Solutions of Nonclassical Ordinary Differential Equations and the Associated Kurzweil Equations

S. A. Bishop, *Member, IAENG,* E. O. Ayoola, H. I. Okagbue

***Abstract*— The nonclassical ordinary differential equation is an equivalent form of the Hudson and Parthasarathy formulation of quantum stochastic differential equation (QSDE). In this paper, we address certain qualitative aspects, the existence, uniqueness and stability of solutions of Kurzweil equations associated with a certain class of quantum stochastic differential equations. We study the dynamics and stability of such solutions in a locally convex topological space of quantum stochastic processes.**

***Index Terms*— Nonclassical ODE, Non-Lipschitz function, Kurzweil equations and integrals, noncommutative stochastic processes.**

I. INTRODUCTION

As with classical differential equations, existence and uniqueness of solution of SDEs have been studied by many authors [8-11]. In [9] a class of stochastic differential equations (SDEs) with non-Lipschitz conditions was studied. A unique strong solution is obtained and other qualitative properties of solution were studied. The results in [9] generalized the classical Lipschitz condition for existence of solutions of SDEs.

It was shown in [2] that a SDE has a unique solution in the L^2 space provided the coefficients satisfy a certain Lipschitz condition. However, these methods are not directly applicable in our own case. For the class of equations considered here, recent methods of establishing existence of solutions have been applied and many interesting phenomena have been established in [1, 3-6].

We consider the following QSDE introduced in [5]:

$$dX(t) = E(X(t),t)d\Lambda_\pi(t) + F(X(t),t)dA_f^+(t) + G(X(t),t)dA_g(t) + H(X(t),t)dt$$
$$X(t_0) = X_0, \quad t \in I, \qquad (1)$$

Manuscript received March 1, 2018; revised June 20, 2018. This work was supported in full by Covenant University Center for Research, Innovation and Development, Ota, Nigeria.

Sheila A. Bishop (sheila.bishop@covenantuniversity.edu.ng; +2347081203200) is with the Department of Mathematics, Covenant University P.M.B. 1023, Km. 10 Idiroko Road, Canaan Land, Ota, Ogun State, Nigeria.

E. O. Ayoola (eoayoola@gmail.com) is with the Department of Mathematics, University of Ibadan, Ibadan, Nigeria.

Hilary. I. Okagbue (hilary.okagbue@covenantuniversity.edu.ng) is with the Department of Mathematics, Covenant University P.M.B. 1023, Km. 10 Idiroko Road, Canaan Land, Ota, Ogun State, Nigeria.

where

(a) $f, g \in L^\infty_\gamma(\mathbb{R}_+)$, $\pi \in L^\infty_{B(\gamma)}(\mathbb{R}_+)$.

(b) Λ_π, A_f^+, A_g are stochastic processes.

(c) $E, F, G, H \in L^2_{loc}([0,T] \times \tilde{\mathcal{A}})_{mvs}$.

(d) t is the Lebesgue measure.

It has been shown that the quantum stochastic differential equation (1) introduced by Hudson and Parthasarathy [6] provide an essential tool in the theoretical description of physical systems, especially those arising in quantum optics, quantum measure theory, quantum open systems and quantum dynamical systems.

Ekhaguere in [5], established that the following nonclassical ODE;

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle = P(X(t),t)(\eta,\xi)$$
$$X(t_0) = X_0, \quad t \in I \qquad (2)$$

is equivalent to (1) and the operator $(x,t) \to P(x,t)(\eta,\xi)$ in (2) is defined in [3, 4],

Where $\eta, \xi \in (\mathbb{D} \underline{\otimes} \mathbb{E})$ is arbitrary, $(x,t) \in I \times \tilde{\mathcal{A}}$. $\tilde{\mathcal{A}}, \mathbb{D}, \mathbb{E}$ and others will be defined later.

In [1], the Kurzweil equation

$$\frac{d}{dt}\langle \eta, X(\tau)\xi\rangle = DF(X(\tau),t)(\eta,\xi) \qquad (3)$$

associated with (2) was established. Approximate solutions were obtained. In arriving at the results in [1], some assumptions were imposed on the Lipschitz function thereby limiting the results only to a class of equations that satisfy these assumptions.

In this paper, our aim is to study existence, uniqueness and stability of solution of (2) associated with Kurzweil equations (3) by generalizing the Lipschitz condition. The results established here will be of more application compared with the results of Ayoola and Ekhaguere [1,5]. The essence for generalizing the results in the literature is to enrich the theory of nonclassical ODE and to obtain similar results obtained in the classical setting. See the references [7-9].

