Mixed-Mode Oscillations and Mixed-Mode Oscillation-Incrementing Bifurcations in an Extended BVP Oscillator with a Diode

Yuta Nagata, Tetsuro Endo, Hiroyuki Kamata, Shimizu Kuniyasu, Takuji Kousaka, Naohiko Inaba

Abstract—**Mixed-mode oscillations (MMOs) are phenomena first discovered in chemical experiments. We investigate MMOs generated by an extended Bonhoeffer–van der Pol (BVP) oscillator with a diode by using a constrained equation with a grazing-sliding region. From this equation, the Poincaré return map is derived rigorously as one-dimensional and explains successive MMO-incrementing bifurcations.**

Index Terms—**Nonlinear circuits, Dynamical oscillators, Bonhoeffer–van der Pol oscillator, Mixed-mode oscillations.**

I. Introduction

MIXED-MODE oscillations (MMOs) are phenomena discovered in 1970's in chemical experiments and have been studied extensively in recent years [1], [2], [3], [4], [5], [8]. MMOs consist of L large oscillations and s small peaks, and customarily, they are denoted by the notation "L^s". MMOs are characteristic phenomena observed in extended slow-fast dynamics that can generate Canards [6], [7], [8], [1], and they have been the subject of intense research in recent years [1], [8], [9], [10], [11], [12], [13], [14], [15], [16].

Kawczyński et al. [10], [11] shows that the period-adding sequences that are complex but have a strong order, emerge in a quite simple three-variable autonomous ordinary differential equation (ODE) generating MMOs. Shimizu et al. [6] discovers the simplest successive period-adding sequences expressed by $1^2(1^1)^n$ for successive n in an extended Bonhoeffer–van der Pol (BVP) oscillator and terms the resulting bifurcations as the mixed-mode oscillation-incrementing bifurcations (MMOIBs). It has been known that MMOIBs occur successively many times in many chemical fields both in autonomous [8], [10], [11] and nonautonomous [17], [18] ODEs. Kousaka et al. [19] attempts to explain the successive generation of MMOIBs in a driven BVP oscillator with a diode using constrained dynamics where the diode essentially operates as a switch. In Kousaka's dynamics, one-dimensional Poincaré return map is derived. They explain successive generation of MMOIBs, because MMOIBs occur in a way similar to the period-adding bifurcations in the circle map and accumulate toward a saddle-node bifurcation point. We call this the MMO increment-terminating tangent bifurcation point [19].

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In this study, we investigate successive MMOIBs generated by an extended BVP oscillator proposed by Yoshinaga et al. [7]. We analyze successive MMOIBs generated by the extended BVP oscillator including a diode with a grazing-sliding region. We assume that the diode in the circuit operates essentially as an ideal switch. In this case, the governing equation of the circuit is represented by a constrained equation and one-dimensional Poincaré return map is derived from this circuit. By using this return map, we can clearly demonstrate that the MMOIBs of $1^2(1^1)^n$ type occur for successive n toward MMO increment-terminating tangent bifurcation point. This fact is explained because the one-dimensional Poincaré return map comprises two upward convex branches.

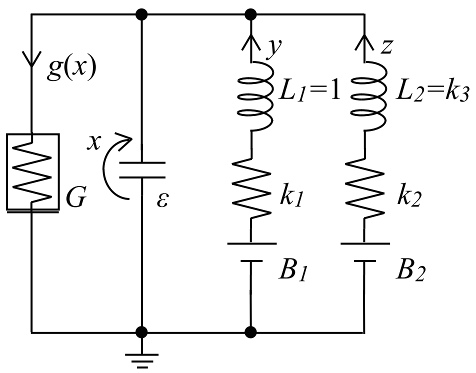

Fig. 1. Circuit diagram of the extended Bonhoeffer–van der Pol oscillator with a diode.

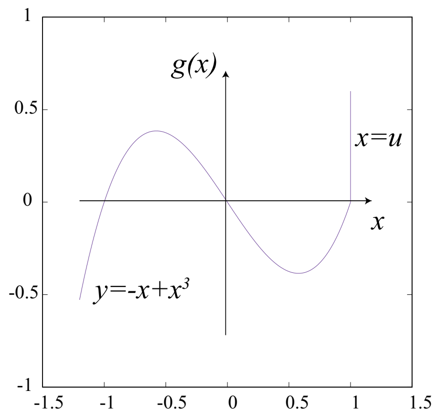

Fig. 2. Nonlinear negative conductance $g(x)$.

