEs of all the probability functions considered can be obtained for the particular values of the distribution. Several analytic, semi-analytic and numerical methods can be applied to obtain the solutions of the respective differential equations. Also comparison with two or more solution methods is useful in understanding the link between ODEs and the probability distributions. Several semi-analytical methods can be applied to solve the ODEs for a given parameter of chosen probability function. The ODEs can be solved using any of the semi-analytical methods as applied in [11-26].

VII. Concluding Remarks

In this work, differentiation was used to obtain some classes of ordinary differential equations for the probability density function (PDF), quantile function (QF), survival function (SF), inverse survival function (ISF) and hazard function (HF) of the Harris extended exponential distribution. The result of the reversed hazard function (RHF) was not obtained because of its complexity. The work was simplified by the application of simple algebraic procedures. The inverse survival function and quantile function generated several ODEs based on their different parameters. In all, the parameters that define the distribution determine the nature of the respective ODEs and the range determines the existence of the ODEs.

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