On New Solutions of Impulsive Quantum Stochastic Differential Inclusion

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Abstract—This paper addresses the existence of solution of Impulsive Quantum Stochastic Differential Inclusion (also known as impulsive non-classical ordinary differential inclusion) with an additional bounded linear operator. The multivalued maps are necessarily compact. By showing that the multivalued maps can actually satisfy some conditions of appropriate fixed-point theorems, the major result is established.

Index Terms—QSDI, Stochastic processes, Bounded Linear Operator, Fixed point.

I. INTRODUCTION

Some interesting results concerning the following quantum stochastic differential equation (QSDE) and inclusion (QSDI) introduced by Hudson and Parthasarathy [12]:

$$\begin{align*}
\frac{d}{dt} & \langle \eta, y(t) \rangle \in P(t, y(t)) \langle \eta, \xi \rangle \\
\langle \eta, y(0) \rangle & = \langle \eta, \gamma_0 \rangle 
\end{align*}$$

have been studied. See [1, 4-6, 10]. However, it has shown by Ekhaguere [10], that the following nonclassical ordinary differential inclusion (NODI):

$$\begin{align*}
\frac{d}{dt} & \langle \eta, y(t) \rangle \in P(t, y(t)) \langle \eta, \xi \rangle \\
\langle \eta, y(0) \rangle & = \langle \eta, \gamma_0 \rangle 
\end{align*}$$

is equivalent to inclusion (1) where the map $P$ in (2) is well defined in [1]. The definitions and notations of some spaces such as $L^2_{loc}(\mathbb{R}_+)$, $L^p_{loc}(\mathbb{R}_+)$, $L^p_{loc}(I \times \mathbb{R}_+)$, $\Gamma(L^2_{loc}(\mathbb{R}_+))$ are adopted from the reference [6] and the references therein.

(i) The stochastic processes are densely defined linear operators on the complex Hilbert space $\mathcal{H}$.

(ii) $\mathcal{H}$ is a topological space and clos($\mathcal{H}$), comp($\mathcal{H}$) is the norm induced by $\mathcal{H}$, respectively. The definitions and notations of some spaces such as $L^2_{loc}(\mathbb{R}_+)$, $L^p_{loc}(\mathbb{R}_+)$, $L^p_{loc}(I \times \mathbb{R}_+)$, $\Gamma(L^2_{loc}(\mathbb{R}_+))$ are well defined in [1]. The definitions and notations of some spaces such as $L^2_{loc}(\mathbb{R}_+)$, $L^p_{loc}(\mathbb{R}_+)$, $L^p_{loc}(I \times \mathbb{R}_+)$, $\Gamma(L^2_{loc}(\mathbb{R}_+))$ are well defined in [1].

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To the best of our knowledge within the consulted literature, there are no results concerning impulsive QSDI in the sense of this paper. However, in the classical setting, several results on differential equations/inclusions with impulse effects have been established. See the references [2, 3] and the references therein. This work generalizes existing results in the literature on impulsive quantum stochastic differential equations since some of these results can be recovered from the result in this paper. The remaining part of this paper will consist of two sections: Section 2 will contain some preliminaries which we adopt from the references [1, 4, 5, 10] while in section 3, the main result will be established.

In this paper, the fixed-point method is used to establish the major result. In recent times, this method has proven to be very efficient in establishing existence of solutions of differential equations and approximate solutions.

II. PRELIMINARIES

Let $\mathcal{A}$ be a topological space and clos($\mathcal{A}$), comp($\mathcal{A}$) be as defined in [6]. Define $I = [0, b]$. Define $I' = I\setminus \{t_1, t_2, ..., t_m\}$ and $0 < t_1 < ... < t_m < t_{m+1} = b$.

Definitions 2.1

(i) $PC(I, \mathcal{A}) = \{y : I \rightarrow \mathcal{A} : y(t) \text{ is continuous and } y(t_k) = y(t_k^-)\}$
We define the sesquilinear equivalent forms $PC(I,A(\eta,\xi))$ associated with $PC(I,sesq(D \otimes E))$ in the same way with the above, where $A$ is a subset of $\mathbb{R}$.

(ii) $PC(I,A(\eta,\xi))$ is a Banach space whose norm is defined by
\[ \|y\|_{R^2,PC} = \sup_{t \in I} |y(t)|(\eta,\xi) : t \in I, \] where $\mathbb{R} \subseteq -A$.

(iii) $\rho(B,C) := \max(\delta(B,C),\delta(C,B))$, $B,C \in clos(C)$ and
\[ d(y,B) = \inf_{y \in B} \|x - y\|, \]
\[ \delta(B,C) = \sup_{y \in B} d(y,B), \]
where $y \in C$, is a complex number, $\rho(B,C)$ the Hausdorff distance, and $\rho$ a metric on $clos(C)$.

Note: (i) in definition 2.1 holds everywhere except for some $t_k$. 

