Valuing double barrier options with time-dependent parameters by Fourier series expansion*

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Abstract

Based upon the Fourier series expansion, we propose a simple and easy-to-use approach for computing accurate estimates of Black-Scholes double barrier option prices with time-dependent parameters. This new approach is also able to provide tight upper and lower bounds of the exact barrier option prices. Furthermore, this approach can be straightforwardly extended to the valuation of standard European options with specified moving boundaries as well.

Keywords: Double-barrier Options; Fourier Series; Moving Boundaries; Time-dependent Parameters; Upper and Lower Bounds.

1. Introduction

In the past decade barrier options have become very popular instruments for a wide variety of hedging and investment in foreign exchange, equity and commodity markets, largely in the over-the-counter markets. As estimated by Hsu[1], the market for barrier options has doubled in size every year since 1992. An advantage of trading barrier options is that they provide more flexibility in tailoring the portfolio returns while lowering the cost of option premiums. Closed-form solutions of European barrier option models with constant model parameters have been developed by Merton[2], Rubinstein and Reiner[3], Kunitomo and Ikeda[4], Rich[5], Carr[6], Geman and Yor[7], and Hui[8,9]. All these derivations assume that the model parameters such as volatility, interest rate and dividend yield are constant. However, the inclusion of time-varying parameters is an important concern because their term structures reflect expectation and dynamics of market factors. Unlike the standard European options, the valuation of barrier options with time-dependent parameters is not a trivial extension, and has been the focus of some recent work. Roberts and Shortland[10] applied the hazard rate tangent approximation to evaluate upper and lower bounds of the option price for parameters with time dependence in the Black-Scholes model. However, their bounds are not in closed form and could not be improved further. By the method of images Lo *et al.*[11] developed a simple and easy-to-use method for computing accurate estimates (in closed form) of single-barrier option prices with time-dependent parameters. Their approach also provided very tight upper and lower bounds (in closed form) for the exact barrier option prices systematically. Rapisarda[12] applied Lo *et al.*'s[11] results to derive in an analytical fashion the approximate prices of various types of barrier options, *e.g.* forward start/early expiry barriers, window barriers, etc. Moreover, in order to further improve these results, Rapisarda[13] proposed a perturbation expansion scheme too.

The purpose of this paper is to provide a valuation technique based upon the Fourier series expansion to price double-barrier options with time-dependent parameters. To our knowledge this is the first application of Fourier series to the valuation of barrier options with time-varying parameters, although Fourier series has enjoyed wide application in several diverse fields for about two centuries. The proposed valuation technique provides exact closed-form price functions of double-barrier options with time-varying parameters in the presence of two parametric moving barriers. These price functions not only enable us to obtain accurate estimates of the prices of options associated with fixed barriers, but we can also determine tight upper and lower bounds of the exact barrier option prices.

*The conclusions herein do not represent the views of the Hong Kong Monetary Authority.

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2. Double-barrier options with time-dependent parameters

Consider the Black-Scholes equation with time-dependent model parameters for a standard European option

$$\frac{\partial P(S,t)}{\partial t} = \frac{1}{2}\sigma(t)^2 S^2 \frac{\partial^2 P(S,t)}{\partial S^2} + [r(t)-d(t)] S \frac{\partial P(S,t)}{\partial S} - r(t) P(S,t) \quad , \tag{1}$$

where P is the option value, S is the underlying asset price, t is the time to maturity, σ is the volatility, r is the risk-free interest rate and d is the dividend. Introducing the new variable $x \equiv \ln(S/S_0)$ where S_0 is a constant, the pricing equation is simplified to

$$\frac{\partial P(x,t)}{\partial t} = \frac{1}{2}\sigma(t)^2 \frac{\partial^2 P(x,t)}{\partial x^2} + \left[r(t)-d(t)-\frac{1}{2}\sigma(t)^2\right]\frac{\partial P(x,t)}{\partial x} - r(t) P(x,t) \quad . \tag{2}$$

