Limiting Fréchet subdifferentials of marginal functions.

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Abstract. This paper investigates calculus rules for the limiting Fréchet-subdifferential of infimum value functions of locally Lipschitzian and non-Lipschitzian functions. It is not required that the infimum is attained.

Key words. marginal function , Fréchet subdifferential, Limiting Fréchet subdifferential.

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Introduction. The study of subdifferentiability of marginal functions is of great interest not only because it is related to the Lagrange multipliers but also because it is connected to the study of the sensitivity of some problems in optimization and optimal control. Generally, the infimum defining the marginal function is required to be attained near the point of interest. The present paper studies, with the help of some recent results by Mordukhovich and Shao [10], the limiting Fréchet subdifferential of the optimal value function without assuming that the infimum is attained. The limiting Fréchet subdifferential is the extension, by Kruger and Mordukhovich [8] to Banach spaces admitting equivalent Fréchet differentiable norms, of the subdifferential introduced by Mordukhovich [9] in finite dimensions. After recalling some notions in the first section, the second one is devoted to the study of functions of the form

$$m(x) := \inf\{g(y) : y \in G(x)\},$$

where g is a real-valued locally Lipschitz function from an Asplund space X into $\Re$, and G is a multivalued mapping from X into an Asplund space Y. The result obtained is strong enough to allow to describe the subdifferential of the partial distance function $x \longrightarrow d(\bar{y}; G(x))$ at any point $\bar{x}$ satisfying $(\bar{x}, \bar{y}) \in gph\ G$, whenever the multivalued mapping G is pseudo-Lipschitz at the point $(\bar{x}, \bar{y})$. In fact, this description was the main motivation of our study here. The third section deals with the study of non necessarily Lipschitz marginal function of the more general form

$$m(x) := \inf\{f(x, y) : y \in G(x)\},$$

where f is a real-valued function defined on $X \times Y$.

1. Preliminaries. In all this paper X and Y denote two Asplund spaces, X^* the topological dual of X and B_X the closed unit ball of X. (We refer the reader to [11] for properties of Asplund spaces). Let

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f be an extended real-valued function $f : X \longrightarrow \Re \bigcup \{-\infty, +\infty\}$. We will use the important notion of Fréchet ϵ-subgradient in the sequel (see [7], [8], [15], and [16]). An element x^* in X^* is said to be a $F_\epsilon-subgradient$ to f at a point $\bar{x}$ where $| f(\bar{x}) | < +\infty$ if there exists a neighbourhood V of $\bar{x}$ such that

$$\langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \epsilon \parallel x - \bar{x} \parallel$$

for every $x \in V$. Note that this inequality means that $\bar{x}$ is a local minimum of the function

$$x \longrightarrow f(x) - \langle x^*, x - \bar{x} \rangle + \epsilon \parallel x - \bar{x} \parallel .$$

We will denote by $\partial_{F,\epsilon} f(\bar{x})$ the set of all F_ϵ-subgradients to f at $\bar{x}$. The limiting Fréchet-subdifferential in the sense of Kruger-Mordukhovich [8] is the set $\partial_F f(\bar{x})$ of all $x^* \in X^*$ for which there exist sequences $(\epsilon_n) \downarrow 0, (x_n) \to \bar{x}$ with $(f(x_n)) \to f(\bar{x})$, $x_n^* \in \partial_{F,\epsilon_n} f(x_n)$ with $(x_n^*) \to^{w^*} x^*$. One of the main calculus rules in Asplund spaces of the limiting Fréchet subdifferential is given by the following theorem established by Mordukhovich and Shao [10] (in a more general setting).