Section 2 of this paper will consist of preliminaries while Section in 3, the major results is studied. The iterative method will be employed to establish results on existence and stability respectively.

II. PRELIMINARY RESULTS, DEFINITIONS AND NOTATIONS

$\tilde{\mathcal{A}}$ is a topological space defined in [1]. The definitions and notations of the spaces Ad($\tilde{\mathcal{A}}$), Ad($\tilde{\mathcal{A}}$)$_{wac}$, $L^p_{loc}(\tilde{\mathcal{A}})$, $L^\infty_{loc}(\mathbb{R}_+)$ are obtained from the references [1, 5].

2.1 Definition: Let $I \subseteq \mathbb{R}_+$

ISBN: 978-988-14049-0-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

(i) $\varphi: I \times \mathcal{A} \to \mathcal{A}$ is GLipschitzian if for any $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, we get a function $K_{\eta\xi}^{\varphi}: I \to (0, \infty)$ lying in $L_{loc}^{1}(I)$ such that,

$\|\varphi(x,t) - \varphi(y,t)\|_{\eta\xi} \le K_{\eta\xi}^{\varphi}(t) W \|x - y\|_{\eta\xi}, W(t) \neq t$

where $x, y \in \mathcal{A}$.

(ii) The above definition also holds when $\Phi: I \times \mathcal{A} \to sesq(\mathbb{D} \underline{\otimes} \mathbb{E})$, that is $(x,t) \to \Phi(x,t)(\eta, \xi)$ is also GLipschitzian (resp. continuous).

2.2 Notation: Here GLipschitz, GLipschitzian denotes the generalized Lipschitz functions and maps respectively.

All through this work $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ is arbitrary.

2.3 Definition: $P: \mathcal{A} \times [t_0, T] \to sesq(\mathbb{D} \underline{\otimes} \mathbb{E})$ is in $C(\mathcal{A} \times [t_0, T], W), W(t) \neq t$ if;

(i) $P(x, .)(\eta, \xi)$ is computable

(ii) There are computable functions $M_{\eta\xi}: [t_0, T] \to \mathbb{R}_+ | \int_{t_0}^{t} M_{\eta\xi}(s) ds < \infty$ and

$$|P(x, .)(\eta, \xi)| \le M_{\eta\xi}(s), (x, .) \in \mathcal{A} \times [t_0, T] \quad (4)$$

(iii) We have computable functions $K_{\eta\xi}^{P}: [t_0, T] \to \mathbb{R}_+$ whose integral is finite and

$$\|P(x,s) - P(y,s)\|_{\eta\xi} \le K_{\eta\xi}^{P}(s) W \|x - y\|_{\eta\xi} \quad (5)$$

where $(x,s), (y,s) \in \mathcal{A} \times [t_0, T]$ and $W(t) \neq t$.

2.4 Notation: $C(\mathcal{A} \times [t_0, T], W), W(t) \neq t$ denotes the class with a generalized Lipschitz function that satisfies the Caratheodory conditions.

2.5 Definition: A stochastic process $x \in Ad(\mathcal{A})_{wac}$ is a solution of (2) equivalently (3) if it satisfies the integral equation

$$X(t) = X_0 + \int_0^t E(X(s), s) d\Lambda_\pi(s) + F(X(s), s) dA_f^{+}(s) + G(X(s), s) dA_g(s) + H(X(s), s) ds \quad (6)$$

The following results established in [5] will be useful.

2.1 Theorem: Assume the following:

(i) $p, q, u, v \in Ad(\mathcal{A})_t$ are simple stochastic processes and M is their stochastic integral.

(ii) If $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, where (ii) If $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, where $\xi = c \otimes e(\alpha)$, $\eta = d \otimes e(\beta), c, d \in \mathbb{D}$, $\alpha, \beta \in L_{\gamma, loc}^{\infty}(\mathbb{R}_+)$ for $t \ge 0$, then

$$\langle \eta, M(t)\xi \rangle = \int_0^t < \eta, \{ \langle \alpha(s), \pi(s)\beta(s)\xi \rangle_\gamma p(s) + \langle f(s), \beta(s)\xi \rangle_\gamma q(s) + \langle \alpha(s), g(s) \rangle_\gamma u(s) + v(s) \} \xi > ds \quad (7)$$

III. MAJOR RESULTS

We consider QSDE (1) introduced in section 1. The results are established under the conditions of *Definition 2.3.*

3.1 Theorem

(i) Let $P(x,t)(\eta, \xi)$ belong to $C(\mathcal{A} \times [t_0, T], W), W(t) \neq t$.

