One and Two-parameters Lindley Distributions: Ordinary Differential Equations

Hilary I. Okagbue, Member, IAENG, Pelumi E. Oguntunde, Abiodun A. Opanuga and Sheila A. Bishop

Abstract — Differential calculus was applied to get the ordinary differential equations (ODE) of the probability functions of the one and two-parameters Lindley distributions. The parameters and support that characterized the distribution inevitably determine the nature, existence, uniqueness and solution of the ODEs. The method is recommended to be applied to other probability distributions and probability functions not considered in this paper. Computer codes and programs can be used for the implementation.

Index Terms — Differential calculus, quantile function, hazard function, reversed hazard function, inverse survival function, probability density function, Lindley distribution.

I. INTRODUCTION

Calculus in general and differential calculus in particular is often used in statistics in parameter and modal estimations. The method of maximum likelihood is an example.

Differential equations often arise from the understanding and modeling of real life problems or some observed physical phenomena. Approximations of probability functions are one of the major areas of application of calculus and ordinary differential equations in mathematical statistics. The approximations are helpful in the recovery of the probability functions of complex distributions [1-4].

Apart from mode estimation, parameter estimation and approximation, probability density function (PDF) of distributions can be transformed as ODE whose solution yields the respective PDF. Some of which are available. See [5-9] for details.

The aim of this research is to obtain homogenous ordinary differential equations for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of the one-parameter and two-parameter Lindley distributions. This will also help to provide the answers as to whether there are discrepancies between the support of the distributions and the conditions necessary for the behavior and existence of the ODEs. Similar results for other distributions have been proposed, see [10-23] for details.

One parameter and two-parameter Lindley distributions were considered in this research. The details of the distribution can be obtained in [24-26]. Krishna and Kumar [27] studied the effects of censored samples in the reliability analysis of the distribution while Gupta and Singh [28] looked at the parameters estimation of the distribution with hybrid censored data.

Some of the generalizations or modification of the distribution include: the discrete Poisson-Lindley distribution [29], 3-parameter generalization of the 1-parameter Lindley distribution [30], size-biased Poisson-Lindley distribution [31], negative binomial-Lindley distribution [32], discrete Lindley distribution [33], two-parameter weighted Lindley distribution [34], an extended Lindley distribution [35], Power Lindley distribution [36] and generalized Lindley distribution [37].

Also available are: transmuted Lindley distribution [38], two parameter quasi Lindley distribution [39], Kumaraswamy Quasi Lindley [40], transmuted Quasi Lindley distribution [41], beta-Lindley distribution [42], inverse Lindley distribution [43], truncated Lindley distribution [44], Pareto Poisson–Lindley distribution [45], Log–Lindley distribution [46], generalized Power Lindley distribution [47], a new generalization of the distribution based on the probabilistic mixture of two gamma distributions [48] and Lindley exponential distribution [49-50].

Also available are: gamma Lindley distribution [51], Lindley-Poisson distribution [52], Marshall-Olkin extended Lindley distribution [53], a new 4-parameter beta Lindley distribution [54], a new three parameter Lindley distribution [55], generalized inverse Lindley distribution [56], extended inverse Lindley distribution [57], truncated Lindley gamma distribution [58], two parameter discrete Lindley distribution [59], five parameter Lindley distribution [60], Lindley slash distribution [61], wrapped Lindley distribution [62], transmuted two-parameter Lindley distribution [63] and multivariate Lindley distribution [64].

Modeling risk of lifetime data by [65] is one of the various applications of the distribution available. Differential calculus was used to obtain the results.

