Marshall-Olkin-Nadarajah-Haghighi Distribution: Ordinary Differential Equations

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***Abstract*— Marshall-Olkin Nadarajah-Haghighi (MONH) distribution is an improved probability model proposed as an extension of the exponential distribution. In this work, differentiation was used to obtain the ordinary differential equations (ODE) of the probability functions of Marshall-Olkin Nadarajah-Haghighi distribution. The parameters and support that characterized the distribution inevitably determine the nature, existence, uniqueness and solution of the ODEs. The method is recommended to be applied to other probability distributions and probability functions not considered in this paper. Computer codes and programs can be used for the implementation.**

***Index Terms*— Differential calculus, quantile function, hazard function, reversed hazard function, survival function, inverse survival function, probability density function, exponential.**

I. Introduction

CALCULUS in general and differential calculus in particular is often used in statistics in parameter and modal estimations. The method of maximum likelihood is an example.

Differential equations often arise from the understanding and modeling of real life problems or some observed physical phenomena. Approximations of probability functions are one of the major areas of application of calculus and ordinary differential equations in mathematical statistics. The approximations are helpful in the recovery of the probability functions of complex distributions [1-4].

Apart from mode estimation, parameter estimation and approximation, probability density function (PDF) of distributions can be transformed as ODE whose solution yields the respective PDF. Some of which are available: see [5-9].

The aim of this research is to obtain homogenous ordinary differential equations for the probability density function (PDF), Quantile function (QF), survival function (SF), hazard function (HF) and reversed hazard function (RHF) of the Marshall-Olkin Nadarajah-Haghighi (MONH) distribution. Inverse survival function (ISF) was not included because of the complexity of the resulting ODE. This will also help to provide the answers as to whether there are discrepancies between the support of the distribution and the conditions necessary for the existence of the ODEs. Similar results for other distributions have been proposed, see [10-23] for details.

Lemonte et al. [24] proposed the three parameter (MONH) distribution as an extension of the exponential distribution and statistical tests were performed to support the claim.

Differential calculus was used to obtain the results.

II. Probability Density Function

The PDF of the MONH distribution is given as;

$$f(x)=\frac{\alpha\beta\lambda(1+\lambda x)^{\alpha-1}\,\mathrm{e}^{[1-(1+\lambda x)^{\alpha}]}}{1-(1-\beta)\,\mathrm{e}^{[1-(1+\lambda x)^{\alpha}]}} \qquad (1)$$

Differentiate equation (1), to obtain the first order ODE;

$$f'(x)=\left\{\begin{array}{l}\dfrac{(\alpha-1)\lambda(1+\lambda x)^{\alpha-2}}{(1+\lambda x)^{\alpha-1}}\\ -\dfrac{\alpha\lambda(1+\lambda x)^{\alpha-1}\,\mathrm{e}^{[1-(1+\lambda x)^{\alpha}]}}{\mathrm{e}^{[1-(1+\lambda x)^{\alpha}]}}\\ +\dfrac{\alpha\lambda(1-\beta)(1+\lambda x)^{\alpha-1}\,\mathrm{e}^{[1-(1+\lambda x)^{\alpha}]}}{1-(1-\beta)\,\mathrm{e}^{[1-(1+\lambda x)^{\alpha}]}}\end{array}\right\}f(x) \qquad (2)$$

The equation can only exists for $\alpha,\beta,\lambda>0, x\geq 0$.

$$f'(x)=\left\{\begin{array}{l}\dfrac{(\alpha-1)\lambda}{1+\lambda x}-\alpha\lambda(1+\lambda x)^{\alpha-1}\\ +\dfrac{(1-\beta)f(x)}{\beta}\end{array}\right\}f(x) \qquad (3)$$

The first order ODE for the PDFof the MONH distribution is given by;

$$\beta(1+\lambda x)f'(x)-(1-\beta)(1+\lambda x)f^{2}(x)-(\beta(\alpha-1)\lambda -\alpha\lambda\beta(1+\lambda x)^{\alpha})f(x)=0 \qquad (4)$$

$$f(0)=\alpha\lambda \qquad (5)$$

See [10-23] for details.