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II. Circuit Set Up

Figure 1 shows the circuit diagram of the extended BVP oscillator proposed by Yoshinaga et al. [7]. This circuit is very simple and belongs to *natural circuits* [20], nevertheless, it exhibits tremendously complex MMO bifurcations. We propose an extended BVP oscillator containing a diode in parallel with a nonlinear negative conductor. We consider the case where the diode in the circuit operates as a switch. In this case, the governing equation of the circuit is represented by the following constrained equation with a grazing-sliding region. In addition, Yoshinaga et al. investigate the relationship between observed complex bifurcation structure and MMOs.

The governing equation of the circuit is expressed by a system of three autonomous ODEs as follows:
1. diode off region

$$\left\{\begin{array}{lcl}\varepsilon\dot{x} & = & y+z-g(x)\\ \dot{y} & = & -x-k_1y+B_1\\ \dot{z} & = & k_3(-x-k_2z+B_2)\end{array}\right. \tag{1}$$

2. diode on region

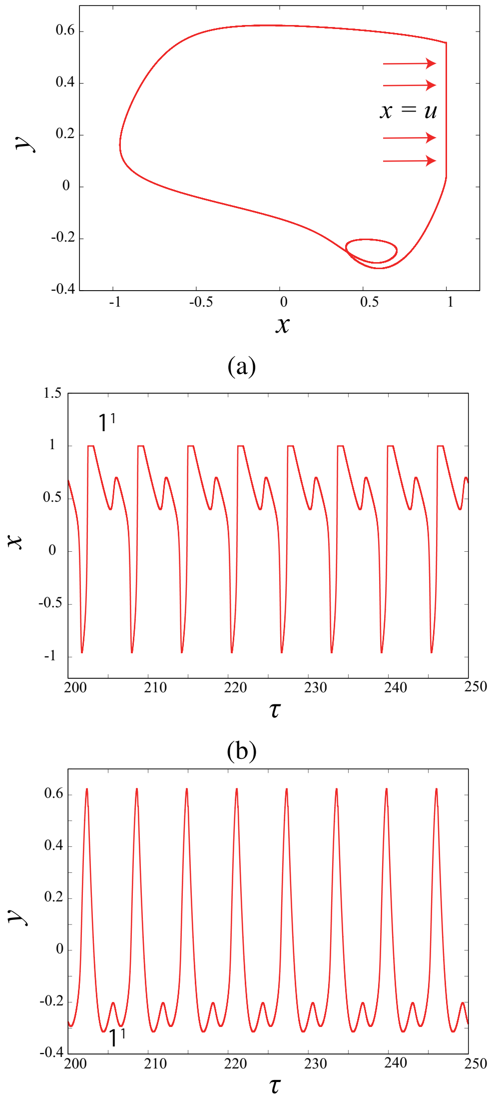
$$\left\{\begin{array}{lcl}x & = & u\\ \dot{y} & = & -u-k_1y+B_1\\ \dot{z} & = & k_3(-u-k_2z+B_2)\end{array}\right. \tag{2}$$

where x represents a variable that corresponds to the voltage across the capacitor C, and y and z denote the variables corresponding to the currents through the inductors L_1 and L_2, respectively. In addition, ε is a parameter that corresponds to the small capacitor C and is assumed to be small. Parameters k_1 and k_2 correspond to the two resistors, respectively. L_1 can be normalized to unity via rescaling. Furthermore, $k_3(>0)$ is the ratio L_2/L_1 and k_3 is assumed to be small. In this case, x is a fast variable, whereas y and z are slow and super-slow variables, respectively. Such circuit dynamics are called a three-time-scale system [9]. A three-time-scale system is one of the representative MMO-generating dynamics [9]. In the following discussion, we assume $B_1 = B_2 (\equiv B)$ and select B as the bifurcation parameter. The constant parameters are set to $\varepsilon = 0.1, k_1 = k_2 = 0.714$, and $k_3 = 0.1$. We assume that the nonlinear v-i characteristics are represented by $g(x)$ in Fig. 2, where $g(x)$ takes the constant value at $x = \alpha$. In this case, the circuit equation is represented by a constrained equation when the diode is on because the voltage across the capacitor is constrained to the threshold voltage of the diode, and the transition condition of these two equations are given by the following equations:

$$\begin{array}{llll}1.\ \text{off} & \rightarrow \ \text{on} & : x = u & (a)\\ 2.\ \text{on} & \rightarrow \ \text{off} & : y+z = -u+u^3 & (b)\end{array} \tag{3}$$

where u is the threshold voltage of the diode and Eq. (3)(a) is derived when x increases to u, and Eq. (3)(b) applies when the current through $g(x)$ decreases to $-u+u^3$. Throughout this study, u is set to 1. Yoshinaga shows that $1^1, 1^2, 1^3$... emerges successively. Figs. 3 (a), (b), and (c) represent an attractor 1^1 projected onto $x-y$ plane, time series waveforms of x and y, respectively. In these figures, the symbol 1^1 is marked. The MMOs 1^n are known to occur for successive n.

