Proceedings of the World Congress on Engineering and Computer Science 2017 Vol II
WCECS 2017, October 25-27, 2017, San Francisco, USA

On Classical Young Measures Generated by Certain Rapidly Oscillating Sequences

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***Abstract*—The paper is devoted to theory of classical Young measures. It focuses on the situation where a sequence of rapidly oscillating functions has uniform representation in a sense that is defined in this article. A proposition is stated which characterises the Young measures generated by such a class of sequences. This characterisation enables one to find an explicit formulae for the density functions of these generated measures as well as the computation of the values of the related Young functionals. Examples of possible applications of the new results are presented as well.**

***Index Terms*—Young measures, Young functionals, fast-oscillating sequences, periodic functions, uniform distribution.**

I. Introduction

NON-convex optimization problems are at the core of various contemporary engineering applications. They arise e.g. in optimal control, nonlinear evolution equation, variational calculus, micromagnetic phenomena in ferromagnetic materials as well as in microstructures theory in continuum mechanics. It appears however that the optimization problems may not possess a classical minimizer especially when the minimizing sequences have rapid oscillations. Such a behavior of the sequences requires a generalization of the notion of a solution for such problems. It often can be achieved by means of Young measures.

Young measures theory has a long history. It starts with the seminal work [7] of L. C. Young who introduced the notion (called by himself "generalized curves") to provide extended solutions for some non-convex problems in variational calculus. He developed these pioneering ideas in [8].

Nowadays we are provided with vast literature where the Young measures are defined under different assumptions about underlying spaces and analysed from different standpoints. However this paper focuses on the classical Young measures related to sequences of rapidly oscillating functions. In the next section we introduce some preliminary definitions and results. In Section III we define some classes of fast-oscillating sequences and state new proposition that allows us to find explicit forms of the density functions of related classical Young measures in various situations. Section IV presents some examples that illustrate the possible applications of the main result stated in Section III. Finally we make some remarks about possible further extensions and applications.

II. Preliminary definitions and results

We now introduce basic notions of the Young measure theory from the point of view of nonlinear elasticity. Our presentation follows the approach taken in [5], where the reader is referred to for detailed information along with necessary notions from functional analysis and further bibliography. Another book treating Young measures thoroughly in the context the optimization theory and variational calculus is [6].

Let Ω be a nonempty, open and bounded subset of $\mathbf{R}^d$ with smooth boundary. Denote by $L^\infty(\Omega)$ the Banach space of essentially bounded functions defined on Ω with values in a compact set $K \subset \mathbf{R}^l$. Let $\{f_n\}$ be a sequence of functions converging to some function f_0 weakly* in L^∞ and denote by φ a continuous real valued function with domain $\mathbf{R}^l$. By the continuity of φ the sequence $\{\varphi(f_n)\}$ is uniformly bounded in L^∞ norm and Banach-Alaoglu theorem yields the existence of the (not relabeled) subsequence such that $\varphi(f_n) \to g$ weakly* in L^∞. However, in general g is not $\varphi(f_0)$, moreover, it is not even a function with domain in $\mathbf{R}^l$. To quote from [5]: 'The Young measure associated with $\{f_n\}$ furnishes the link among $\{f_n\}$, f_0, g and φ.' We now state the basic existence theorem for Young measures in its full generality.

Theorem 2.1: (see Theorem 2.2 in [5]) Let $\Omega \subset \mathbf{R}^d$ be a measurable set and let $z_n \colon \Omega \to \mathbf{R}^l$ be measurable functions such that

$$\sup_n \int_\Omega h(|z_n|)dx < \infty,$$

where $h\colon [0,\infty) \to [0,\infty)$ is a continuous, nondecreasing function such that $\lim_{t\to\infty} h(t) = \infty$. There exist a subsequence, not relabeled, and a family of probability measures $\nu = \{\nu_x\}_{x\in\Omega}$ (the associated Young measure) depending measurably on x with the property that whenever the sequence $\{H(x, z_n(x))\}$ is weakly convergent in $L^1(\Omega)$ for any Carathéodory function $H(x,\lambda)\colon \Omega \times \mathbf{R}^l \to \mathbf{R} \cup \{\infty\}$, the weak limit is the function

$$\overline{H}(x) = \int_{\mathbf{R}^l} H(x,\lambda)d\nu_x(\lambda).$$

The family of probability measures $\nu = \{\nu_x\}_{x\in\Omega}$ is called the Young measure associated with the sequence $\{z_n\}$.