1.1 Theorem([10]). Let $f_1 : X \longrightarrow \Re \bigcup \{-\infty, +\infty\}$ be lower semicontinuous near $\bar{x}$ with $| f_1(\bar{x}) | < \infty$ and let $f_2 : X \longrightarrow \Re$ be locally Lipschitz near $\bar{x}$. Then

$$\partial_F (f_1 + f_2)(\bar{x}) \subset \partial_F f_1(\bar{x}) + \partial_F f_2(\bar{x}).$$

2. Subdifferentials of marginal functions defined by Lipschitz parametric functions.

In this section we consider the marginal function

$$m(x) := \inf\{g(y) : y \in G(x)\},$$

where $g : Y \longrightarrow \Re$ is a real-valued function and G is a multivalued mapping from X into Y. We will denote the graph of G by $gph\ G := \{(x, y) \in X \times Y : y \in G(x)\}$. It is known that a lot of apparently different types of marginal functions can be reduced to this form. In Clarke [2], Hiriart-Urruty [4], Rockafellar [12], Thibault [14] and references therein (for examples) one can find some reductions and several applications to the study of optimization problems. Recall that for $\bar{y} \in G(\bar{x})$ the limiting Fréchet coderivative $D_F^* G(\bar{x}, \bar{y})$ is the multivalued mapping from Y^* into X^* defined by

$$x^* \in D_F^* G(\bar{x}, \bar{y})(y^*) \iff$$

$$(x^*, -y^*) \in \Re_+ \partial_F d(.; gph\ G)(\bar{x}, \bar{y}).$$

We will write $\partial_F d(\bar{x}, \bar{y}; gph\ \ G)$ in place of $\partial_F d(.; gph\ G)(\bar{x}, \bar{y})$. Here $d(.; S)$ denotes the distance function to a set S. Note that for $\bar{x} \in S \subset X$ and $S_{3r} := S \cap (\bar{x} + 3rB_X)$ with $r > 0$ it is not difficult to see that for $x \in \bar{x} + rB_X$

$$d(x, S) = d(x, S_{3r})$$

and hence

$$\partial_F d(\bar{x}, S) = \partial_F d(\bar{x}, S_{3r}). \qquad (2.1)$$

The following theorems are the main results of this section.

2.1 Theorem. Suppose that g is locally Lipschitz and m is finite and lower semicontinuous at $\bar{x}$ with $| m(\bar{x}) | < \infty$. Then for every $x^* \in \partial_F m(\bar{x})$, there exist sequences $((x_n, y_n)) \longrightarrow (\bar{x}, \bar{y})$ with $y_n \in G(x_n)$ and $(g(y_n)) \longrightarrow m(\bar{x})$, $(\epsilon_n) \downarrow 0$, $y_n^* \in \partial_F g(y_n) + \epsilon_n B_{Y^*}$, $x_n^* \in D_F^* G(x_n, y_n)(y_n^*)$ such that $x^* = \lim x_n^*$, with respect to the weak-star topology.

Proof. Fix $x^* \in \partial_F m(\bar{x})$. By definition of the subdifferential in the sense of Kruger-Mordukhovich, [7], [8] there exist $(u_n) \longrightarrow \bar{x}$, with $m(u_n) \longrightarrow m(\bar{x})$, $\epsilon_n \downarrow 0$ and $u_n^* \in \partial_{F,\epsilon_n} m(u_n)$ with $u_n^* \longrightarrow^{w^*} x^*$. By definition of Fréchet ϵ-subgradients, the point u_n is a local minimum over some ball $u_n + r_n B_X$ of the function

$$x \longrightarrow m(x) - \langle u_n^*, x - u_n \rangle + \epsilon_n \parallel x - u_n \parallel . \qquad (2.2)$$

Set $\bar{\epsilon}_n := \min(\frac{r_n}{6}, \epsilon_n)$ and choose $v_n \in G(u_n)$ such that

$$g(v_n) \leq m(u_n) + \bar{\epsilon}_n^2 \qquad (2.3).$$

If we set

$$f(x, y) := g(y) - \langle u_n^*, x - u_n \rangle + \epsilon_n \parallel x - u_n \parallel, \quad (2.4)$$

and $E_n := gph\ G \cap [(u_n, v_n) + r_n B_{X \times Y}]$. We deduce from (2.2) and (2.3)

$$f(u_n, v_n) \leq \inf_{(x,y) \in E_n} f(x, y) + \bar{\epsilon}_n^2.$$

Applying the Ekeland variational principle [3] to f on E_n we get the existence of $(x_n, y_n) \in E_n$ satisfying for all $(x, y) \in E_n$

$$\parallel x_n - u_n \parallel + \parallel y_n - v_n \parallel \leq \bar{\epsilon}_n, \quad f(x_n, y_n) \leq f(u_n, v_n) \qquad (2.5)$$