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II. ONE-PARAMETER LINDLEY DISTRIBUTION

A. Probability Density Function

The PDF of the one-parameter Lindley distribution is given as;
\[ f(x) = \frac{\theta^2}{\theta + 1}(x + 1)e^{-\theta x} \]  
(1)

Differentiate equation (1);
\[ f'(x) = \left\{ \frac{1}{x + 1} - \frac{\theta e^{-\theta x}}{e^{-\theta x}} \right\} f(x) \]  
(2)

The equation can only exists for \( x, \theta > 0 \).
\[ f'(x) = \left\{ \frac{1}{x + 1} - \theta \right\} f(x) \]  
(3)

The first order ODE for the PDF of the one-parameter Lindley distribution is given as;
\[ (x + 1)f'(x) + (\theta(x + 1) - 1)f(x) = 0 \]  
(4)
\[ f(1) = \frac{2\theta^2 e^{-\theta}}{\theta + 1} \]  
(5)

See [10-23] for details.

B. Quantile Function

The QF of the one-parameter Lindley distribution can be obtained from its cumulative distribution function (CDF);
\[ F(t) = 1 - \frac{(\theta t + \theta + 1)e^{-\theta t}}{\theta + 1} \]  
(6)

\[ p = 1 - \frac{(\theta Q(p) + \theta + 1)e^{-\theta Q(p)}}{\theta + 1} \]  
(7)

\[ (\theta Q(p) + \theta + 1)e^{-\theta Q(p)} = 1 - p \]  
(8)

\[ (\theta Q(p) + \theta + 1)e^{-\theta Q(p)} = (\theta + 1)(1 - p) \]  
(9)

Taking logarithmic on both sides;
\[ -\theta Q(p) + \ln(\theta Q(p) + \theta + 1) = \ln(\theta + 1) + \ln(1 - p) \]  
(10)

Differentiate equation (10);
\[ -\theta Q'(p) + \frac{\theta Q'(p)}{(\theta Q(p) + \theta + 1)} = -\frac{1}{1 - p} \]  
(11)

The equation can only exists for \( \theta > 0, 0 < p < 1 \).
\[ \theta Q'(p) - \frac{\theta Q'(p)}{(\theta Q(p) + \theta + 1)} = \frac{1}{1 - p} \]  
(12)

\[ (\theta Q(p) + \theta + 1)\theta Q'(p) - \theta Q'(p) \]  
(13)

\[ \theta Q(p) + \theta + 1 - \theta Q'(p) = \frac{\theta Q(p) + \theta + 1}{1 - p} \]  
(14)

\[ (Q(p) + 1)\theta^2 Q'(p) = \frac{\theta Q(p) + \theta + 1}{1 - p} \]  
(15)

\[ (1 - p)(Q(p) + 1)\theta^2 Q'(p) = \theta Q(p) + \theta + 1 \]  
(16)

The first order ODE for the QF of the one-parameter Lindley distribution is given as;
\[ (1 - p)(Q(p) + 1)\theta^2 Q'(p) - \theta Q(p) - \theta - 1 = 0 \]  
(17)

\[ \ln(\theta Q(0.1) + \theta + 1) - \theta Q(0.1) = \ln(\theta + 1) + \ln 0.9 \]  
(18)

See [10-23] for details.

C. Survival Function

The SF of the one-parameter Lindley distribution is given by;
\[ S(t) = \frac{(\theta t + \theta + 1)e^{-\theta t}}{\theta + 1} \]  
(19)

Differentiate equation (19), to obtain the first order ODE;
\[ S'(t) = -\frac{\theta^2}{\theta + 1}(t + 1)e^{-\theta t} \]  
(20)

The equation can only exists for \( t, \theta > 0 \).
\[ e^{-\theta t} = \frac{(\theta + 1)S(t)}{\theta t + \theta} \]  
(21)

Substitute equation (21) into equation (20);
\[ S'(t) = -\frac{\theta^2(t + 1)S(t)}{\theta t + \theta + 1} \]  
(22)

The first order ODE for the SF of the one-parameter Lindley distribution is given as;
\[ (\theta t + \theta + 1)S'(t) + \theta^2(t + 1)S(t) = 0 \]  
(23)

\[ S(1) = \frac{(2\theta + 1)e^{-\theta}}{\theta + 1} \]  
(24)

See [10-23] for details.