Without loss of generality, we assume that $P(x,t)$ is given by

$$\begin{aligned} P(x,t) &= \exp\{c_3(t)\}\exp\left\{-x^*(t)\frac{\partial}{\partial x}\right\}\tilde{P}(x,t) \\ &= \exp\{c_3(t)\}\tilde{P}(x-x^*(t),t) \end{aligned} \tag{3}$$

where

$$\begin{aligned} x^*(t) &= -c_1(t) - \beta c_2(t) \\ c_1(t) &= \int_0^t \left\{r(t') - d(t') - \frac{\sigma(t')^2}{2}\right\} dt' \\ c_2(t) &= \frac{1}{2}\int_0^t \sigma(t')^2 dt' \\ c_3(t) &= -\int_0^t r(t') dt' \end{aligned} \tag{4}$$

with β being a real adjustable parameter. Then it can be easily shown that $\tilde{P}(x,t)$ satisfies the partial differential equation:

$$\frac{\partial \tilde{P}(x,t)}{\partial t} = \frac{1}{2}\sigma(t)^2 \frac{\partial^2 \tilde{P}(x,t)}{\partial x^2} - \frac{1}{2}\beta\sigma(t)^2 \frac{\partial \tilde{P}(x,t)}{\partial x} \quad . \tag{5}$$

Next, by introducing $\tau = c_2(t)$ and $\bar{P}(x,t) = \exp\{\beta^2\tau/4\}\exp\{-\beta x/2\}\tilde{P}(x,t)$, Eq.(5) can be cast in the canonical form of the diffusion equation

$$\frac{\partial \bar{P}(x,\tau)}{\partial \tau} = \frac{\partial^2 \bar{P}(x,\tau)}{\partial x^2} \quad . \tag{6}$$

By direct substitution, it is straightforward to show that

$$\bar{P}(x,\tau) = \sqrt{\frac{L_0}{L(\tau)}}\exp\left\{-\frac{\gamma L_0}{4L(\tau)}x^2\right\} \times \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L(\tau)}\right) \times \exp\left\{-\frac{n^2\pi^2\tau}{L_0 L(\tau)}\right\} \tag{7}$$

is the general solution of the diffusion equation for $0 \le x \le L(\tau) \equiv L_0(1+\gamma\tau)$ and $\tau \ge 0$, subject to the absorbing boundary conditions: $\bar{P}(x,\tau) = 0$ at both $x = 0$ and $x = L(\tau)$. Here γ is a real adjustable parameter, L_0 denotes the interval at $t = 0$, and A_n's are the expansion coefficients to be determined. As a result, the price function $P(x,t)$ of the corresponding double knock-out option is given by

$$\begin{aligned} P(x,t) = &\sqrt{\frac{L_0}{L(c_2(t))}}\exp\left\{c_3(t) - \frac{1}{4}\beta^2 c_2(t)\right\} \times \\ &\exp\left\{\frac{1}{2}\beta[x - x^*(t)]\right\} \times \\ &\exp\left\{-\frac{\gamma L_0}{4L(c_2(t))}[x-x^*(t)]^2\right\} \times \\ &\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi[x-x^*(t)]}{L(c_2(t))}\right) \times \\ &\exp\left\{-\frac{n^2\pi^2 c_2(t)}{L_0 L(c_2(t))}\right\} \quad . \end{aligned} \tag{8}$$

This double barrier option has two moving boundaries specified by

$$\begin{aligned} S_L^*(t) &= S_0 \exp\{x^*(t)\} \\ &= S_0 \exp\{-c_1(t) - \beta c_2(t)\} \end{aligned} \tag{9}$$

and

$$\begin{aligned} S_U^*(t) &= S_0 \exp\{x^*(t) + L(c_2(t))\} \\ &= S_1 \exp\{-c_1(t) - \alpha c_2(t)\} \end{aligned} \tag{10}$$

where $\alpha \equiv \beta - \gamma L_0$ and $S_1 = S_0 \exp(L_0)$. It is obvious that the two real adjustable parameters α and β are responsible for controlling the movement of the two barriers.

