Parametric Vibration of a Cardan Shaft and Sensitivity Analysis

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Abstract—In this paper a new model for analyzing self-excited torsional vibrations of a heavy duty ground vehicle cardan shaft is proposed. The model considers two heavy inertias articulated by a circuit of torsional springs and a damper. The inertias of the system at either ends are driven by a gearbox through a Hooke’s joint. Traction resistance torques at road wheels and engine are projected onto the shaft through gearing transmission. The model has been used to study sensitivity of the transmission to rigid body motion, elastic deformation and stiffness perturbation of the Hooke’s joint. It is shown that, cardan shaft vibration excitations are higher at start-up and dampen away in after a short duration. Modelled results reveal that the sensitivity of stiffness to velocity and displacement inputs is higher at start-up than steady-state motion of the shaft.

Index Terms—Cardan shaft, Coupling, Nonlinear, Torsional Vibration

I. INTRODUCTION

A cardan shaft of a vehicle driveline is essential for transmitting torque from the gearbox to the final drive [1-2]. Torsion is the most prevalent type of loading in ground vehicle drivelines. Vibration of driveline systems is one of the principal causes of noise in ground vehicle transmissions [4-6]. Analysis of torsional vibration of a ground vehicle driveline is a basic and an important step in safe design of vehicle power transmission systems.

Identification of dynamic load spectra of a driveline assembly is critical in evaluating the load carrying capacity, structural integrity and maintenance of vehicle transmission [2]. A theoretical study of driveline torsional vibration was reported by Cathpole and Healy et al [7]. The research used vibration histories taken from a number of stations along the driveline, and passenger compartment noise levels recorded in a series of road tests. Application of digital analysis to the data revealed that, torsional resonances cause high noise levels in a car. In [8], acceleration vibration response of a rear driving axle caused by common excitation forces was investigated based on multibody modeling of a car taking into account flexibility of major components of the powertrain. In [9], transient characteristics of a vehicle powertrain system were investigated. The investigation compared free and forced torsional vibration of the complex test rig with that of an existing car.

This paper, presents a model for analyzing partial vibration of a driveline system. The system comprises, an elastic cardan shaft subsystem with a Hooke’s joint. The model is developed from vehicle dynamics and vibration theory. Parametric excitation due to Hooke’s joint is modelled as a perturbation of small order. The model has been used to determine fundamental torsional modes at low frequencies.

II. SYSTEM DESCRIPTION

The system model in figure 1b is developed by adopting basic dynamics of a Hooke’s coupling. The model considers a simple cardan shaft whose own inertia is represented by four discrete rigid inertia disks, \( J_m1 \), \( J_m2 \), \( J_m3 \), \( J_m4 \) mounted on two massless torsional springs, \( k_1 \), \( k_2 \).

Inertia \( J_1 \) comprises \( J_m1 \), inertias of the, parts attached to and rotating with the gearbox output shaft, inertias of the flywheel and of the parts of the engine rotating with the gearbox output shaft. \( J_4 \) comprises \( J_m4 \), and inertias of the parts attached to and rotating with the final drive shaft including the road wheels.

![Fig. 1 a Kinematic sketch of the cardan shaft system](image-url)
Rigid-body rotation of $J_4$, $\theta_2$, has been determined by substituting, $\psi_1 = 0$ in equations (3) and (7), whereupon,

$$\theta_{2s} = \theta - \frac{an}{(\gamma n^2 + 1)}$$

Where, $n = \tan \theta$.

IV. SYSTEM KINETIC ENERGY

This comprises the kinetic energy of the parts of the driveline coupled to and rotating with the cardan shaft, and is expressed as

$$G = \frac{1}{2} J_1 \dot{\theta}^2 + \frac{1}{2} J_2 (\theta + \psi_1 )^2 +$$

$$+ \frac{1}{2} J_3 \left[ (\theta + \psi_1 ) - \frac{d}{dt} \left( \mu (\theta + \psi_1 ) \right) \right]^2$$

$$+ \frac{1}{2} J_4 (\theta_s + \psi_2 )^2$$

Differentiating equation (10.1), assigning and performing substitutions as follows:

$$\frac{d}{dt} \left( \mu (\theta + \psi_1 ) \right) = \frac{d}{dt} \left( \frac{an}{(\gamma n^2 + 1)} (\theta + \psi_1 ) \right)$$

$$= \Gamma (\dot{\theta} + \dot{\psi}_1 )$$

$$\theta_{2s} = \frac{d}{dt} \left[ \theta - \frac{an}{(\gamma n^2 + 1)} \right] = \dot{\theta}(1 - g)$$

