Introducing the Kumaraswamy Perks Distribution

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Abstract— **In this paper, we introduced the Kumaraswamy Perks distribution using the Kumaraswamy family of distributions. We established some statistical properties of the new model and we estimated the unknown parameters using the method of maximum likelihood estimation.**

Index Terms— **Estimation, Generalized Models, Kumaraswamy family of distributions, Mathematical Statistics, Perks distribution**

I. INTRODUCTION

The Perks distribution was developed by [1] and it is an extension or modification of the Gompertz-Makeham distribution. Its application in actuarial science has been identified in [2-3]. [4] have used the Markov Chain Monte Carlo (MCMC) simulation method to obtain the parameter estimates of the Perks distribution based on a complete sample. Other authors who have worked on the Perks distribution recently include [5-7] and many more.

In particular, [7] extended the Perks distribution and arrived at the Exponentiated Perks distribution and its performance has been compared with respect to the Perks distribution, Exponentiated Weibull distribution, Exponentiated Exponential distribution and Generalized Rayleigh distribution. Three real life applications were presented but the Exponentiaed Perks distribution performed better in one, while the Exponentiated Weibull distribution and the Exponentiated Exponential distribution were preferred respectively in the remaining two applications based on their Akaike Information Criteria (AIC) values.

However, this present study focuses on extending the Perks distribution using the Kumaraswamy generalized family of distributions proposed by [8]. In existence are other families of distributions like the Beta generalized family of distribution [9], Transmuted family of distributions [10] and many more which can be found in [11-16]. Others works that involve the use of Kumaraswamy family of distributions include but not limited to [17-19] and the references.

The proposed four-parameter Kumaraswamy Perks distribution is defined in the next section and its various statistical properties are also established.

II. THE KUMARASWAMY PERKS DISTRIBUTION

According to [8], [17] and [19], the cumulative distribution function (cdf) and probability density function (pdf) of the kumaraswamy generalized family of distribution are:

$$F(x)=1-\left\{1-\left[G(x)\right]^{a}\right\}^{b} \tag{1}$$

and

$$f(x)=abg(x)\left[G(x)\right]^{a-1}\left\{1-\left[G(x)\right]^{a}\right\}^{b-1} \tag{2}$$

respectively

Where a and b are extra shape parameters

Also, the cdf and pdf of the Perks distribution are:

$$G(x)=1-\left(\frac{1+c}{1+ce^{dx}}\right) \quad ; \quad x\geq 0, c>0, d>0 \tag{3}$$

and

$$g(x)=cde^{dx}\frac{(1+c)}{\left(1+ce^{dx}\right)^{2}} \quad ; x\geq 0, c>0, d>0 \tag{4}$$

respectively. Where c and d are shape parameters

Now, we derive the cdf of the Kumaraswamy Perks distribution by substituting Equation (3) into Equation (1) as follows:

$$F(x)=1-\left\{1-\left[1-\left(\frac{1+c}{1+ce^{dx}}\right)\right]^{a}\right\}^{b} \tag{5}$$

Its associated pdf is:

$$f(x)=abcde^{dx}\frac{(1+c)}{\left(1+ce^{dx}\right)^{2}}\left[1-\left(\frac{1+c}{1+ce^{dx}}\right)\right]^{a-1}\left\{1-\left[1-\left(\frac{1+c}{1+ce^{dx}}\right)\right]^{a}\right\}^{b-1} \tag{6}$$

Where $a>0, b>0, c>0, d>0$ are shape parameters.

From now on, the random variable that follows the Kumaraswamy Perks distribution is denoted by $X \sim KumPerks(a,b,c,d)$.

Some possible plots for the pdf of the Kumaraswamy Perks distribution are as shown in Figure 1:

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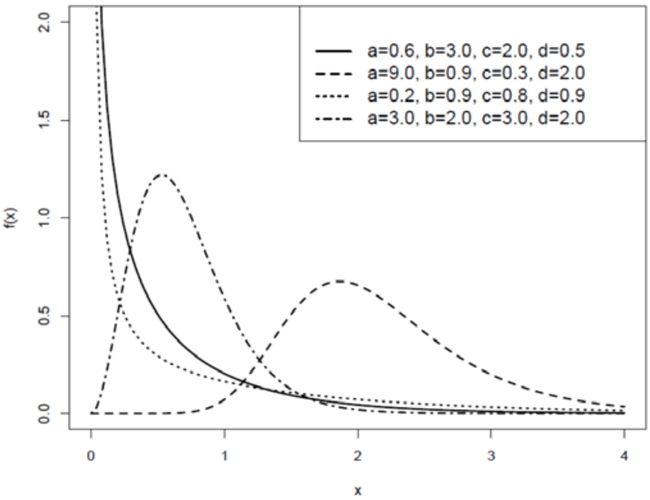

Figure 1: PDF of the KumPerks Distribution

As we have indicated in Figure 1, the shape of the model could exhibit unimodality (inverted bathtub) and decreasing shapes.