Manuscript received February 9, 2018; revised March 14, 2018.
This work was sponsored by Covenant University, Ota, Nigeria.
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III. Quantile Function

The QF of the MONH distribution is given as;

$$Q(p)=\frac{1}{\lambda}\left\{\left[1-\ln\left(\frac{1-p}{1-(1-\beta)p}\right)\right]^{\frac{1}{\alpha}}-1\right\} \quad (6)$$

Differentiate equation (6), to obtain the first order ODE;

$$Q'(p)=\frac{1}{\alpha\lambda}\left[\left(\frac{1}{1-p}-\frac{1-\beta}{1-(1-\beta)p}\right)\right] \left\{1-\ln\left(\frac{1-p}{1-(1-\beta)p}\right)\right\}^{\frac{1}{\alpha}-1} \quad (7)$$

The equation can only exists for $\alpha,\beta,\lambda>0, 0<p<1.$
Equation (6) can be further simplify as;

$$\lambda Q(p)+1=\left[1-\ln\left(\frac{1-p}{1-(1-\beta)p}\right)\right]^{\frac{1}{\alpha}} \quad (8)$$

Substitute equation (8) into equation (7);

$$Q'(p)=\frac{1}{\alpha\lambda}\left[\left(\frac{1}{1-p}-\frac{1-\beta}{1-(1-\beta)p}\right)\right] \frac{\lambda Q(p)+1}{\left\{1-\ln\left(\frac{1-p}{1-(1-\beta)p}\right)\right\}} \quad (9)$$

Equation (8) can be further be broken down as;

$$(\lambda Q(p)+1)^{\alpha}=1-\ln\left(\frac{1-p}{1-(1-\beta)p}\right) \quad (10)$$

Substitute equation (10) into equation (9);

$$Q'(p)=\frac{1}{\alpha\lambda}\left[\left(\frac{1}{1-p}-\frac{1-\beta}{1-(1-\beta)p}\right)\right]\frac{\lambda Q(p)+1}{(\lambda Q(p)+1)^{\alpha}} \quad (11)$$

$$Q'(p)=\frac{(\lambda Q(p)+1)^{1-\alpha}}{\alpha\lambda}\left(\frac{\beta(1-2p)}{(1-p)(1-(1-\beta)p)}\right) \quad (12)$$

The ODE can be obtained for the particular values of $\alpha,\beta,\lambda.$ Table 1 contains some examples.

Table 1: ODE for the PDF for different given parameters

α	β	λ	Ordinary Differential Equations
1	1	1	$(1-p)Q'(p)+2p-1=0$
1	1	2	$2(1-p)Q'(p)+2p-1=0$
1	2	1	$(1-p)(1+p)Q'(p)+4p-2=0$
1	2	2	$(1-p)(1+p)Q'(p)+2p-1=0$

See [10-23] for details.

IV. Survival Function

The SF of the MONH distribution is given as;

$$S(t)=\frac{\beta e^{[1-(1+\lambda t)^{\alpha}]}}{1-(1-\beta)e^{[1-(1+\lambda t)^{\alpha}]}} \quad (13)$$

Obtain the derivative of equation (13) in order to obtain the first order ODE;

$$S'(t)=-\frac{\alpha\beta\lambda(1+\lambda t)^{\alpha-1}e^{[1-(1+\lambda t)^{\alpha}]}}{1-(1-\beta)e^{[1-(1+\lambda t)^{\alpha}]}} \quad (14)$$

The equation can only exists for $\alpha,\beta,\lambda>0, t\geq 0.$
Use equation (13) in equation (14);

$$S'(t)=-\alpha\beta\lambda(1+\lambda t)^{\alpha-1}S(t) \quad (15)$$

The ODEs can be obtained for any given values of $\alpha,\beta,\lambda.$. Table 2 contains some examples;

Table 2: ODE for the SF for different given parameters

α	β	λ	Ordinary Differential Equations
1	1	1	$S'(t)+S(t)=0$
1	1	2	$S'(t)+2S(t)=0$
1	2	1	$S'(t)+2S(t)=0$
1	2	2	$S'(t)+4S(t)=0$
2	1	1	$S'(t)+2(1+t)S(t)=0$
2	1	2	$S'(t)+4(1+2t)S(t)=0$
2	2	1	$S'(t)+4(1+t)S(t)=0$
2	2	2	$S'(t)+8(1+2t)S(t)=0$

See [10-23] for details.