**Definition 2.2**

For the rest of this paper, $\Phi$ indexed by $I = [0,T] \subseteq \mathbb{R}$, is multivalued stochastic process.

(i) $\Phi$ is $L^1$-measurable in the sense of [2]

(ii) A selection of $\Phi$ is a stochastic process $y : I \rightarrow A$ such that $y(t) \in \Phi(t)$ for almost all $t \in I$.

(iii) $\Phi$ is said to be lower semicontinuous if for every open set $V \subset A$, $\Phi^{-1}(V)$ is open.

(iv) $\Phi$ is adapted if $\Phi(t) \subseteq A_t$, $t \in I$ and its denoted by $Ad(A)$

(v) A member $y \in Ad(A)$ is said to be weakly absolutely continuous (AC) if $t \rightarrow (\eta, y(t)\xi), t \in I$ is AC for $\eta, \xi \in D \otimes E$. Denote all such sets by $A(\eta,\xi)$.

(vi) $\Phi$ is locally absolutely $p$-integrable.

Also define the set $S^\Phi$ by
\[ S^\Phi = \{ g \in L^1(\mathbb{R}, I) : g(t) \in \Phi(y(t), t), a.e. t \in I \} \neq \emptyset \]

Note: Subsequently, $\eta, \xi \in D \otimes E$ is arbitrary.

Next, we introduce the following QSDI with an additional bounded linear operator:

\[
C(t) = \begin{cases} 
A(t)y(t) + B(y(t)) + P(t, y)(\eta, \xi) 
\end{cases}
\]

\[
\Delta y(t) = f_k(y(t_k)), t = t_k, k = 1,...,m
\]

Observe that, the following inclusion is equivalent to inclusion (3).

\[
\frac{d}{dt}(\eta, y(t) - A(t)y(t)) \in B(\eta, y(t)) + C(t, y)(\eta, \xi)
\]

for some $t \in I, t \neq t_k, k = 1,...,m$.

\[
\eta, y(0) = (\eta, y_0) = (\eta, y_0)
\]

where the multivalued map $P : \mathbb{R} \times I \rightarrow clos(A)$ is assumed to be compact and convex, $y \rightarrow \Phi(y(t), t), \Phi \in \{ A, G, F, G, H \}$ is a multivalued stochastic process, $y_0 \in A$ and $B : A \rightarrow A$, $J_k \in C(\mathbb{R}, A)$.

**Definition 2.3**

(i) $\Delta y(t) = y(t_k^2) - y(t_k^2), t = t_k, y(t_k) = \lim_{h \rightarrow 0^+} y(t_k + h)$

(ii) $y(t_k^2) = \lim_{h \rightarrow 0} y(t_k - h), k = 1,...,m$.

(iii) Let $y \in L(I, A \otimes \mathbb{R}), y(t) \in \Phi(y(t), t)$ then, $y(t) \in (I, Ad(A), w) \cap PC(I, A)$ is a solution of (3) if it satisfies the following integral inclusion

\[
y(t) = \int_0^T t(s)B(y(s))(\eta, \xi) + \int_0^T t(s)P(y(s))(\eta, \xi) ds
\]

\[
+ \int_0^T t(s)G(y(s))(\eta, \xi) ds
\]

\[
+ \int_0^T t(s)F(y(s))(\eta, \xi) ds
\]

\[
+ \int_0^T t(s)H(y(s))(\eta, \xi) ds
\]

Note: (i) in definition 2.2 holds everywhere except for some $t_k$.

**Theorem 3.1:** Assume conditions (S1) - (S4) hold, then (3) has at least a weak solution.

Proof: Firstly, (4) has to be transformed into a fixed-point problem.

Define $N : PC(I, A(\eta, \xi)) \rightarrow D$, $N(\eta, T) = y_0(\eta, \xi) + \int_0^T T(t-s)g_\eta d(s) ds$

\[
+ \int_0^T t(s)B(y(s))(\eta, \xi) ds
\]

where $w_\eta(t) = \max(M\|B\|, M\|P\|)$

\[
d = M\|y_0\|_\eta + \sum_{k=1}^m c_k
\]

$B \subseteq PC(\mathbb{R}, I)$, and $sup_{t \in I} \|y(t)\|_\eta < \infty$.