and

$$f(x_n,y_n) \leq f(x,y) + \bar{\epsilon_n}(\| x_n - x \| + \| y_n - y \|).$$

Note that f admits a Lipschitz constant λ'_n around (x_n, y_n) and hence, for $\lambda_n := \lambda'_n + \bar{\epsilon_n}$, by Proposition 2.4.3 in Clarke [2] (x_n, y_n) is a local minimum (without constraint) of the function

$$(x,y) \longrightarrow f(x,y) + \bar{\epsilon_n}(\| x_n - x \| + \| y_n - y \|)+$$

$$\lambda_n d(x,y;E_n). \tag{2.6}$$

Note also that

$$\| x_n - u_n \| + \| y_n - v_n \|) \leq \bar{\epsilon}_n \leq \frac{1}{3} r_n$$

and hence by (2.1) we have

$$\partial_F d(x_n, y_n; gph\ G) = \partial_F d(x_n, y_n; E).$$

Therefore, by (2.6) and subdifferential calculus rules in Asplund spaces for the limiting Fréchet subdifferential (see Theorem 1.1 and [10]) we get the following relation

$$(u_n^*, 0) \in \{0\} \times \partial_F g(y_n) + \lambda_n \partial_F d(x_n, y_n; gph\ G)+$$

$$2\epsilon_n (B_{X^*} \times B_{Y^*}). \tag{2.7}$$

Since the second inequality in (2.5) means

$$g(y_n) - \langle u_n^*, x_n - u_n \rangle + \epsilon_n \| x_n - u_n \| \leq g(v_n)$$

it follows from (2.3) that

$$m(x_n) \leq g(y_n) \leq$$

$$\langle u_n^*, x_n - u_n \rangle - \epsilon_n \| x_n - u_n \| + m(u_n) + \bar{\epsilon}_n^2$$

and hence (since $(u_n^*)_n$ is bounded and m is lower semicontinuous at $\bar{x}$)
$(g(y_n)) \longrightarrow m(\bar{x})$. By (2.7) we may choose $a_n^* \in B_{X^*}, b_n^* \in B_{Y^*}, (x_n^*, -y_n^*) \in \lambda_n \partial_F d(x_n, y_n; gph\ G)$ and $w_n^* \in \partial_F g(y_n)$ such that

$$u_n^* = x_n^* + 2\epsilon_n a_n^* \tag{2.8}$$

and

$$0 = w_n^* - y_n^* + 2\epsilon_n b_n^*. \tag{2.9}$$

Hence $(x_n)_n$ weak-star converges to x^* and $x_n^* \in D_F^* G(x_n, y_n)(y_n^*)$. As $y_n^* \in \partial_F g(y_n) + 2\epsilon_n B_{Y^*}$ the proof of the theorem is complete.

Before stating the next theorem recall that for a multivalued mapping M from a topological space T into X^* and for $S \subset T$ the sequential limit superior $\limsup_{t \to \bar{t}, t \in S} M(t)$ is defined by $x^* \in \limsup_{t \to \bar{t}, t \in S} M(t)$ iff there exist sequences $(t_n) \to \bar{t}$ with $t_n \in S$ and $(x_n^*) \to^{w^*} x^*$ with $x_n^* \in M(t_n)$ for all n.

2.2 Theorem. Suppose that m is lower semicontinuous at $\bar{x}$ with $| m(\bar{x}) | < \infty$ and there exists some neighborhood V of $\bar{x}$ such that g is β-Lipschitz over some neighborhood of $G(V)$. Then

$$\partial_F m(\bar{x}) \subset \bigcup \{ \limsup_{x \to \bar{x}, g(y) \to m(\bar{x}), y \in G(x)} D_F^* G(x,y)(y^*) :$$

$$y^* \in \limsup_{g(y) \to m(\bar{x})} \partial_F g(y) \}.$$

Proof. We follow the proof of Theorem 2.1 and we fix some real number γ with $\| u_n^* \| \leq \gamma$ for all n. Then we obtain that the Lipschitz constant λ_n of f around (x_n, y_n) may be chosen equal to $\lambda_n := max(\beta, \gamma) + \bar{\epsilon_n}$ which is bounded with respect to n. Moreover, since $\partial_F g(y_n) \subset \beta B_{Y^*}$ and since the closed unit ball of Y^* is weak star sequentially compact, we may suppose that $(y_n^*)_n$ weak-star converges to some y^*. Then we conclude by (2.8) and (2.9) $(x_n^*, -y_n^*) \in \lambda_n \partial_F d(x_n, y_n; gph\ G)$ and $(x_n^*) \to^{w^*} x^*$.