D. Inverse Survival Function

The ISF of the one-parameter Lindley distribution can be obtained from the survival function;
\[ p = \frac{(\theta Q(p) + \theta + 1)e^{-\theta Q(p)}}{\theta + 1} \]  
(25)

\[ (\theta Q(p) + \theta + 1)e^{-\theta Q(p)} = (\theta + 1)p \]  
(26)

Taking logarithmic on both sides;
\[ -\theta Q(p) + \ln(\theta Q(p) + \theta + 1) = \ln(\theta + 1) + \ln p \]  
(27)

Differentiate equation (27);
\[ -\theta Q'(p) + \frac{\theta Q'(p)}{(\theta Q(p) + \theta + 1)} = \frac{1}{p} \]  
(28)

\[ \theta Q'(p) - \frac{\theta Q'(p)}{(\theta Q(p) + \theta + 1)} = -\frac{1}{p} \]  
(29)

The equation can only exists for \( \theta > 0, 0 < p < 1 \).
\[ (\theta Q(p) + \theta + 1)\theta Q'(p) - \theta Q'(p) = \frac{1}{p} \]  
(30)

\[ (\theta Q(p) + \theta + 1)(\theta Q'(p) - \theta Q'(p)) = -\frac{(\theta Q(p) + \theta + 1)}{p} \]  
(31)
\[(Q(p)+1)\theta^{2}Q'(p) = -\left(\theta Q(p) + \theta + 1\right)p\]  \(\text{(32)}\)

\[\theta^{2}p(Q(p)+1)Q'(p) = -\left(\theta Q(p) + \theta + 1\right)\]  \(\text{(33)}\)

The first order ODE for the ISF of the one-parameter Lindley distribution is given as:

\[\theta^{2}p(Q(p)+1)Q'(p) + \theta Q(p) + \theta + 1 = 0\]  \(\text{(34)}\)

\[\ln(\theta Q(0.1) + \theta + 1) - \theta Q(0.1) = \ln(\theta + 1) + \ln 0.1\]  \(\text{(35)}\)

See [10-23] for details.

### E. Hazard Function

The HF of the one-parameter Lindley distribution is given by:

\[h(t) = \frac{\theta^{2}(t+1)}{\theta t + \theta + 1}\]  \(\text{(36)}\)

Differentiate equation (36), to obtain the first order ODE;

\[h'(t) = \left\{\frac{1}{t+1} - \frac{\theta(\theta t + \theta + 1)^{-2}}{(\theta t + \theta + 1)^{-1}}\right\}h(t)\]  \(\text{(37)}\)

\[h'(t) = \left\{\frac{1}{t+1} - \frac{\theta}{(\theta t + \theta + 1)}\right\}h(t)\]  \(\text{(38)}\)

The equation can only exists for \(t, \theta > 0\).

Equation (36) can also be written as:

\[h(t) = \frac{\theta}{\theta t + \theta + 1}\]  \(\text{(39)}\)

Substitute equation (39) into equation (38);

\[h'(t) = \left\{\frac{1}{t+1} - \frac{h(t)}{\theta(t+1)}\right\}h(t)\]  \(\text{(40)}\)

The first order ODE for the HF of the one-parameter Lindley distribution is given as:

\[(t+1)h'(t) + h'(t) - \theta h(t) = 0\]  \(\text{(41)}\)

\[h(1) = \frac{2\theta^{2}}{2\theta + 1}\]  \(\text{(42)}\)

### F. Reversed Hazard Function

The RHF of the one-parameter Lindley distribution is given by:

\[j(t) = \frac{\theta^{2}(t+1)e^{-\theta t}}{\theta + 1} - \left(\theta t + \theta + 1\right)e^{-\theta t}\]  \(\text{(43)}\)