III. MAIN RESULTS

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where $w_\eta(t) = \max(M\|B\|, M\|P\|)$

\[
d = M\|y_0\|_\eta + \sum_{k=1}^m c_k
\]

$B \subseteq PC(\mathbb{R}, I)$, and $sup_{t \in I} \|y(t)\|_\eta < \infty$.
Assume \( K^* = \sup \{ \| y \|_{\mathbb{R}^m} : y \in K \} \).
We show that \( N(K) \subset K \) and is also relatively compact.
Fix \( t \in I \) and let \( g \in S^\Phi \) such that,
\[
h(t)(\eta, \xi) = \langle \eta, T(t) y_0 \rangle + \int_0^t T(t-s)B(y(s)) \eta, \xi \rangle ds + \int_0^t T(t-s)g_H(s) ds + \sum_{0 < t_k < t} T(t-s)J_k(y(t_k)) (\eta, \xi) \]
+ \( T(\varepsilon) \int_0^{t-\varepsilon} T(t-s)g(s) (\eta, \xi) ds \)
+ \( \sum_{0 < t_k < t} T(t-t_k-\varepsilon) J_k(y(t_k)) (\eta, \xi) \), \( g \in S^\Phi \).
Since the operator \( T(t) \) is compact, it implies that \( H_T(t) = [h_t(t) : h \in N(y)] \) is relatively compact in \( A \), \( 0 < \varepsilon < t \).
Now, \( \forall h \in N(y) \), we get
\[
|h_t(t)(\eta, \xi) - h(t)(\eta, \xi)| \leq \| B \|_{K^*} \int_0^t \| T(t-s) \| ds
\]
\[
+ \int_0^t \| T(t-s) \| \| g(s) \| (\eta, \xi) \| ds
\]
\[
+ \sum_{0 < t_k < t} \| c_k \| \| T(t-t_k) \| ds
\]
Showing that there are relatively compact sets close to the set \{ \( h(t) : h \in N(y) \) \} and so, the set \{ \( h(t) : h \in N(y) \) \} is relatively compact in \( A \).
Let \( y_0 \rightarrow y^* \), \( h_n \rightarrow h \). We establish that \( h_n \in N(h_n) = g_n \in S^\Phi \).
For \( t \in I \),
\[
h_n(t)(\eta, \xi) = \langle \eta, T(t) y_0 \rangle + \int_0^t T(t-s)B(y_n(s)) (\eta, \xi) ds
\]
+ \( T(\varepsilon) \int_0^{t-\varepsilon} T(t-s)g(s) (\eta, \xi) ds \)
+ \( \sum_{0 < t_k < t} T(t-t_k-\varepsilon) J_k(y_n(t_k)) (\eta, \xi) \).
(15)
We show that \( \exists h \in S^\Phi \) from which we get,
\[
h_n(t)(\eta, \xi) = \langle \eta, T(t) y_0 \rangle + \int_0^t T(t-s)B(y_n(s)) (\eta, \xi) ds
\]
+ \( T(\varepsilon) \int_0^{t-\varepsilon} T(t-s)g(s) (\eta, \xi) ds \)
+ \( \sum_{0 < t_k < t} T(t-t_k-\varepsilon) J_k(y_n(t_k)) (\eta, \xi) \), \( g \in S^\Phi \).
By the continuity of \( J_k \) and \( B \), we get
\[
|h_n(t)(\eta, \xi) - \langle \eta, T(t) y_0 \rangle |
\]
\[
- \int_0^t T(t-s)B(y_n(s)) (\eta, \xi) ds
\]
\[
- \int_0^t T(t-s)g(s) (\eta, \xi) ds
\]
\[
- \sum_{0 < t_k < t} T(t-t_k-\varepsilon) J_k(y_n(t_k)) (\eta, \xi) \rightarrow 0
\]
as \( n \rightarrow \infty \).
Now, by considering the operator
\[
|f(t)(\eta, \xi) - f(t)(\eta, \xi)| \leq \| B \|_{K^*} \int_0^t \| T(t-s) \| ds
\]
\[
+ \int_0^t \| T(t-s) \| \| g(s) \| (\eta, \xi) \| ds
\]
\[
+ \sum_{0 < t_k < t} \| c_k \| \| T(t-t_k) \| ds
\]
we get,

\[ h_\ast (t_\ast (\eta, \xi)) = \langle \eta, T(t_\ast) y_\ast, \xi \rangle - \int_0^t T(t - s) B(y_\ast(s))(\eta, \xi) \, ds \]

\[ - \sum_{0 < t_k < t} T(t - t_k) J_k(y_\ast(t_k))(\eta, \xi) \in \Gamma(S^\ast) \]

Since \( y_\ast \to y_\ast \), it follows that

\[ h_\ast (t)(\eta, \xi) - \langle \eta, T(t)y_\ast, \xi \rangle - \sum_{0 < t_k < t} T(t - t_k) J_k(y_\ast(t_k))(\eta, \xi) - \int_0^t T(t - s) B(y_\ast(s))(\eta, \xi) \, ds = \int_0^t T(t - s) g_\ast R(s) \, ds \]

for some \( g_\ast \in S^\ast \). And the desired result is obtained.

IV. CONCLUSION
Having satisfied the conditions of Schaefer’s theorem, we conclude that \( N \) has a fixed point which is a solution of (3) respectively (4). If the operator \( B \) in equation (3) is zero, i.e., \( B = 0 \), then we obtain some results in [4, 6]. Hence, the result in this paper is a generalization of the results in the existing literature on impulsive quantum stochastic differential inclusion (equations).

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