Where,

$$\Gamma = \frac{aN}{\gamma N^2 + 1} \left( \frac{N^2 + 1}{N} - \frac{2\gamma N(N^2 + 1)}{\gamma N^2 + 1} \right)$$

$$g = \frac{an}{\gamma n^2 + 1} \left( \frac{n^2 + 1}{n} - \frac{2\gamma n(n^2 + 1)}{\gamma n^2 + 1} \right)$$

leads to equation (12) below

$$G = \frac{1}{2} J_1 \dot{\theta}^2 + \frac{1}{2} J_2 (\theta + \psi_1 )^2 + \frac{1}{2} J_3 [1 - \Gamma ](\theta + \psi_1 )^2 +$$

$$+ \frac{1}{2} J_4 (\theta + \psi_1 )^2$$

V. SYSTEM POTENTIAL ENERGY

This comprise the torsional strain energy between points A and B and points C and D in figure 1b and is expressed as,

$$V = \frac{1}{2} k_1 \psi_1^2 + \frac{1}{2} k_2 [1 - \mu ](\psi_1 - \psi_2 )^2$$

VI. THE RAYLEIGH DISSIPATION FUNCTION

The Rayleigh’s dissipation function as follows:

$$D = \frac{1}{2} c_1 \psi_1^2 + \frac{1}{2} c_2 \left[ \frac{d}{dt} \left(1 - \mu \right) \psi_1 - \psi_2 \right]^2$$
VII. THE EQUATION OF MOTION

The Lagrangian equation in an \( i \) generalized coordinate frame is

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} + V = \{ T_0 \}, \quad q_i = \theta, \psi_1, \psi_2
\]

(15)

Upon substitution of equations (13), (14) into equation (15), performing requisite differentiation and manipulation, the system dynamic equation reads,

\[
\begin{bmatrix}
    m_{\theta \theta} & m_{\theta \psi_1} & m_{\theta \psi_2} \\
    m_{\psi_1 \theta} & m_{\psi_1 \psi_1} & m_{\psi_1 \psi_2} \\
    m_{\psi_2 \theta} & m_{\psi_2 \psi_1} & m_{\psi_2 \psi_2}
\end{bmatrix}
\begin{bmatrix}
    \ddot{\theta} \\
    \ddot{\psi}_1 \\
    \ddot{\psi}_2
\end{bmatrix}
+ \begin{bmatrix}
    c_{\theta \theta} & c_{\theta \psi_1} & c_{\theta \psi_2} \\
    c_{\psi_1 \theta} & c_{\psi_1 \psi_1} & c_{\psi_1 \psi_2} \\
    c_{\psi_2 \theta} & c_{\psi_2 \psi_1} & c_{\psi_2 \psi_2}
\end{bmatrix}
\begin{bmatrix}
    \dot{\psi}_1 \\
    \dot{\psi}_2
\end{bmatrix}
= \begin{bmatrix}
    \ddot{\psi}_1 \\
    \ddot{\psi}_2
\end{bmatrix}
- \begin{bmatrix}
    P_{\theta} \\
    P_{\psi_1} \\
    P_{\psi_2}
\end{bmatrix}
\]

(16)

Where, the elements of matrices in their final forms are,

\[
m_{\theta \theta} = J_1 + J_2 + J_3 (1 - \Gamma)^2 + J_4 (1 - g)^2
\]

\[
m_{\psi_1 \theta} = m_{\psi_1 \psi_1} = m_{\psi_1 \psi_2} = J_2 + J_3 (1 - \Gamma)^2
\]

\[
m_{\psi_2 \theta} = m_{\psi_2 \psi_1} = m_{\psi_2 \psi_2} = 0
\]

\[
m_{\psi_2 \psi_1} = m_{\psi_2 \psi_2} = k_2
\]

\[
k_{\psi_1 \psi_1} = k_{\psi_1 \psi_2} = k_2 (1 - \mu)
\]

\[
k_{\psi_2 \psi_1} = k_{\psi_2 \psi_2} = k_1 + k_2 (1 - \mu)^2
\]

\[
c_{\theta \theta} = c_2 \left( \psi_1 \left( \frac{\Gamma - \mu}{\theta + \psi_1} \right)^2 \right)
\]

\[
c_{\theta \psi_1} = c_2 \left( \psi_1 \left( \frac{\Gamma - \mu}{\theta + \psi_1} \right) (1 - \mu) \right)
\]

\[
c_{\theta \psi_2} = c_2 \left( \psi_1 \left( \frac{\Gamma - \mu}{\theta + \psi_1} \right) \right)
\]

\[
c_{\psi_1 \theta} = c_2 \left( \left( \frac{\Gamma - \mu}{\theta + \psi_1} \right) (1 - \mu) \right)
\]

\[
c_{\psi_1 \psi_1} = c_2 \left( \left( \frac{\Gamma - \mu}{\theta + \psi_1} \right)^2 \right)
\]

\[
c_{\psi_1 \psi_2} = c_2 \left( \left( \frac{\Gamma - \mu}{\theta + \psi_1} \right) (1 - \mu) \right)
\]

\[
c_{\psi_2 \theta} = c_2 \left( \left( \frac{\Gamma - \mu}{\theta + \psi_1} \right) \right)
\]

\[
c_{\psi_2 \psi_1} = c_2 \left( \left( \frac{\Gamma - \mu}{\theta + \psi_1} \right) (1 - \mu) \right)
\]

\[
c_{\psi_2 \psi_2} = c_2 \left( \left( \frac{\Gamma - \mu}{\theta + \psi_1} \right)^2 \right)
\]