Let us recall that the Carathéodory function is a function that is measurable with respect to the first and continuous with respect to the second variable.

It often happens that the Young measure $\nu = \{\nu_x\}_{x\in\Omega}$ does not depend on $x \in \Omega$. In this case we denote it merely by ν; such Young measure is called homogeneous.

One may also look at the Young measure as at object associated with *any* measurable function defined on a nonempty, open, bounded subset Ω of $\mathbf{R}^d$ with values in a compact subset K of $\mathbf{R}^l$. Such a conclusion can be derived from the theorem 3.6.1 in [6]. Due to this theorem it can be proved that the Young measure associated with a simple

Manuscript received June 6, 2017; revised June 15, 2017.
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function is homogeneous and is the convex combination of Dirac measures. These Dirac measures are concentrated at the values of the simple function under consideration while coefficients of the convex combination are proportional to the Lebesgue measure of the sets on which the respective values are taken on by the function; see [4] for details and more general results concernig simple method of obtaining explicit form of Young measures associated with oscillating functions (similar, although mathematically more complicated situation, is met in elasticity when the deformed body has a laminate structure; see e.g. Section 4.6 in [3]). On the basis of this concept a more general characterisation of a Young measure associated with any Borel function was introduced in [2]. The main result stated there provides direct link between the Young measure concepts and the probability theory. Namely, the following theorem holds:

Theorem 2.2: Let $f\colon \mathbf{R}^d \supset \Omega \to K \subset \mathbf{R}^l$ be a Borel function with Young measure ν. Then ν is the probability distribution of the random variable $Y = f(U)$, where U has a uniform distribution on Ω.

Now, let us recall the notion of classical Young measure associated with a sequence of oscillating functions $\{f_k\}$, see e.g. [6].

Definition 2.1: The classical Young measure generated by the sequence $\{f_k\}$ is a family of probability measures $\nu = \{\nu_x\}_{x\in\Omega}$ satysfying the condition:

for any Carathéodory function H

$$\int\limits_{\Omega} H(x, f_k(x))dx \overset{k\to\infty}{\longrightarrow} \int\limits_{\Omega}\int\limits_{K} H(x,y)d\nu_x(y)dx \qquad (1)$$

The application that assignes to any Carathéodory function H the integral given on the right-hand side of the above equation is called *Young functional*. Its values on H will be denoted here as $\mathrm{YF}(H)$ while the integrals on the left-hand side of 1 will be denoted as $C(f_k, H)$

Basically, the above definition presents the original understanding of the Young measure, as introduced in his work [8]. These measures are of our main concern in this paper.

III. Main results

A. Rapidly oscillating sequences with uniform representation

Let function $f\colon [a,b) \to K \subset \mathbf{R}$ be a Borel function defined on the interval $[a,b),\ b > a$, and let $f^e\colon \mathbf{R} \to K$ be the periodic extension of f (with the period equal to $T = b - a$).

Let Ω be a given interval. A sequence $\{f_k\}$ of functions $f_k\colon \Omega \to K,\ k = 1, 2, ...$ defined by the formula

$$f_k(x) = f^e(kx),\ x \in \Omega \qquad (2)$$

will be called a Rapidly Oscillating Sequence with Uniform representation f , and denoted as ROSU(f). In such a case we will also say that f generates rapidly oscillating sequence $\{f_k\}$. Note, that the interval Ω - the domain of elements of ROSU(f) - does not have to be the same as $[a,b)$ i.e. the domain of f.

Example 1 In this example we present illustrative plots of some elements of ROSU(f), where

$$f(x) = 2 - 2\sin(x),\ x \in [0, 3\pi/2) \qquad (3)$$

Its purpose is not only to illustrate the behaviuor of ROSU (which is quite obvious, in fact) but also to illustrate the concept of classical Young measure associated with the sequence $\{f_k\}$. The plots of the function f given by (3) as well as of examplary elements of ROSU(f) with the domain $\Omega = [2, 6)$ are presented in Fig.1. Namely it shows the plots of f_1, f_5 and f_{50}.