Provided the final payoff condition of a double knock-out call option, $P(S,0) = \max(\mathrm{S-K},0)$, where K is the strike price, we can easily show that the expansion coefficients A_n's are determined by

$$\begin{aligned} A_n &= \frac{2}{L_0}\int_{\ln(K/S_0)}^{L_0} dx\, \sin\left(\frac{n\pi x}{L_0}\right) \times \\ & \exp\left\{-\frac{1}{2}\beta x + \frac{1}{4}\gamma x^2\right\} \times \\ & (S_0 \exp(x) - K) \quad . \end{aligned} \tag{11}$$

Unfortunately, the integral in Eq.(11) cannot be evaluated in closed form and numerical quadrature is needed for the evaluation. Nevertheless, according to our calculations the numerical quadrature can be very efficiently performed by Mathcad running on a PC with Window 98. Furthermore, if we choose $\gamma = 0$, *i.e.* the distance between the two barriers are being kept constant, then the integral can be performed analytically and the expansion coefficients A_n's are given by

$$\begin{aligned} A_n &= \frac{2}{L_0}\int_{\ln(K/S_0)}^{L_0} dx\, \sin\left(\frac{n\pi x}{L_0}\right) \times \\ & \exp\left\{-\frac{1}{2}\beta x\right\}(S_0 \exp(x) - K) \\ &= \frac{2S_0/L_0}{(1-\beta/2)^2 + (n\pi/L_0)^2} \times \\ & \left\{\frac{n\pi}{L_0}\cos\left(\frac{n\pi \ln(K/S_0)}{L_0}\right)\left(\frac{K}{S_0}\right)^{1-\beta/2} - \right. \\ & \left(1-\frac{\beta}{2}\right)\sin\left(\frac{n\pi \ln(K/S_0)}{L_0}\right)\left(\frac{K}{S_0}\right)^{1-\beta/2} - \\ & \left.(-1)^n\left(\frac{n\pi}{L_0}\right)\left(\frac{S_1}{S_0}\right)^{1-\beta/2}\right\} - \\ & \frac{2K/L_0}{(\beta/2)^2 + (n\pi/L_0)^2} \times \\ & \left\{\frac{n\pi}{L_0}\cos\left(\frac{n\pi \ln(K/S_0)}{L_0}\right)\left(\frac{K}{S_0}\right)^{-\beta/2} + \right. \\ & \frac{\beta}{2}\sin\left(\frac{n\pi \ln(K/S_0)}{L_0}\right)\left(\frac{K}{S_0}\right)^{-\beta/2} - \\ & \left.(-1)^n\left(\frac{n\pi}{L_0}\right)\left(\frac{S_1}{S_0}\right)^{-\beta/2}\right\} \quad . \end{aligned} \tag{12}$$

To simulate **fixed upper and lower barriers**, we choose the optimal values of the adjustable parameters β and α in such a way that both of the integrals

$$\int_0^T \left\{\ln\left(\frac{S_L^*(t)}{S_0}\right)\right\}^2 dt$$

and

$$\int_0^T \left\{\ln\left(\frac{S_U^*(t)}{S_1}\right)\right\}^2 dt$$

are minimum. In other words, we try to minimize the deviations from the fixed barriers by varying the parameters β and α. Here T denotes the time at which the option price is evaluated. Simple algebraic manipulations then yield the optimal values of β and α as follows:

$$\beta_{opt} = \alpha_{opt} = -\frac{\int_0^T c_1(t)c_2(t)dt}{\int_0^T [c_2(t)]^2\, dt} \quad . \tag{13}$$

It is obvious that in the special case of constant parameters the price function $P(x,t)$ in Eq.(8) will be reduced to the well-known exact result, and both β_{opt} and α_{opt} are equal to $1 - 2(r-d)\sigma^{-2}$. Furthermore, within the framework of this new approach, we can also determine the upper and lower bounds for the exact barrier option prices. It is not difficult to show[1] that if the moving barriers stay outside the region bounded by the fixed barriers, *i.e.* $S_U^*(t) > S_1$ and $S_L^*(t) < S_0$, for the duration of interest, then the corresponding option price will provide an **upper bound** for the exact value. On the other hand, if the moving barriers are embedded inside the bounded region, *i.e.* $S_U^*(t) < S_1$ and $S_L^*(t) > S_0$, then the corresponding option price will serve as a **lower bound**.