Special Cases

We found that some models are special cases (or sub-models) of the KumPerks distribution. See Table 1 for example:

Table 1: Related Distributions

Distributions	**Parameters**				**Author**
Kumaraswamy Perks	a	b	c	d	Proposed
Generalized Perks	1	b	c	d	New
Exponentiated Perks	a	1	c	d	[7]
Perks	1	1	c	d	[1]

A possible plot for the cdf is as shown in Figure 2:

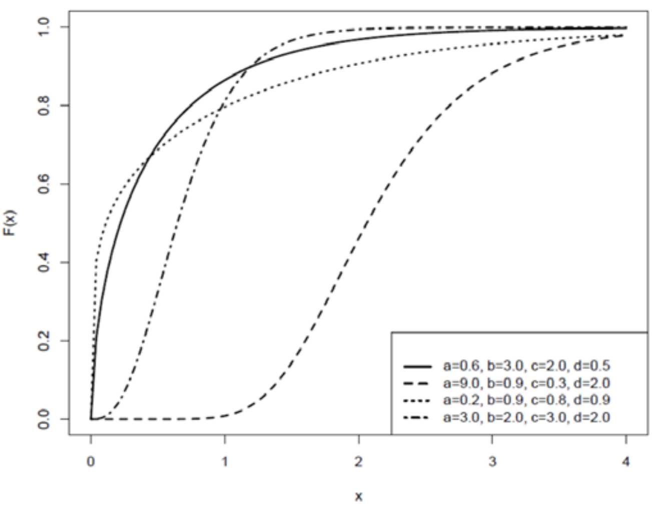

Figure 2: CDF of the KumPerks Distribution

III. SOME STATISTICAL PROPERTIES OF THE KUMARASWAMY PERKS DISTRIBUTION

Some basic statistical properties of the KumPerks distribution are derived and established in this section.

Survival Function: The mathematical expression for survival function which can otherwise be called reliability function is:

$$S(x)=1-F(x) \tag{7}$$

So, we obtain the survival function of the KumPerks distribution as:

$$S(x)=\left\{1-\left[1-\left(\frac{1+c}{1+ce^{dx}}\right)\right]^{a}\right\}^{b} \tag{8}$$

For $x>0, a>0, b>0, c>0, d>0$

The Hazard Function: The mathematical formula for hazard function which is otherwise called failure rate is:

$$h(x)=\frac{f(x)}{S(x)} \tag{9}$$

So, we obtain the hazard function of the KumPerks distribution as:

$$h(x)=\frac{abcde^{dx}\dfrac{(1+c)}{\left(1+ce^{dx}\right)^{2}}\left[1-\left(\dfrac{1+c}{1+ce^{dx}}\right)\right]^{a-1}}{\left\{1-\left[1-\left(\dfrac{1+c}{1+ce^{dx}}\right)\right]^{a}\right\}} \tag{10}$$

For $x>0, a>0, b>0, c>0, d>0$

The Reversed Hazard Function: The mathematical formula for reversed hazard function is:

$$r(x)=\frac{f(x)}{F(x)} \tag{11}$$

So, we obtain the reversed hazard function of the KumPerks distribution as:

$$r(x)=\frac{abcde^{dx}\dfrac{(1+c)}{\left(1+ce^{dx}\right)^{2}}\left[1-\left(\dfrac{1+c}{1+ce^{dx}}\right)\right]^{a-1}\left\{1-\left[1-\left(\dfrac{1+c}{1+ce^{dx}}\right)\right]^{a}\right\}^{b-1}}{1-\left\{1-\left[1-\left(\dfrac{1+c}{1+ce^{dx}}\right)\right]^{a}\right\}^{b}} \tag{12}$$

For $x>0, a>0, b>0, c>0, d>0$

The Odds Function: The mathematical formula for odds function is:

$$O(x)=\frac{F(x)}{S(x)} \tag{13}$$

So, we obtain the odds function of the KumPerks distribution as:

$$O(x)=\frac{1-\left\{1-\left[1-\left(\dfrac{1+c}{1+ce^{dx}}\right)\right]^{a}\right\}^{b}}{\left\{1-\left[1-\left(\dfrac{1+c}{1+ce^{dx}}\right)\right]^{a}\right\}^{b}} \tag{14}$$