It can be easilly seen that the graphs of f_k are getting denser when k tends to infinity. Unfortunately, a conventional weak* cluster point of $\{f_k\}$ loses most of the information about the fast oscillations in $\{f_k\}$ because, in some sense, it takes into account only the mean values of $\{f_k\}$ - as integrals do. That is why we need a new concept of the limit and here the theory of Young measures helps us. If ν_x is the Young measure associated with $\{f_k\}$ then, roughly speaking, for any measurable set $A \subset K$ the intuitive meaning of $\nu_x(A)$ is the probability that for an *infinitesimally* small neighbourhood S of $x \in \Omega$ and sufficintly large k's we can "find" $f_k(s)$ in A, when s changes within S. The "density" of the values in K can be observed in the plot (d) (for f_{50}) where the "more probable values" create darker straps in the figure.

B. Classical Young measures generated by the ROSU(f)

It results directly from the definition of ROSU(f) that its behaviour, as k tends to infinity, is exactly the same in every neighbourhood of any $x \in \Omega$. In other words, its asymptotic behaviour in an arbitrarilly small interval $I \subset \Omega$ does not depend on where the interval is placed within the domain. Consequently, it is obvious that the classical Young measure generated by the ROSU(f) is the homogeneous Young measure (i.e. it does not depend on $x \in \Omega$).

Now, let U_S denote a random variable uniformly distributed on S. Let us consider two functions $g_1 : \Omega_1 \to K$ and $g_2 : \Omega_2 \to K$. We will say that the two functions *identically transform a uniform distribution* if the distributions of the random variables $g_1(U_{\Omega_1})$ and $g_2(U_{\Omega_2})$, are the same. This fact will be denoted by $g_1 \approx g_2$. Obviously the "$\approx$" is the equivalence relation.

Note that if the ROSU(f) is defined on the same interval as the generating function f, then *any* of its elements transform a uniform distribution identically as the function f, i.e. $f_k \approx f$ for any $k = 1, 2, ...$ Now let us consider the case where the ROSU(f) is defined on interval Ω that is different than the domain $[a,b)$ of the generating function f. In such a case it can be seen that $f_k(U_\Omega) \overset{D}{\to} f(U_{[a,b)})$, where $\overset{D}{\to}$ denotes the *convergence in distribution*, see [1]. In other words the distribution of $f(U_{[a,b)})$ is a vague limit of the distributions of $f_k(U_\Omega)$. Indeed, for k's large enough so the length of interval Ω is greater than $(b-a)/k$ we have for *any* measurable subset $A \subset K$:

$$|P(f_k(U_\Omega) \in A) - P(f(U_{[a,b)}) \in A)| < 1/k$$

Finally, by Theorem 2.2 we know, that the distribution of $Y = f(U_{[a,b)})$ is the Young measure associated with the function f. Thus we can formulate the following result concerning classical Young measures.

Proposition 3.1: Classical Young measure generated by ROSU(f) is the homogeneous Young measure. This measure is identical with the distribution of the random variable $Y = f(U_{[a,b)})$.