The vectors \( \left[ R_{\theta \theta} \ R_{\theta \psi_1} \ R_{\theta \psi_2} \right]^T \), \( \left[ H_{\theta} \ H_{\psi_1} \ H_{\psi_2} \right]^T \), \( \left[ P_{\theta} \ P_{\psi_1} \ P_{\psi_2} \right]^T \) correspondingly are, Hooke’s joint quadratic-velocity excited torque, Hooke’s joint elastic excitation torque (signifying elastic coupling elastic deformations across the Hooke’s joint) and the \( \frac{\partial T}{\partial \dot{q}_i} \) term of equation (15). The vectors are:

\[
\begin{bmatrix}
    R_{\theta \\
    R_{\psi_1} \\
    R_{\psi_2}
\end{bmatrix}
= -2J_4 (1 - \Gamma) \left[ \dot{\theta} + \psi_2 \right] \frac{dg}{d\theta}
\]

(21)

\[
\begin{bmatrix}
    H_{\theta} \\
    H_{\psi_1} \\
    H_{\psi_2}
\end{bmatrix}
= k_2 \begin{bmatrix}
    \psi_2 \psi_1 (1 - \mu) \psi_1^2 \delta \mu \\
    \psi_2 \psi_1 (1 - \mu) \psi_1^2 \delta \mu \\
    0
\end{bmatrix}
\]

(22)

The terms \( \frac{\partial \Gamma}{\partial \theta} \), \( \frac{\partial \Gamma}{\partial \dot{q}_i} \) in their final forms are,

\[
\frac{\partial \Gamma}{\partial \theta} = \frac{2aN(2N^2 + 1)[3 - \gamma(\Gamma - \mu)]}{(\gamma N^2 + 1)^3}
\]

\[
\frac{\partial \Gamma}{\partial \dot{q}_i} = \frac{\partial \Gamma}{\partial \theta} \frac{\partial \dot{q}_i}{\partial \theta} = 0, \quad \frac{\partial \mu}{\partial \theta} \left[ \frac{\Gamma - \mu}{\theta + \psi_1} \right]
\]

(23)

VIII. NUMERICAL SIMULATION AND RESULTS

The system was numerical solutions of equation (16) were performed for: \( c_1 = c_2 = 200 \text{ N.m.s.rad}^{-1} \), \( \omega_b = 20\pi \text{ rad.sec}^{-1} \), \( k_1 = k_2 = 8 \times 10^8 \text{ N.m}^{-1} \), \( \beta_1 = 3^\circ, \beta_2 = 6^\circ, q_1 = 4 \pi \text{ rad.s}^{-2}, \zeta = 0.5, r = 0.95 \text{ m} \), \( \theta_2 = \theta_4 = 0, \theta_2 = \theta_4 = 0.1 \text{ rad.s}^{-1}, \epsilon = 0.4 \).

![Fig.3 Time-displacement history of gear box output shaft](image-url)
Fig. 13 Variation of quadratic-velocity dependent torque excited by Hooke’s joint with gearbox output shaft displacement

Fig. 14 Variation of quadratic-velocity dependent torque with gearbox output shaft displacement

Fig. 15 Variation of quadratic-velocity dependent torque with gearbox output shaft displacement

Fig. 16 Variation of quadratic-velocity dependent torque gearbox output shaft displacement

Fig. 17 Variation of quadratic-velocity dependent torque with the gearbox output shaft displacement

Fig. 18 Variation of dimensionless stiffness gearbox output shaft displacement

Fig. 19 Sensitivity of nondimensional stiffness to perturbation parameter

Fig. 20 FFT of the primary shaft perturbation
IX. CONCLUSION

A new model for analyzing self-excited torsional vibrations of a heavy-duty ground vehicle cardan shaft system has been developed. Quadratic-Velocity dependent torques excited by the coupling of kinematic anisotropy of the Hooke’s joint and the elastic motion of the primary shaft have been analyzed. Elastic-torque excited by the coupling of the primary and secondary shaft and due to change in rigid motion across the Joint manifest a significant number of excitation features in the power density and FFT spectrum of the cardan shaft.

The cardan shaft vibration excitation is higher at start-up and dampen in a short after start-up. Modeled results reveal that the sensitivity of stiffness to velocity and displacement inputs are higher at start-up than steady-state motion of the shaft.

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REFERENCES