Differentiate equation (43);

\[j'(t) = \left\{\frac{1}{t+1} - \frac{\theta e^{-\theta t}}{\theta + 1} - \frac{e^{-\theta t}}{\theta + 1} \right\}j(t)\]  \(\text{(44)}\)

\[j'(t) = \left\{\frac{1}{t+1} - \frac{\theta e^{-\theta t}}{\theta + 1} - \frac{e^{-\theta t}}{(\theta + 1)^{-1}} \right\}j(t)\]  \(\text{(45)}\)

The equation can only exists for \(t, \theta > 0\).

\[j'(t) = \left\{\frac{1}{t+1} - \theta - j(t)\right\}j(t)\]  \(\text{(46)}\)

The first order ODE for the RHF of the one-parameter Lindley distribution is given as:

\[(t+1)j'(t) - (1 - (t+1)\theta)j(t) + (t+1)j^{2}(t) = 0\]  \(\text{(47)}\)

\[j(1) = \frac{2\theta^{2}e^{-\theta}}{(\theta + 1) - (2\theta + 1)e^{-\theta}} = \frac{2\theta^{2}}{(\theta + 1)e^{-\theta} - (2\theta + 1)}\]  \(\text{(48)}\)

### III. TWO-PARAMETER LINDLEY DISTRIBUTION

#### A. Probability Density Function

The PDF of the two-parameter Lindley distribution is given as;

\[f(x) = \frac{\theta^{2}}{\theta + \beta} \left(\beta x + 1\right)e^{-\beta x}\]  \(\text{(49)}\)

Differentiate equation (49);

\[f'(x) = \left\{\frac{\beta}{\beta x + 1} - \frac{\theta e^{-\beta x}}{e^{-\beta x}}\right\}f(x)\]  \(\text{(50)}\)

\[f'(x) = \left\{\frac{\beta}{\beta x + 1} - \theta\right\}f(x)\]  \(\text{(51)}\)

The equation can only exists for \(x, \beta, \theta > 0\).

The first order ODE for the PDF of the two-parameter Lindley distribution is given as:

\[(\beta x + 1)f'(x) - (\beta - \theta(\beta x + 1))f(x) = 0\]  \(\text{(52)}\)

\[f(1) = \frac{\theta^{2}(1+\beta)e^{-\beta}}{\theta + \beta}\]  \(\text{(53)}\)

#### B. Quantile Function

The QF of the two-parameter Lindley distribution can be obtained from the cumulative distribution function;

\[F(t) = \frac{\theta\beta t + \theta + \beta e^{-\theta t}}{\theta + \beta}\]  \(\text{(54)}\)

\[p = 1 - \frac{(\theta\beta Q(p) + \theta + \beta)e^{-\theta Q(p)}}{\theta + \beta}\]  \(\text{(55)}\)

\[\frac{(\beta\theta Q(p) + \theta + \beta)e^{-\theta Q(p)}}{\theta + \beta} = 1 - p\]  \(\text{(56)}\)

\[\frac{(\beta\theta Q(p) + \theta + \beta)e^{-\theta Q(p)}}{\theta + \beta} = (\theta + \beta)(1 - p)\]  \(\text{(57)}\)
Taking logarithmic on both sides;
\[-\theta Q(p) + \ln(\beta \theta Q(p) + \theta + \beta) = \ln(\theta + \beta) + \ln(1 - p) \tag{58}\]

Differentiate equation (58);
\[-\theta Q'(p) + \frac{\beta \theta Q(p)}{\beta \theta Q(p) + \theta + \beta} = -\frac{1}{1 - p} \tag{59}\]
\[
\theta Q'(p) - \frac{\beta \theta Q'(p)}{\beta \theta Q(p) + \theta + \beta} = -\frac{1}{1 - p} \tag{60}\]
\[
(\beta \theta Q(p) + \theta + \beta) \theta Q'(p) - \beta \theta Q'(p) = \frac{1}{1 - p} \tag{61}\]