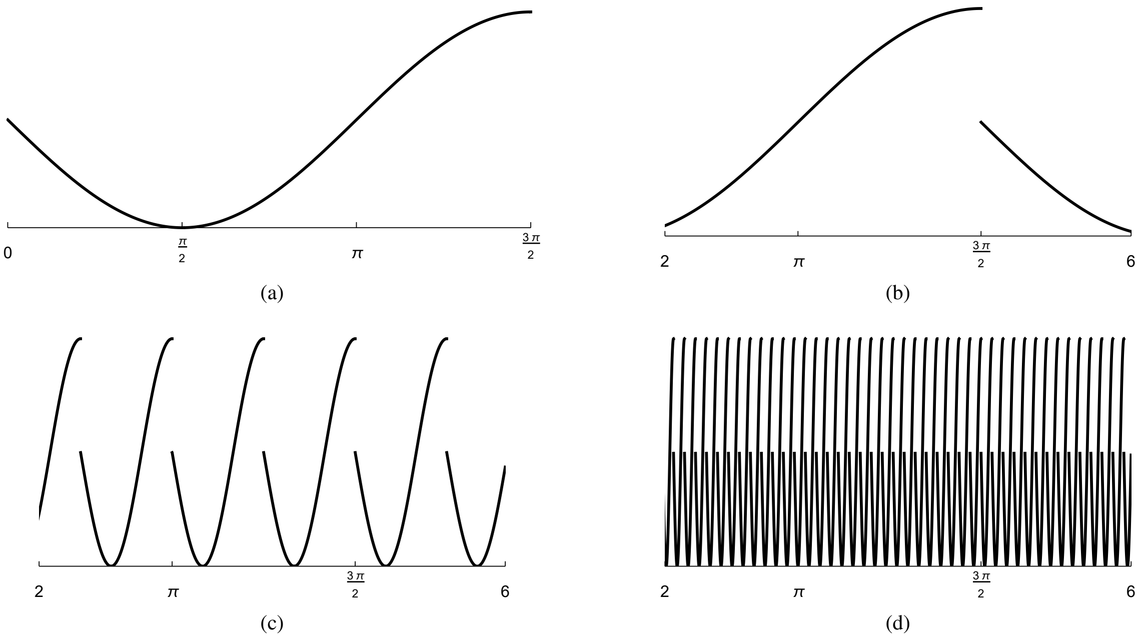

Fig. 1: A function f given by Eq. (3) and functions $f_1,\ f_5$ and f_{50} belonging to ROSU(f) with the domain $\Omega = [2, 6)$. Plots of the functions $f,\ f_1,\ f_5$ and f_{50} are labeled as a, b, c, and d, respectively

IV. APPLICATIONS AND EXAMPLES

On the basis of the above general description of the classical Young measure generated by ROSU(f) we can obtain a number of rules which allow one to find an explict form of the classical Young measure in various specific cases. For example, let us consider the following situation.

Let $[a, b)$ be the interval-domain of the function f. Let us consider an open partition of $[a, b)$ into a number of open intervals $I_1, I_2, \ldots, I_n$ such that the intervals are pairwise disjoint and $\bigcup_{i}^{n} \overline{I}_i = [a, b]$, where $\overline{A}$ denotes the closure of the set A.

Let function f be continuously differentiable on each interval of the partition and let $f'(x) \neq 0$ for all $x \in \bigcup_{l=1}^{n} I_l$

Using the well-known probabilistic result concernig the distributions of such functions of random variables we can obtain the following corollary of the Proposition 3.1.

Corollary 4.1: The classical Young measure generated by *any* ROSU(f) is a homogeneous one and its density g with respect to the Lebesgue measure on K is of the following form

$$g(y) = \frac{1}{b-a} \sum_{l=1}^{n} |h_l'(y)| \mathbf{1}_{D_l}(y) \tag{4}$$

where h_l is the inverse of f on the interval I_l, while $D_l = f(I_l)$ is the domain of h_l. The symbol $\mathbf{1}$ stands for the charactertistic function of the set indicated in its subscript.

To show how Proposition 3.1 works in practice, let us consider a specific function f and an exemplary Carathéodory function H for which both sides of the Eq. (1) can be computed precisely.

Example 2

Let us consider a function $f(x) = \sin x$ defined on the interval $[0,\ 2\pi)$. Now, let the ROSU(f) be defined on the interval $[2, 4)$.

Let us compute the integrals $C(f_k, H)$ that appears on the left-hand side of the Eq. 1 for the exemplary Carathéodory function $H(x, y) = xy^2$. We get

$$C(f_k, H) = \int_2^4 H(x, f_k(x))dx = \int_2^4 x \sin^2(kx)dx$$

$$= \frac{1}{8k^2}[\cos(4k) - \cos(8k) + 4k(6k + \sin(4k) - 2\sin(8k))]$$

The left hand side of the Eq. 1 is a limit of the above expression when k tends to infinity, so it equals 3.

In order to compute the value of the Young functional YF(H), i.e. the integral given by right-hand side of the Eq. 1 we need to know the classical Young measure generated by the ROSU(f). For this purpose we make use of the Corollary 4.1 and receive the following formula:

$$g(y) = \frac{1}{\pi\sqrt{1-y^2}} \mathbf{1}_{(-1,1)}(y)$$

Thus the value of the Young functional in the considered case is the following (recall that the Young measure ν_x is homogeneous in this case, so subscript "x" is ommited) :

$$\text{YF}(H) = \int_{\Omega} \int_{K} H(x, y) d\nu(y) dx =$$

$$\int_{[2,4)} \int_{(-1,1)} H(x, y) g(y) dy dx = \int_{2}^{4} \int_{-1}^{1} \frac{xy^2}{\pi\sqrt{1-y^2}} dy dx = 3$$