Figures 4(a),(b), and (c) show the time series waveforms of 1^2, 1^3, and 1^4, respectively. In addition, the extended BVP oscillator exhibits successive MMOIBs of types $1^2(1^1)^n$ for successive n between 1^1- and 1^2-generating region as shown in Fig. 5. In Fig. 5, 1^1 appears once and 1^2s emerge six times per sequence. Furthermore, our numerical calculation shows that the sequences $1^3(1^2)^n$, $1^4(1^3)^n$, and $1^5(1^4)^n$ for successive n are generated in the areas between 1^2- and 1^3, 1^3- and 1^4-, and 1^4- and 1^5-generating regions, i.e., areas marked C_1, C_2, C_3, and C_4, respectively (see Fig. 7(a)).



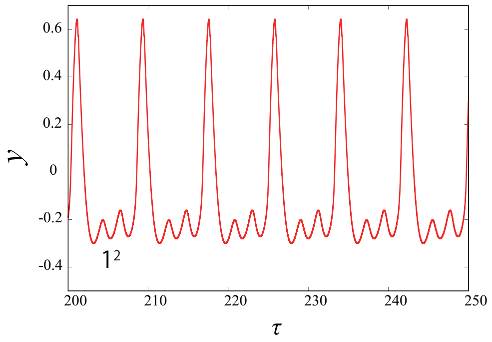
(a) (b) (c)

Fig. 3. MMOs 1^1 $(B = 0.372)$. (a) The solution projected in the $x-y$ plane, (b) Time-series waveform x (c) Time-series waveform y.

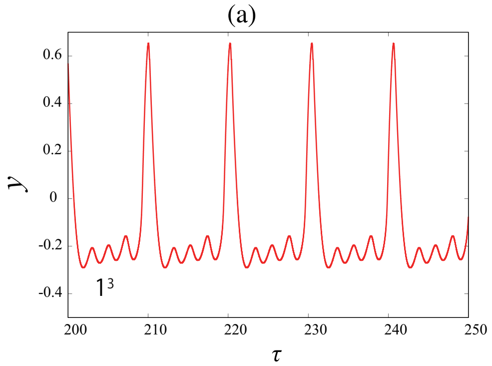
We clarify the mechanism causing complex MMOIBs by defining the Poincaré return map. Because the governing equation is two-dimensional when the diode is on, the Poincaré return map is derived as a one-dimensional one. To construct the Poincaré return map, we introduce some symbols:

$$\begin{array}{l}\Pi = \{(x,y,z)|x = u\}\\ \Sigma_1 = \{(x,y,z)|x = u, z = -u+u^3-y\}\end{array} \tag{4}$$

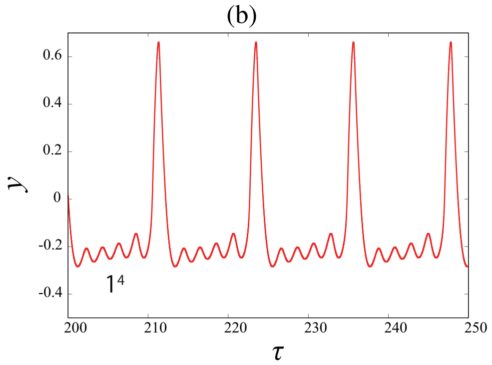
where Π is the plane when the diode is on and where Σ_1 corresponds to the transition condition Eq. (3)(b). Figure. 6 shows the geometric structure of the vector fields. Let us consider the flow initially located on Σ_1. The initial point is



(a)



(b)



(c)

Fig. 4. MMOs (a) 1^2 ($B = 0.382$), (b) 1^3 ($B = 0.39$), (c) 1^4 ($B = 0.395$).