(ii) Assume the coefficients $E, F, G, H \in L_{loc}^{2}(\mathcal{A} \times I)$ satisfy the general Lipschitz condition.

Then for any $(x_0, t_0) \in \mathcal{A} \times I$ there exists a unique solution $\varphi \in L_{loc}^{2}(\mathcal{A}) \cap Ad(\mathcal{A})_{wac}$ of equation (1) equivalently (2) satisfying $\varphi(t_0) = x_0$.

Proof. Let $\{\varphi_n(t)\}_{n \ge 0}$ be a Cauchy sequence in $\mathcal{A}$. All through except otherwise stated $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ is arbitrary. Let $\varphi_0(t) = x_0, t \in [t_0, T]$ and for $n \ge 0$

$$\varphi_{n+1}(t) = x_0 + \int_{t_0}^{t} (E(\varphi_n(s), s) d\wedge_\pi(s) + F(\varphi_n(s), s) d(s)$$

$$G(\varphi_n(s), s) dA_f(s) + H(\varphi_n(s), s) d(s))$$

Let $\varphi_n(t) \in L_{loc}^{2}(\mathcal{A}), n \ge 1$.

By hypothesis, $E(x_0, s), F(x_0, s), G(x_0, s), H(x_0, s) \in \mathcal{A} \times [t_0, T]$, so also $E(x_0, .), F(x_0, .), G(x_0, .), H(x_0, .) \in L_{loc}^{2}(\mathcal{A})$. Therefore $\varphi_1(t), t \in [t_0, T]$ is well defined and the associated quantum stochastic integral exists.

By (7), $\varphi_1(t) \in L_{loc}^{2}(\mathcal{A}) \cap Ad(\mathcal{A})_{wac}$. Assume the same holds for $\varphi_n(t)$, then each $E(\varphi_n(s), s), F(\varphi_n(s), s), G(\varphi_n(s), s)$ *and* $H(\varphi_n(s), s)$ is adapted and lie in $L_{loc}^{2}(\mathcal{A})$. Thus, the same holds for $\varphi_{n+1}(t)$.

Again using (7), $\varphi_{n+1}(t) \in L_{loc}^{2}(\mathcal{A}) \cap Ad(\mathcal{A})_{wac}$. Hence our claim is established by induction. Next, we show the convergence of these successive approximations.

The definition of the map P and (7) yields,

$$\|\varphi_{n+1}(t) - \varphi_n(t)\|_{\eta\xi} = |\langle \eta, (\varphi_{n+1}(t) - \varphi_n(t))\xi \rangle| = = \left| \int_{t_0}^{t} (P(\varphi_n(s), s)(\eta, \xi) - P(\varphi_{n-1}(s), s)(\eta, \xi)) ds \right| \quad (8)$$

Hence since the coefficients E, F, G, H satisfy the general Lipschitz condition, the same is true for

$(x,t) \to (x,t)(\eta, \xi)$ and hence

$$|P(x,t)(\eta, \xi) - P(y,t)(\eta, \xi)| \le K_{\eta\xi}^{P}(t) W \|x - y\|_{\eta\xi} \quad (*)$$

where $x, y \in \mathcal{A}$ *and* $t \in [t_0, T]$.

Hence using the last inequality (*) in (8), we get

$$\|\varphi_{n+1}(t) - \varphi_n(t)\|_{\eta\xi} \le \int_{t_0}^{t} K_{\eta\xi}^{P}(s) W(\|\varphi_n(s) - \varphi_{n-1}(s)\|_{\eta\xi}) ds \quad (9)$$

Because $s \to \|\varphi_1(s) - x_0\|_{\eta\xi}$ is continuous on $[t_0, T]$, let $R_{\eta\xi} = sup_{s \in [t_0, T]} \|\varphi_1(s) - x_0\|_{\eta\xi}$, $s \in [t_0, T]$ which yields $\|\varphi_1(s) - x_0\|_{\eta\xi} \le R_{\eta\xi}$ also $W(\|\varphi_1(s) - x_0\|_{\eta\xi}) \le W(R_{\eta\xi})$ since $W(t) \neq t$. Also let

$$M_{\eta\xi}(t) = \int_{t_0}^{t} K_{\eta\xi}^{P}(s) ds$$

From (9) we obtain

$$\|\varphi_{n+1}(t) - \varphi_n(t)\|_{\eta\xi} \le \frac{W(R_{\eta\xi})(M_{\eta\xi}(t))^n}{n!}, n = 1, 2, \quad (10)$$

The following is established by induction.