For $x>0, a>0, b>0, c>0, d>0$

Distribution of Order Statistics

The pdf of the ith order statistic for a random sample of size n from a distribution function $F(x)$ and an associated pdf $f(x)$ is given by:

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$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}f(x)\left[F(x)\right]^{i-1}\left[1-F(x)\right]^{n-i} \tag{15}$$

So, we obtain the pdf for the ith order statistics for the KumPerks distribution as:

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}abcde^{dx}\frac{(1+c)}{\left(1+ce^{dx}\right)^2}\left[1-\left(\frac{1+c}{1+ce^{dx}}\right)\right]^{a-1}\left\{1-\left[1-\left(\frac{1+c}{1+ce^{dx}}\right)\right]^{a}\right\}^{b-1}\times$$
$$\left[1-\left\{1-\left[1-\left(\frac{1+c}{1+ce^{dx}}\right)\right]^{a}\right\}^{b}\right]^{i-1}\left[\left\{1-\left[1-\left(\frac{1+c}{1+ce^{dx}}\right)\right]^{a}\right\}^{b}\right]^{n-i}$$

The distribution of minimum order statistics for the KumPerks distribution is:

$$f_{1:n}(x)=abcdne^{dx}\frac{(1+c)}{\left(1+ce^{dx}\right)^2}\left[1-\left(\frac{1+c}{1+ce^{dx}}\right)\right]^{a-1}\times$$
$$\left\{1-\left[1-\left(\frac{1+c}{1+ce^{dx}}\right)\right]^{a}\right\}^{b-1}\left[\left\{1-\left[1-\left(\frac{1+c}{1+ce^{dx}}\right)\right]^{a}\right\}^{b}\right]^{n-1} \tag{16}$$

The corresponding distribution of maximum order statistics is:

$$f_{n:n}(x)=abcdne^{dx}\frac{(1+c)}{\left(1+ce^{dx}\right)^2}\left[1-\left(\frac{1+c}{1+ce^{dx}}\right)\right]^{a-1}\times \tag{17}$$
$$\left\{1-\left[1-\left(\frac{1+c}{1+ce^{dx}}\right)\right]^{a}\right\}^{b-1}\left[1-\left\{1-\left[1-\left(\frac{1+c}{1+ce^{dx}}\right)\right]^{a}\right\}^{b}\right]^{n-1}$$

Random Sample Generation and Median

Random samples can be generated for the KumPerks distribution using the quantile function. Quantile function (q) is mathematically denoted by:

$$q=F^{-1}(u) \tag{18}$$

Hence, the quantile function for the KumPerks distribution is derived as:

$$q=d^{-1}\ln\left\{c^{-1}\left[\frac{1+c}{1-\left[1-(1-u)^{1/b}\right]^{1/a}}-1\right]\right\} \tag{19}$$

The median is simply derived by substituting u=0.5 into Equation (19) as:

$$q_{0.5}=d^{-1}\ln\left\{c^{-1}\left[\frac{1+c}{1-\left[1-(0.5)^{1/b}\right]^{1/a}}-1\right]\right\} \tag{20}$$

Other quantiles can also be derived by substituting the appropriate values for u.

Parameter Estimation

Let $x_1,x_2,\ldots,x_n$ denote random samples of size 'n' drawn from the densities of the KumPerks distribution. Using the method of maximum likelihood estimation (MLE), the likelihood function is:

$$f(x_1,x_2,\ldots,x_n;a,b,c,d)=$$
$$\prod_{i=1}^{n}\left[abcde^{dx_i}\frac{(1+c)}{\left(1+ce^{dx_i}\right)^2}\left[1-\left(\frac{1+c}{1+ce^{dx_i}}\right)\right]^{a-1}\left\{1-\left[1-\left(\frac{1+c}{1+ce^{dx_i}}\right)\right]^{a}\right\}^{b-1}\right]$$

The log-likelihood function denoted by L is:

$$L=n\log(a)+n\log(b)+n\log(c)+n\log(d)+d\sum_{i=1}^{n}x_i+n\log(1+c)$$
$$-2\sum_{i=1}^{n}\log\left(1+ce^{dx_i}\right)+(a-1)\sum_{i=1}^{n}\log\left[1-\left(\frac{1+c}{1+ce^{dx_i}}\right)\right]+$$
$$(b-1)\sum_{i=1}^{n}\log\left\{1-\left[1-\left(\frac{1+c}{1+ce^{dx_i}}\right)\right]^{a}\right\} \tag{21}$$

The log-likelihood function is differentiated with respect to parameters a,b,c and d, the resulting equations are equated to zero and solved simultaneously by a closed form solution could not be obtained. Software can however be adopted to estimate the parameters using numerical datasets.