V. Hazard Function

The HF of the MONH distribution is given as;

$$h(t)=\frac{\alpha\lambda(1+\lambda t)^{\alpha-1}}{\beta e^{[1-(1+\lambda t)^{\alpha}]}} \quad (16)$$

Find the derivative of equation (16) in order to obtain the first order ODE;

$$h'(t)=\left\{\begin{array}{l}\dfrac{\lambda(\alpha-1)(1+\lambda t)^{\alpha-2}}{(1+\lambda t)^{\alpha-1}}\\ +\dfrac{\alpha\lambda(1+\lambda t)^{\alpha-1}e^{[1-(1+\lambda t)^{\alpha}]}(e^{[1-(1+\lambda t)^{\alpha}]})^{-2}}{(e^{[1-(1+\lambda t)^{\alpha}]})^{-1}}\end{array}\right\}h(t) \quad (17)$$

$$h'(t)=\left\{\frac{\lambda(\alpha-1)}{1+\lambda t}+\alpha\lambda(1+\lambda t)^{\alpha-1}\right\}h(t) \quad (18)$$

The equation can only exists for $\alpha,\beta,\lambda>0, t\geq 0.$

$$(1+\lambda t)h'(t)-(\lambda(\alpha-1)+\alpha\lambda(1+\lambda t)^{\alpha})h(t)=0 \quad (19)$$

The ODEs can be obtained for any given values of $\alpha,\lambda.$
Table 3 contains some cases.

Table 3: ODE for the HF for different given parameters

α	λ	Ordinary Differential Equations
1	1	$(1+t)h'(t)-(1-t)h(t)=0$
1	2	$(1+2t)h'(t)-2(1-2t)h(t)=0$
1	3	$(1+3t)h'(t)-3(1-3t)h(t)=0$
2	1	$(1+t)h'(t)-(1+2(1-t)^2)h(t)=0$
2	2	$(1+2t)h'(t)-(2+4(1-t)^2)h(t)=0$
2	3	$(1+3t)h'(t)-(3+6(1-t)^2)h(t)=0$
3	1	$(1+t)h'(t)-(2+3(1-t)^3)h(t)=0$
3	2	$(1+2t)h'(t)-(4+6(1-2t)^3)h(t)=0$
3	3	$(1+3t)h'(t)-(6+9(1-3t)^3)h(t)=0$

See [10-23] for details.

VI. Reversed Hazard Function

The RHF of the MONH distribution is given as;

$$j(t)=\frac{\alpha\beta\lambda(1+\lambda t)^{\alpha-1}\,\mathrm{e}^{[1-(1+\lambda t)^{\alpha}]}}{1-\mathrm{e}^{[1-(1+\lambda t)^{\alpha}]}} \tag{20}$$

Differentiate equation (20) to obtain the first order ODE;

$$j'(t)=\left\{\begin{array}{l}\dfrac{\lambda(\alpha-1)(1+\lambda t)^{\alpha-2}}{(1+\lambda t)^{\alpha-1}} \\ -\dfrac{\alpha\lambda(1+\lambda t)^{\alpha-1}\,\mathrm{e}^{[1-(1+\lambda t)^{\alpha}]}}{\mathrm{e}^{[1-(1+\lambda t)^{\alpha}]}}- \\ \dfrac{\alpha\lambda(1+\lambda t)^{\alpha-1}\,\mathrm{e}^{[1-(1+\lambda t)^{\alpha}]}(1-\mathrm{e}^{[1-(1+\lambda t)^{\alpha}]})^{-2}}{(1-\mathrm{e}^{[1-(1+\lambda t)^{\alpha}]})^{-1}}\end{array}\right\}j(t) \tag{21}$$