Before prooving the next corollary, we recall that G is pseudo-Lipschitz at $(\bar{x}, \bar{y}) \in gph\ G$ if (see Aubin [1]) there exist $r > 0, s > 0$ such that for any
$x_1, x_2 \in \bar{x} + sB_X$

$$G(x_1) \cap (\bar{y} + sB_Y) \subset G(x_2) + r \| x_1 - x_2 \| B_X$$

Rockafellar [13] showed that G is pseudo-Lipschitz at $(\bar{x}, \bar{y}) \in gph\ G$ iff $d(.;G(.))$ is Lipschitz over a neighborhood of $(\bar{x}, \bar{y})$. We can now state the second corollary which is in the line of some results in Thibault [14] and Jourani and Thibault [6].

2.3 Corollary Let G be a multivalued mapping between X and Y which is pseudo-Lipschitz at $(\bar{x}, \bar{y}) \in gph\ G$. Then

$$\partial_F d(\bar{y}, G(.))(\bar{x}) \subset \bigcup_{y^* \in B_{Y^*}} \{x^* \in X^* :$$

$$(x^*, -y^*) \in \partial_F d(.; gph\ G)(\bar{x}, \bar{y})\}.$$

Proof. Put $g(y) =\| y - \bar{y} \|$ and $m(x) := d(\bar{y}, G(x))$. Then m is Lipschitz around $\bar{x}$ and one may suppose that the constants λ_n in the proof of Theorem 2.2 satisfy $\lambda_n \longrightarrow 1$ as $n \longrightarrow \infty$. Moreover, $g(y) \longrightarrow m(\bar{x})$ means here that $y \longrightarrow \bar{y}$, which implies

$$\limsup_{g(y)\to m(\bar{x})} \partial_F g(y) = \limsup_{y\to\bar{y}} \partial_F g(y) = \partial_F g(\bar{y}) = B_{Y^*}$$

and

$$\limsup_{x\to\bar{x}, g(y)\to m(\bar{x}), y\in G(x)} \partial_F d(.; gph\ G)(x, y) =$$

$$\partial_F d(.; gph\ G)(\bar{x}, \bar{y}).$$

So the corollary follows from Theorem 2.2 .

3. Subdifferentials of general marginal function.

We consider in this section the marginal value function m given by

$$m(x) := \inf\{f(x, y) : y \in G(x)\},$$

where $f : X \times Y \longrightarrow \Re$ is supposed to be lower semicontinuous, G to be a multivalued mapping from X into Y with closed graph. We assume that m is lower semicontinuous around a given point $\bar{x}$ with $| m(\bar{x}) |< \infty$. In order to state our main theorem we need for $C_1 := epif$ and $C_2 := (gph\, G)\times\Re$, the following hypothesis (H): $\exists k > 0, \forall(\eta_n) \to 0^+$, $\forall(x_n) \to \bar{x}$ with $(m(x_n)) \to m(\bar{x})$, $\exists y_n \in G(x_n), \exists\alpha > 0$ such that for n large enough one has

$$f(x_n, y_n) \le m(x_n) + \eta_n$$

and

$$d(x, y, r; [C_1 \cap C_2]) \le k\, [d(x, y, r; C_1) + d(x, y, r; C_2)] \tag{3.0}$$

for all $x \in x_n + \alpha B_X$, $y \in y_n + \alpha B_Y$ and $r \in [m(\bar{x}) - \alpha, m(\bar{x}) + \alpha]$.

The hypothesis (H) is obviously satisfied whenever there exists some $s > 0$ such that (3.0) holds for all $x \in \bar{x} + sB_X$, $r \in [m(\bar{x}) - s, m(\bar{x}) + s]$ and $y \in G(\bar{x} + sB_X) + sB_Y$. This last condition corresponds to the metric regularity between C_1 and C_2 near $(\bar{x}, m(\bar{x}))$ and uniformly with respect to y in a neighborhood of the image by G of a neighborhood of $\bar{x}$.