For n=1, inequality (10) holds by (9). Suppose (10) is true when n=k which implies

$$\|\varphi_{k+1}(t) - \varphi_k(t)\|_{\eta\xi} \le \frac{W(R_{\eta\xi})(M_{\eta\xi}(t))^k}{k!}, n = 1, 2, \quad (11)$$

using (9)

$$\|\varphi_{k+2}(t) - \varphi_{k+1}(t)\|_{\eta\xi} \le$$

ISBN: 978-988-14049-0-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$\int_{t_0}^{t} K_{\eta\xi}^{P}(s)\, W(\|\varphi_{k+1}(s) - \varphi_k(s)\|_{\eta\xi})ds$
$= \int_{t_0}^{t} K_{\eta\xi}^{P}(s) \frac{W(R_{\eta\xi})(M_{\eta\xi}(t))^k}{k!}\, ds \le$
$\frac{W(R_{\eta\xi})(M_{\eta\xi}(t))^k}{k!} \int_{t_0}^{t} K_{\eta\xi}^{P}(s)\, (M_{\eta\xi}(s))^k ds$ by (11).

By applying integration by parts, we obtain

$$\int_{t_0}^{t} K_{\eta\xi}^{P}(s)\, (M_{\eta\xi}(s))^k ds = \frac{(M_{\eta\xi}(t))^{k+1}}{k+1} \qquad (12)$$

So that,

$$\|\varphi_{k+2}(t) - \varphi_{k+1}(t)\|_{\eta\xi} \le \frac{W(R_{\eta\xi})(M_{\eta\xi}(t))^{k+1}}{(k+1)!}$$

and (10) is true for n= k+1, n= 1,2,3,…
This implies that when n > k, we get

$$\|\varphi_{n+1}(t) - \varphi_{k+1}(t)\|_{\eta\xi}$$

$$\|\sum_{m=k+1}^{n}(\varphi_{m+1}(t) - \varphi_m(t))\|_{\eta\xi}$$

$$\le \sum_{m=k+1}^{n} \|\varphi_{m+1}(t) - \varphi_m(t)\|_{\eta\xi}$$

$$\le \sum_{m=k+1}^{n} \frac{W(R_{\eta\xi})(M_{\eta\xi}(T))^m}{m!}$$

Therefor the Cauchy sequence $\{\varphi_n(t)\}$ is in $\tilde{\mathcal{A}}$ and $\varphi_n(t) \to \varphi(t)$ uniformly. Now if $\varphi_n(t) \in Ad(\tilde{\mathcal{A}})_{wac}$, it implies that $\varphi \in Ad(\tilde{\mathcal{A}})_{wac}$. Now we establish that φ satisfies QSDE (1).

Surely $\varphi(t_0) = x(t_0) = x_0$. Again, by equation (7),

$$\| \int_{t_0}^{t}[E(\varphi_n(s), s)d\Lambda_\pi(s) + F(\varphi_n(s), s)dA_g^+(s) +$$
$$+G(\varphi_n(s), s)dA_f(s) + H(\varphi_n(s), s)ds \;\|_{\eta\xi}$$
$$-\| \int_{t_0}^{t} E(\varphi(s), s)d\Lambda_\pi(s) + F(\varphi(s), s)dA_g^+(s) +$$
$$+G(\varphi(s), s)dA_f(s) + H(\varphi(s), s)ds \;\|_{\eta\xi}$$
$$= \left|\int_{t_0}^{t} [P(\varphi_n(s), s)(\eta, \xi) - P(\varphi(s), s)(\eta, \xi)]ds\right|$$

$$\le \int_{t_0}^{t} K_{\eta\xi}^{P}(s)\, W(\|\varphi_n(s) - \varphi(s)\|_{\eta\xi})ds \to 0$$
$$as\ n \to \infty$$