As we can see, the "whole information" about the rapid oscillations in this case is contained in the clasical Young measure and - due to the Proposition 3.1 - can be revealed with the help of the Corollary 4.1. In this example the considered generating function f has continuously differentiable extension f^e and the integrations needed to compute the $C(f_k, H)$ were easy to perform. But although this example is intentionally simple, it perfectly shows the benefits resulting

ISBN: 978-988-14048-4-8
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

from our proposition. It is quite clear that even in this case, where the oscillations have such smooth nature, the computation of the limit of $C(f_k, H)$ for more complex Carathéodory functions could be much more difficult task than the calculation of the value of the Young functional $\mathrm{YF}(H)$. Moreover, the problem is getting even more difficult if the extension f^e of the generating function f is piecewise continuous with countably many discontinuity points. The next example deals with such a case.

Example 3 (continuation of Example 1)

Let us consider the generating function f given by Eq. 3 and ROSU(f) introduced in Example 1. Although we use again sinus function as the "basis" for the definition of f, it appears that due to the discontinuity of its extension f^e the general symbolic form for the integrals $C(f_k, H)$ cannot be computed, as long as k is unspecified. They can be only computed for given values of k (and not for large values) or approximated numerically, but even this can be very challenging task for such a functions. Moreover it is certainly insufficient for calculation of the limiting value, which is our aim. With the help of Wolfram Mathematica 10.4 software we computed numerically the integrals in the considered case assuming the Carathéodory functions $H(x, y) = x^2y^3$. The exemplary computed values are as follows (recall that the domain of ROSU(f) is the interval $[2, 6)$): $C(f_5, H) = 941.71$, $C(f_{10}, H) = 891.05$, $C(f_{50}, H) = 942.32$, $C(f_{100}, H) = 956.47$, $C(f_{500}, H) = 953.92$, $C(f_{600}, H) = 955.75$

As we can see it is not easy to guess what is the limit value of the sequence $\{C(f_k, H)\}$.

Now to compute this limit let us make use of the Young concept. For this purpose we need the density function g of the classical Young measure genereted by the ROSU(f). With the help of Corollary 4.1 in this case we obtain:

$$\begin{aligned} g(y) &= 4/(3\pi\sqrt{4-(y-2)^2})\mathbf{1}_{(0,2)}(y) \\ &+2/(3\pi\sqrt{4-(y-2)^2})\mathbf{1}_{[2,4)}(y) \end{aligned}$$

Thus the value of the Young functional on H is the following:

$$\begin{aligned} \mathrm{YF}(H) &= \int_\Omega \int_K H(x, y) d\nu(y) dx \\ &= \int_2^6 \int_0^2 \frac{4x^2y^3}{3\pi\sqrt{4-(y-2)^2}} dy dx \\ &+ \int_2^6 \int_2^4 \frac{2x^2y^3}{3\pi\sqrt{4-(y-2)^2}} dy dx \\ &= \frac{832(45\pi - 44)}{27\pi} \end{aligned}$$

The above limit value could hardly be guessed on the basis of the approximately computed values of the elemnents of $\{C(f_k, H)\}$. The decimal form of the limit, which is 955.09, differs from the computed numerically value of $C(f_{600}, H) = 955.92$. It is worth mentioning at this point that the numerical integration of $C(f_k, H)$ for $k > 600$ failed to converge due to highly oscillatory integrand.

V. Final remarks

The probability theory provides us with a number of different versions of the theorems concerning the shapes of distributions of functions of random variables/vectors. Consequently various other rules for computing explicit formulae for the density functions of classical Young measures generated by ROSU(f) can also be developed on the basis of the new result stated in Proposition 3.1. We are also sure that the same approach enables development of analogous results related to rapidly oscillating sequences with uniform representation which are defined on open and bounded subsets of $\mathbf{R}^d$. It is promising direction of future research.

The possibility of derivation of explicit formulae for the density functions of classical Young measures is not the only benefit resulting from our Proposition 3.1. In many interesting cases explicit formulae for these densities cannot be found. For instance, if one wants to make use of Corollary 4.1 they have to obtain the inverses of f on particular subintervals of its domain, but it is not always possible. However in all such cases thanks to Proposition 3.1 one may use directly Monte Carlo simulations in order to compute values of the Young functionals. That fact significantly broaden the range of possible applications of our main result.

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