IV. CONCLUSION

We have successfully defined the KumPerks distribution. The shape of the KumPerks distribution could either be unimodal or decreasing. Explicit expressions for the survival, hazard, reversed hazard and odds functions were given. The shapes of the failure rate indicate that the KumPerks distribution would be useful for modeling data sets with decreasing, increasing and inverted bathtub failure rates (though, the plot is not shown in this article because of space conservation).

It is expected that the model would be useful and applicable in modeling data sets in actuarial sciences and the model is also expected to be more flexible than the Exponentiated Perks distribution proposed by [7] because of the extra shape parameter that we introduced.

Further research would involve application to real life data sets with the aim of assessing the potentials of the proposed KumPerks distribution. Simulation studies can also be conducted to investigate the behavior of the model parameters.

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REFERENCES

[1] W. Perks (1932). On some experiments in the graduation of mortality statistics, *Journal of the Institute of Actuaries*, 43, 12-57

[2] S. J. Richards (2008). Applying survival models to pensioner mortality data, *British Actuarial Journal*, 14(2), 257-303

[3] S. Habeman, A. Renshaw (2011). A comparative study of parametric mortality projection models, *Insurance: Mathematics and Economics*, 48(1), 35-55.

[4] A. K. Chaudhary, V. Kumar (2013). A bayesian analysis of Perks distribution via markov chain monte carlo simulation, *Nepal Journal of Science and Technology*, 14(1), 153-166.

[5] S. Nadarajah, S. A. A. Bakar (2012). ActuDistns: A new R package for actuarial survival models. http:/cran.r-project.org/web/packages/ActuDistns/index.html.

[6] T. W. Yee (2012). The VGAM Package, *R News*, 8, 28-39.

[7] B. Singh, N. Choudhary (2017). The exponentiated Perks distribution, International *Journal of System Assurance and Engineering Management*, 8(2), 468-478.

[8] G. M. Cordeiro, M. de Castro (2011). A new family of generalized distributions, *Journal of Statistical Computation and Simulation*, 81, 883-898.

[9] N. Eugene, C. Lee, F. Famoye (2002). Beta-Normal distribution with applications, *Communications in Statistics: Theory and Methods*, 31, 497-512.

[10] W. T. Shaw, I. R. Buckley (2007). The alchemy of probability distributions: beyond Gram-Charlier expansions and a skew kurtotic-normal distribution from a rank transmutation map, *Research Report*.

[11] E. A. Owoloko, P. E. Oguntunde, A. O. Adejumo (2015). Performance rating of the transmuted exponential distribution: an analytical approach, *SpringerPlus*, 4:818.

[12] P. E. Oguntunde, A. O. Adejumo, Okagbue H. I., M. K. Rastogi (2016). Statistical Properties and Applications of a New Lindley Exponential Distribution, *Gazi University Journal of Science*, 29(4), 831-838.

[13] G. M. Cordeiro, M. Alizadeh, A. D. C. Nascimento, M. Rasekhi (2016). The exponentiated Gompertz generated family of distributions: properties and applications, *Chilean Journal of Statistics*, 7(2), 29-50.

[14] M. Alizadeh, G. M. Cordeiro, L. G. Bastos Pinho, I. Ghosh (2017). The Gompertz-G family of distributions, *Journal of Statistical Theory and Practice*, 11(1), 179-207.

[15] P. E. Oguntunde, M. A. Khaleel, M. T. Ahmed, A. O. Adejumo, O. A. Odetunmibi (2017). A new generalization of the Lomax distribution with increasing, decreasing and constant failure rate, *Modelling and Simulation in Engineering*, Article ID 6043169, 6 pages

[16] F. Merovci, M. A. Khaleel, N. A. Ibrahim, M. Shitan, The Beta Type X distribution: Properties and Application, *SpringerPlus*, 5: 697.

[17] P. E. Oguntunde, O. A. Odetunmibi, H. I. Okagbue, P. O. Ugwoke (2015). The Kumaraswamy-Power Distribution: A Generalization of the Power Distribution, *International Journal of Mathematical Analysis*, 9(13), 637-645.

[18] P. E. Oguntunde, O. S. Babatunde, A. O. Ogunmola (2014). Theoretical Analysis of the Kumaraswamy Inverse Exponential Distribution, *International Journal of Statistics and Applications*, 4(2), 113-116.

[19] P. E. Oguntunde, A. O. Adejumo, E. A. Owoloko (2017). Application of Kumaraswamy Inverse Exponential Distribution to Real Lifetime Data, *International Journal of Applied Mathematics and Statistics*, 56(5), 34-47.

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