Because $\varphi_n(s) \to \varphi(s)$ in $\tilde{\mathcal{A}}$ uniformly for $t \in [t_0, T]$.
Therefore,

$$\varphi(t) = \lim_{n\to\infty} \varphi_{n+1}(t)$$
$$x_0 + \lim_{n\to\infty}(\int_{t_0}^{t} (E(\varphi(s), s)d\Lambda_\pi(s) +$$
$$+F(\varphi_n(s), s)dA_g^+(s) + G(\varphi_n(s), s)dA_f(s)$$
$$+H(\varphi_n(s), s)ds)$$
$$= x_0 + \int_{t_0}^{t} (E(\varphi(s), s)d\Lambda_\pi(s) + F(\varphi(s), s)dA_g^+(s))$$
$$+G(\varphi(s), s)dA_f(s) + H(\varphi(s), s)d$$

Therefore $\varphi(t)$ is a solution of equation (1).

Uniqueness

Let $y(t) \in Ad(\tilde{\mathcal{A}})_{wac}$, $t \in [t_0, T]$ be another solution with $y(t_0) = x_0$.Then, by equation (7), we get

$$\|\varphi(t) - y(t)\|_{\eta\xi}$$
$$= \int_{t_0}^{t} [P(\varphi(s), s)(\eta, \xi) - P(y(s), s)(\eta, \xi)]ds$$
$$\le \int_{t_0}^{t} K_{\eta\xi}^{P}(s)\, W(\|\varphi(s) - y(s)\|_{\eta\xi})ds$$

Since the integral $\int_{t_0}^{t} K_{\eta\xi}^{P}(s)\, ds$ exists on the given interval, it is also essentially bounded. Then, we get a constant $C_{\eta\xi,t}$ such that ess sup$K_{\eta\xi}^{P}(s) = C_{\eta\xi,t}$, $s \in [t_0, T]$.

Thus

$$\|\varphi(t) - y(t)\|_{\eta\xi}$$
$$\le C_{\eta\xi,t} \int_{t_0}^{t} W(\|\varphi(s) - y(s)\|_{\eta\xi})\, ds$$

By the Gronwall's inequality, we conclude that $\varphi(t) = y(t)$, $t \in [t_0, T]$. Hence the solution is unique.

Stability of solution

The next theorem establishes that the solutions to the nonclassical ODE (2) associated with (3) are stable. Let E, F, G and H satisfy the conditions of theorem 3.1 and let $x(t), y(t)$ be solutions of equation (2) corresponding to the initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$, respectively where x_0, $y_0 \in \tilde{\mathcal{A}}$.

The next result shows that the solution $x(t)$ is stable under changes in the initial condition $x(t_0) = x_0$.

Theorem 3.2: Given $T > t_0$ and $\varepsilon > 0$, $\exists\ \delta > 0$ such that if $\|x_0 - y_0\|_{\eta\xi} < \delta$,
then $\|x(t) - y(t)\|_{\eta\xi} < \varepsilon$ for all $t \in [t_0, T]$, $\eta, \xi \in \mathbb{D} \otimes \mathbb{E}$.

Proof: Let $x_n(t),\ y_n(t), n = 0, 1, \ldots$ be the iterates corresponding to $x_0 and\ y_0$ respectively, and let $x(t_0) = x_0$, $y(t_0) = y_0$ for $t_0 \le T$, then we obtain the following estimate by employing the definition of P and equation (7) as in the proof of uniqueness of solution.

$$\|x_{n+1}(t) - y_{n+1}(t)\|_{\eta\xi} \le \|x_0 - y_0\|_{\eta\xi}$$
$$+ \int_{t_0}^{t} [P(x_n(s), s)(\eta, \xi) - P(y_n(s), s)(\eta, \xi)]ds$$
$$= \|x_0 - y_0\|_{\eta\xi} +$$
$$+ W(C_{\eta\xi,t} \int_{t_0}^{t}(\|x_n(s) - y_n(s)\|_{\eta\xi})\, ds),$$

where $C_{\eta\xi,t}$ is the essential supremum of $K_{\eta\xi}^{P}(t)$ on $[t_0, T]$.
Therefore, iterating yields

$$\|x_{n+1}(t) - y_{n+1}(t)\|_{\eta\xi} \le$$
$$\|x_0 - y_0\|_{\eta\xi} +$$
$$+W(C_{\eta\xi,t_1} \int_{t_0}^{t_1} (\|x_{n-1}(t_2) - y_{n-1}(t_2)\|_{\eta\xi}\, dt_2)$$
$$\le \|x_0 - y_0\|_{\eta\xi} + W(l_{\eta\xi}\|x_0 - y_0\|_{\eta\xi}) +$$
$$+W\left(l_{\eta\xi}^2 \int_{t_0}^{t} \int_{t_0}^{t_1} \|x_{n-1}(t_2) - y_{n-1}(t_2)\|_{\eta\xi})\, dt_2\right),$$