The equation can only exists for \( \theta, \beta > 0, 0 < p < 1. \)
\[
(\beta \theta Q(p) + \theta + \beta - \beta) \theta Q'(p) = \frac{\beta \theta Q(p) + \theta + \beta}{1 - p} \tag{62}\]
\[
(\beta \theta Q(p) + 1) \theta^2 Q'(p) = \frac{\beta \theta Q(p) + \theta + \beta}{1 - p} \tag{63}\]
\[
(1 - p)(\beta \theta Q(p) + 1) \theta^2 Q'(p) = \beta \theta Q(p) + \theta + \beta \tag{64}\]

The first order ODE for the QF of the two-parameter Lindley distribution is given as;
\[
(1 - p)(\beta \theta Q(p) + 1) \theta^2 Q'(p) - \beta \theta Q(p) - \theta - \beta = 0 \tag{65}\]
\[
\ln(\beta \theta Q(0.1) + \theta + \beta) - \theta Q(0.1) = \ln(\theta + \beta) + \ln 0.9 \tag{66}\]

C. Survival Function

The SF of the two-parameter Lindley distribution is given by;
\[
S(t) = (\beta \theta t + \theta + \beta)e^{-\theta t} \tag{67}\]

Differentiate equation (67) to obtain the first order ODE;
\[
S'(t) = -\frac{\theta^2}{\theta + \beta}(\beta t + 1)e^{-\theta t} \tag{68}\]

Equation (67) can also be written as;
\[
S(t) = \frac{e^{-\theta t}}{\beta \theta t + \theta + \beta} \tag{69}\]

The equation can only exists for \( t, \beta, \theta > 0. \)

Substitute equation (69) into equation (68);
\[
S'(t) = -\frac{\theta^2}{\beta \theta t + \theta + \beta}S(t) \tag{70}\]

The first order ODE for the SF of the two-parameter Lindley distribution is given as;
\[
(\beta \theta t + \theta + \beta)S'(t) + \theta^2(\beta t + 1)S(t) = 0 \tag{71}\]
\[
S(1) = \frac{(\theta + \beta + \beta \theta)e^{-\theta t}}{\theta + \beta} \tag{72}\]

D. Inverse Survival Function

The ISF of the two-parameter Lindley distribution can be obtained from the survival function;
\[
p = \frac{(\beta \theta Q(p) + \theta + \beta)e^{-\theta Q(p)}}{\theta + \beta} \tag{73}\]
\[
(\beta \theta Q(p) + \theta + \beta)e^{-\theta Q(p)} = (\theta + \beta)p \tag{74}\]

Taking logarithmic on both sides;
\[-\theta Q(p) + \ln(\beta \theta Q(p) + \theta + \beta) = \ln(\theta + \beta) + \ln p \tag{75}\]

Differentiate equation (75);
\[-\theta Q'(p) + \frac{\beta \theta Q(p)}{\beta \theta Q(p) + \theta + \beta} = \frac{1}{p} \tag{76}\]
\[
\theta Q'(p) - \frac{\beta \theta Q'(p)}{\beta \theta Q(p) + \theta + \beta} = \frac{1}{p} \tag{77}\]
\[
(\beta \theta Q(p) + \theta + \beta) \theta Q'(p) - \beta \theta Q'(p) = \frac{1}{p} \tag{78}\]

The equation can only exists for \( \theta, \beta > 0, 0 < p < 1. \)
\[
(\beta \theta Q(p) + \theta + \beta - \beta) \theta Q'(p) = -\frac{\beta \theta Q(p) + \theta + \beta}{p} \tag{79}\]
\[
(\beta \theta Q(p) + 1) \theta^2 Q'(p) = -\frac{\beta \theta Q(p) + \theta + \beta}{p} \tag{80}\]
\[
\theta^2 p(\beta \theta Q(p) + 1) \theta Q'(p) = -\frac{\beta \theta Q(p) + \theta + \beta}{p} \tag{81}\]