3. Illustrative examples

For illustration, we apply the approximation method to the following example: $K = 50$, $S_0 = 30$, $S_1 = 70$, $r = 0.1$ and $d = 0.05$. The volatility $\sigma(t)$ is assumed to have the term structure: $\sigma^2(t) = 0.05 - 0.02t$. This example represents the special case where the variance $\sigma^2(t)$ decreases linearly with time. We now try to evaluate the double-barrier option price $P(S,t)$ associated with the current underlying asset price $S = 50$ at time $t = 1.0$. First of all, we determine the optimal values of the adjustable parameters α and β:

$$\alpha_{opt} = \beta_{opt} = -1.348066 \quad . \tag{14}$$

Then an estimate of the exact barrier option price can be evaluated using Eq.(8):

$$P(S = 50, t = 1) = 2.709590 \quad . \tag{15}$$

As a check, the Crank-Nicolson method is used to numerically solve the pricing equation, and the (numerically) exact value of the barrier option price is given by

$$P_{exact}(S = 50, t = 1) = 2.699919 \quad . \tag{16}$$

[1] The proof is based upon the maximum principle for parabolic partial differential equations (see the appendix of Lo *et al.*[11] for the relevant proof).

The approximate estimate is indeed very close to the exact result with an error of 0.36% only. Moreover, the numerical results for the corresponding upper and lower bounds are also evaluated as follows:[2]

$$\begin{aligned} \text{Upper bound} &= 2.716256 \\ \text{Lower bound} &= 2.693913 \ . \end{aligned} \tag{17}$$

Clearly, the new approach is able to give very tight upper and lower bounds for the exact barrier option price with percentage error of less than 1%. It should also be noted that as the time to maturity t gets smaller, the accuracy of the estimate will further improve and the bounds will become tighter, as shown in Table 1a and Table 1b. Furthermore, by means of the multi-stage approximation scheme proposed by Lo *et al.*'s[11], the upper and lower bounds can be efficiently improved in a systematic manner.

4. Conclusion

In this paper we have presented a simple and easy-to-use method in terms of the Fourier series for computing accurate estimates of Black-Scholes double barrier option prices with time-dependent parameters. This new approach is also able to provide very tight upper and lower bounds for the exact barrier option prices. Unlike previous attempts (see, for example, Roberts and Shortland[10]), the evaluation is very efficient and the exact barrier option prices are always within the bounds, as demonstrated by the illustrative examples shown above. It is also natural that by tuning the parameters α and β the approach can be applied to capture the valuation of standard European options with specified moving barriers.

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[2] Each of the moving barriers associated with the upper and lower bounds could be determined by requiring that either the moving barrier returns to its initial position and merges with the fixed barrier at time $t = \tau$, *i.e.* $S_U^*(t=\tau) = S_U^*(t=0)$ or $S_L^*(t=\tau) = S_L^*(t=0)$, or the instantaneous rate of change of $S_U^*(t)$ or $S_L^*(t)$ must be zero at time $t = 0$. Both of the criteria are to ensure the deviation from the fixed barrier to be minimum.

Table 1(a):

T(yrs)	Estimate of option price	β_{opt}	α_{opt}	Percentage error of estimate	Exact result	Δt for Crank-Nicolson method
0.25	2.405360	-1.077712	-1.077712	0.002017	2.405312	0.000025
0.5	2.929754	-1.161215	-1.161215	0.035988	2.928700	0.00005
0.75	2.891853	-1.251110	-1.251110	0.149086	2.887548	0.000075
1.0	2.709590	-1.348066	-1.348066	0.358202	2.699919	0.0001

Comparison of estimates of the option prices with the (numerically) exact results by the Crank-Nicolson (CN) method. Percentage error is defined as (estimate-CN's result)/CN's result $\times$ 100%.

Other input parameters are $K = 50$, $S_0 = 30$, $S_1 = 70$ $S = 50$, $\sigma^2 = 0.05 - 0.02t$, $d = 0.05$ and $r = 0.1$. In all Crank-Nicolson calculations, $\Delta x = 0.0001$.

Table 1(b):

T(yrs)	Lower bound of option price	β_L	α_L	Percentage error of lower bound	Upper bound of option price	Percentage error of upper bound
0.25	2.405316	-1.10526	-1	0.000197	2.405376	0.002661
0.5	2.928146	-1.22222	-1	-0.018911	2.930357	0.056584
0.75	2.885004	-1.35294	-1	-0.088081	2.894584	0.243674
1.0	2.693913	-1.5	-1	-0.222430	2.716256	0.605112

Comparison of the lower and upper bounds of the option prices with the (numerically) exact results by the Crank-Nicolson method. *Note* that in the above calculations $\beta_U = \alpha_L$ and $\alpha_U = \beta_L$.