where $t_0 \le T$, $L_{\eta\xi} = \max\{C_{\eta\xi}, C_{\eta\xi,t_1}\}$.
Continuing the iteration and putting $l_{\eta\xi} = \max\{C_{\eta\xi,t}, C_{\eta\xi,t_j}, j = 1, 2, \ldots, n\}$,
we obtain the estimate

$$\|x_{n+1}(t) - y_{n+1}(t)\|_{\eta\xi} \le \|x_0 - y_0\|_{\eta\xi}$$
$$+WL_{\eta\xi}(t - t_0)\|x_0 - y_0\|_{\eta\xi} + \cdots$$
$$WL_{\eta\xi}^n \frac{(t - t_0)^n}{n!} \|x_0 - y_0\|_{\eta\xi} + W(L_{\eta\xi}^{n+1} \int_{t_0}^{t} \int_{t_0}^{t_1} \ldots$$
$$\int_{t_0}^{t_n} \|x_0 - y_0\|_{\eta\xi})\, dt_1 dt_2 \ldots dt_{n+1}$$

ISBN: 978-988-14049-0-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$\leq W \sum_{m=0}^{n+1} \frac{L_{\eta\xi}^{m}}{m!} (t - t_0)^n \|x_0 - y_0\|_{\eta\xi}$$
$$\leq W \sum_{m=0}^{n+1} \frac{(l_{\eta\xi} T)^m}{m!} \|x_0 - y_0\|_{\eta\xi}$$

Letting $n \to \infty$, yields
$$\|x(t) - y(t)\|_{\eta\xi} \leq W \|x_0 - y_0\|_{\eta\xi} e^{(l_{\eta\xi} T)},$$
$$t_0 \leq t \leq T.$$

IV. CONCLUSION

We have established existence, uniqueness and stability of solution of QSDE (2) associated with Kurzweil equation (3) via equation (1). This is possible since equivalence of these equations have been established in [1, 5]. Hence existence of solution of (1) equivalently (2) will imply existence of solution of (3). The results in this paper generalize existing results on existence of solution of Eq. (1). If the condition W(t) = t, we obtain the results in [1].

ACKNOWLEDGMENT

The authors will like to appreciate the management of Covenant University for the financial support given to them for the completion of this work.

REFERENCES

[1] E. O. Ayoola, Lipschitzian Quantum Stochastic Differential Equations and the Associated Kurzweil Equations. Stochastic Analysis And Applications, 19(4)(2001), 581-603
[2] C. Barnett, R. F. Streater and I. F. Wilde, The Ito-Clifford integral II- stochastic differential equations. J. London Math. Soc. 27(2)(1983), 373-384.
[3] S. A. Bishop and E. O. Ayoola, Variational Stability for Kurzweil Equations associated with Quantum Stochastic Differential Equations. Australian J. of Basic and Applied Sciences, 7(2013), 787-798.
[4] G.O. S. Ekhaguere, Lipschitzian Quantum Stochastic Differential Inclusions. Int. journal of theoretical physics, **31(**11), 2003-2034 (1992).
[5] R. L. Hudson and K. R. Parthasarathy, Quantum Ito's formulae and stochastic evolutions. Comm. math. phys., **93**, 301-324 (1984).
[6] J. Kurzweil, Ordinary Differential Equations. Elsevier, Amsterdam, 1986.
[7] F. Shizan, Recent Developments in Stochastic Differential Equations. Mathematics Subject Classification. Pry 60H10; Sec. 60J60, 60J65,(1991).
[8] F. Shizan and T. Zhang, A Study of a Class of Stochastic Differential Equations with Non-Lipschitzian Coefficients. Springer-Verlag, 132(2005), 356-390.
[9] K. Max, V. Renesse and M. Scheutzow, Existence and uniqueness of solutions of stochastic functional differential equations. "Technische Universit"at Berlin, November 20, (2009).
[10] F. Wang and J. Wang, Finite and Infinite Dimensional Stochastic Differential equations with Non-Lipschitzian coefficients. Chinese J. Applied Prob. and Statistics, 25(2) (2009).

ISBN: 978-988-14049-0-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)