The first order ODE for the ISF of the two-parameter Lindley distribution is given as;
\[
\theta^2 p(\beta \theta Q(p) + 1) \theta Q'(p) + \beta \theta Q(p) + \theta + \beta = 0 \tag{82}\]
\[
\ln(\beta \theta Q(0.1) + \theta + \beta) - \theta Q(0.1) = \ln(\theta + \beta) + \ln 0.1 \tag{83}\]

E. Hazard Function

The HF of the two-parameter Lindley distribution is given by;
\[
h(t) = \frac{\theta^2(\beta t + 1)}{\beta \theta t + \theta + \beta} \tag{84}\]

Differentiate equation (84), to obtain the first order ODE;
\[
h'(t) = \left\{ \frac{\beta}{\beta t + 1} - \frac{\theta \beta(\beta \theta t + \theta + \beta)^{-2}}{(\beta \theta t + \theta + \beta)^{-1}} \right\} h(t) \tag{85}\]
\[
h'(t) = \left\{ \frac{\beta}{\beta t + 1} - \frac{\theta \beta}{(\beta \theta t + \theta + \beta)} \right\} h(t) \tag{86}\]

The condition necessary for the existence of the equation is \( t, \beta, \theta > 0. \)

Alternatively, equation (84) is given as;
\[
\frac{\beta h(t)}{\theta(\beta t + 1)} = \frac{\theta \beta}{\beta \theta t + \theta + \beta} \tag{87}\]
Substitute equation (87) into equation (86);

\[ h'(t) = \frac{\beta}{t + 1} - \frac{\beta h(t)}{\theta(t + 1)} \]

(88)

The first order ODE for the HF of the two-parameter Lindley distribution is given as:

\[(t + 1)h'(t) + \beta h'(t) - \beta \theta h(t) = 0\]

(89)

\[ h(l) = \frac{\theta^2(\beta + 1)}{\beta \theta + t + \beta} \]

(90)

F. Reversed Hazard Function

The RHF of the two-parameter Lindley distribution is given by;

\[ j(t) = \frac{\theta^2(\beta + 1)e^{-\theta t}}{(\theta + \beta) - (\beta \theta t + \theta + \beta)e^{-\theta t}} \]

(91)

Differentiate equation (91) to obtain the first order ODE;

\[ j'(t) = \frac{\beta}{\beta t + 1} - \frac{\theta e^{-\theta t}}{(\theta + \beta) - (\beta \theta t + \theta + \beta)e^{-\theta t}} \]

\[ j'(t) = \frac{\theta^2(\beta + 1)e^{-\theta t}}{(\theta + \beta) - (\beta \theta t + \theta + \beta)e^{-\theta t}} j(t) \]

(92)

The equation can only exists for \( t, \beta, \theta > 0 \).

\[ j'(t) = \frac{\beta}{\beta t + 1} - \theta - j(t) \]

(93)

The first order ODE for the RHF of the two-parameter Lindley distribution is given as;

\[(\beta + 1)j'(t) + (\beta - \theta)(\beta t + 1)j'(t) + (\beta t + 1)j^2(t) = 0\]

(95)

\[ j(l) = \frac{\theta^2(\beta + 1)e^{-\theta l}}{(\theta + \beta) - (\beta \theta + \theta + \beta)e^{-\theta l}} \]

\[ = \frac{\theta^2(\beta + 1)}{(\theta + \beta)e^{\theta l} - (\beta \theta + \theta + \beta)} \]

(96)

IV. CONCLUDING REMARKS

Ordinary differential equations (ODEs) have been obtained for the probability functions of one-parameter and two-parameter Lindley distributions. This differential calculus and efficient algebraic simplifications were used to derive the various classes of the ODEs. The parameter and the supports that characterize the distributions determine the nature, existence, orientation and uniqueness of the ODEs. The results are in agreement with those available in scientific literature. Furthermore several methods can be used to obtain desirable solutions to the ODEs [66-78]. This method of characterizing distributions cannot be applied to distributions whose PDF or CDF are either not differentiable or the domain of the support of the distribution contains singular points.

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