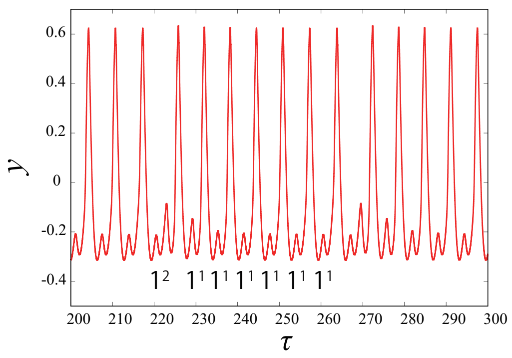


Fig. 5. MMOs $[1^2 1^1 \times 6]$ ($B = 0.3733$).

denoted by y_1. The solution leaving y_1 enters the diode off region and strikes a point marked by P. Then, the solution is constrained onto the diode on plane Π and returns back Σ_1 again. Therefore, one-dimensional Poincaré return map T is rigorously defined as follows:

$$T : \Sigma_1 \to \Sigma_1,\ y_1 \mapsto y_2 = T(y_1) \tag{5}$$

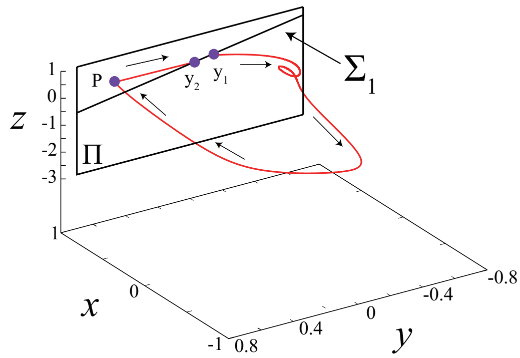
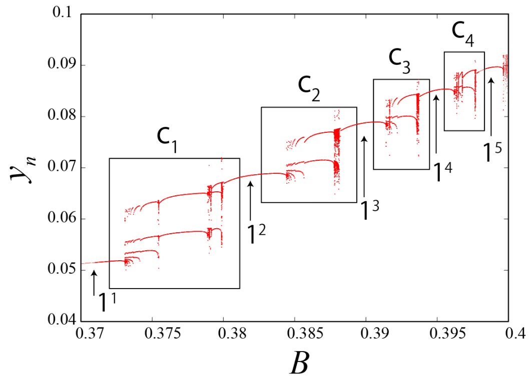
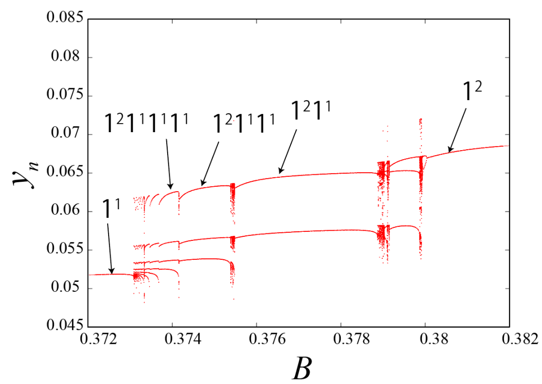


Fig. 6. Geometric structure of the vector fields.

Figures. 7 (a) and (b) show the one-parameter bifurcation



(a)



(b)

Fig. 7. (a) One-parameter bifurcation diagram. (b) Magnified view of (a).

diagrams obtained using the plots of T. In Fig. 7 (a), the generation of 1^1, 1^2, 1^3, and 1^4 is confirmed, respectively. Between 1^1 and 1^2, 1^2 and 1^3, and 1^3 and 1^4, MMOIBs occur successively. Figure 7 (b) shows a magnified view of the one-parameter diagram shown in Fig. 7 (a). In the figure, the MMOIB-generated MMOs $1^2(1^1)^n$ can be clearly observed. To explain why and how MMOIBs occur successively, the one-dimensional Poincaré return map T for $B = 0.3733$ is shown in Fig. 8. The MMOIBs are generated in a similar way to the period-adding bifurcations generated by the circle map. Therefore, the successive MMOIBs are well explained by this return map.

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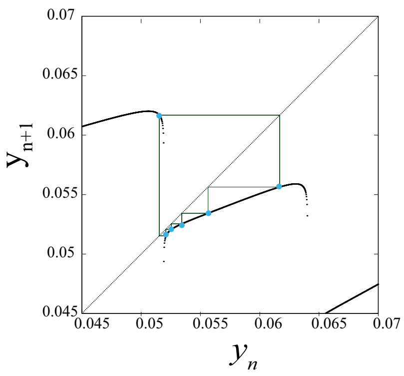

Fig. 8. Poincaré return map T for $B = 0.3734$.

III. Conclusion

We investigate MMOs and MMOIBs generated by the extended BVP oscillator with a diode. It is an extremely simple three-variable autonomous ODEs, nevertheless, exhibits complex bifurcations. MMOIBs occur successively many times as the bifurcation parameter is varied. We explain these results by using the one-dimensional Poincaré return map. Namely, MMOIBs are generated in a similar manner to those of period-adding bifurcations in the circle map terminating by a saddle-node bifurcations.

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