$$j'(t)=\left\{\begin{array}{l}\dfrac{\lambda(\alpha-1)}{1+\lambda t}-\alpha\lambda(1+\lambda t)^{\alpha-1} \\ -\dfrac{\alpha\lambda(1+\lambda t)^{\alpha-1}\,\mathrm{e}^{[1-(1+\lambda t)^{\alpha}]}}{1-\mathrm{e}^{[1-(1+\lambda t)^{\alpha}]}}\end{array}\right\}j(t) \tag{22}$$

The equation can only exists for $\alpha,\beta,\lambda>0, t\geq 0.$

$$j'(t)=\left\{\frac{\lambda(\alpha-1)}{1+\lambda t}-\frac{\alpha\lambda(1+\lambda t)^{\alpha}}{1+\lambda t}-\frac{j(t)}{\beta}\right\}j(t) \tag{23}$$

$$\begin{aligned}&\beta(1+\lambda t)\,j'(t)+(1+\lambda t)\,j^2(t)+(\alpha\beta\lambda(1+\lambda t)^{\alpha}\\&-\lambda\beta(\alpha-1))\,j(t)=0\end{aligned} \tag{24}$$

The ODE can be obtained for any given values of $\alpha,\beta,\lambda.$ Table 4 contains some cases.

Table 4: ODE for the RHF for different given parameters

α	β	λ	Ordinary Differential Equations
1	1	1	$j'(t)+j^2(t)+j(t)=0$
1	1	2	$j'(t)+j^2(t)+2j(t)=0$
1	2	1	$2j'(t)+j^2(t)+2j(t)=0$
1	2	2	$2j'(t)+j^2(t)+4j(t)=0$

When $\alpha=2,\beta=1,\lambda=1,$ equation (24) becomes;

$$(1+t)\,j'(t)+(1+t)\,j^2(t)+(2(1+t)^2-1)\,j(t)=0 \tag{25}$$

When $\alpha=2,\beta=1,\lambda=2,$ equation (24) becomes;

$$(1+2t)\,j'(t)+(1+2t)\,j^2(t)+(4(1+2t)^2-2)\,j(t)=0 \tag{26}$$

When $\alpha=2,\beta=2,\lambda=1,$ equation (24) becomes;

$$2(1+t)\,j'(t)+(1+t)\,j^2(t)+(4(1+t)^2-2)\,j(t)=0 \tag{27}$$

When $\alpha=2,\beta=2,\lambda=2,$ equation (24) becomes;

$$2(1+2t)\,j'(t)+(1+2t)\,j^2(t)+(8(1+2t)^2-4)\,j(t)=0 \tag{28}$$

See [10-23] for details.

VII. Concluding Remarks

Ordinary differential equations (ODEs) have been obtained for the probability functions of Marshall-Olkin Nadarajah-Haghighi (MONH) distribution. Inverse survival function (ISF) was excluded from the study because of the complexity of the resulting ODE. This differential calculus and efficient algebraic simplifications were used to derive the various classes of the ODEs. Different forms of ODEs can be obtained for the any given values of the parameters that defined the distribution. The parameter and the supports that characterize the distribution determine the nature, existence, orientation and uniqueness of the ODEs. The results are in agreement with those available in scientific literature. Furthermore several methods can be used to obtain desirable solutions to the ODEs [25-29]. This method of characterizing distributions cannot be applied to distributions whose PDF or CDF are either not differentiable or the domain of the support of the distribution contains singular points.

Acknowledgment

The comments of the reviewers were very helpful and led to an improvement of the paper. This research benefited from sponsorship from the Statistics sub-cluster of the *Industrial Mathematics Research Group* (TIMREG) of Covenant University and *Centre for Research, Innovation and Discovery* (CUCRID), Covenant University, Ota, Nigeria.

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