3.1. Lemma. Assume the above hypothesis is fulfilled. Then, for every
$u^* \in \partial_F m(\bar{x})$ there exist $\lambda > 0, b \ge 0$, and sequences $\epsilon_n \downarrow 0$, $(x_n^*, -\lambda_n) \to \lambda(u^*, -1)$,
$x_n \to \bar{x}, (\tilde{x}_n, \tilde{y}_n, \tilde{s}_n) \in C_1 \cap C_2$ with $\| \tilde{y}_n - y_n \|< \sqrt{\eta_n}, \tilde{x}_n \to \bar{x}, f(\tilde{x}_n, \tilde{y}_n) \to m(\bar{x})$,
$\tilde{s}_n \to m(\bar{x})$ such that

$$(x_n^*, 0, -\lambda_n) \in kb\partial_F d(\tilde{x}_n, \tilde{y}_n, \tilde{s}_n; epif) + kb\partial_F d(\tilde{x}_n, \tilde{y}_n; gph\ G) \times \{0\} + ((1 + \lambda)\sqrt{\eta_n} + \epsilon_n)B_{X^*} \times B_{Y^*} \times B_\Re.$$

Proof. If we set $g(x, r) := d(x, r; epi\ m)$, then, for $P =]0, +\infty[$, we have the following equivalences

$$u^* \in \partial_F m(\bar{x}) \iff (u^*, -1) \in P\partial_F g(\bar{x}, m(\bar{x}))$$

$$\iff \exists\lambda > 0; (\lambda u^*, -\lambda) \in \partial_F g(\bar{x}, m(\bar{x})).$$

Let $x^* := \lambda u^*$. Then, by [14], there exist sequences $(x_n, r_n) \longrightarrow (\bar{x}, m(\bar{x}))$, with $(x_n, r_n) \in epi\ m$, $\epsilon_n \to 0^+(\epsilon_n < 1)$ and $(x_n^*, -\lambda_n) \to (x^*, -\lambda)$ such that

$$(x_n^*, -\lambda_n) \in \partial_{F,\epsilon_n} g(x_n, r_n). \tag{3.1}$$

By definition of the ϵ -Fréchet subdifferential, (x_n, r_n) is a local minimum of the function

$$(x, r) \longrightarrow g(x, r) - \langle x_n^*, x - x_n\rangle + \lambda_n(r - r_n)+$$

$$\epsilon_n(\| x - x_n \| + | r - r_n |)$$

and hence there exists $\alpha_n > 0$ such that

$$-\langle x_n^*, x-x_n\rangle+\lambda_n(r-r_n)+\epsilon_n(\| x-x_n \| + | r-r_n |) \ge 0 \tag{3.2}$$

for all $(x, r) \in epim \cap (B_X(x_n, \alpha_n) \times B_\Re(r_n, \alpha_n))$. Let $\eta_n \to 0^+$. By the hypothesis (H) there exists $y_n \in G(x_n)$ such that $f(x_n, y_n) \le m(x_n)+\eta_n$. As $m(x_n) \le r_n$, then $f(x_n, y_n) \le r_n + \eta_n$. Set $s_n := r_n + \eta_n$ and observe that $(x_n, y_n, s_n) \in C_1 \cap C_2$. Using (3.2) we obtain

$$-\langle x_n^*, x - x_n\rangle + \lambda_n(r - s_n) - \lambda_n(r_n - s_n)+$$

$$\epsilon_n(\| x - x_n \| + | r - s_n | + | s_n - r_n |) \geq 0$$

and hence for n large enough

$$-\langle x_n^*, x - x_n\rangle + \lambda_n(r - s_n) + \epsilon_n(\| x - x_n \| + | r - s_n |) +$$

$$(1+\lambda)\eta_n \geq 0$$

Let

$$h(x,y,r) := -\langle x_n^*, x - x_n\rangle + \lambda_n(r - s_n) +$$

$$\epsilon_n(\| x - x_n \| + | r - s_n |)$$

and

$$E := C_1 \cap C_2 \cap (B_X(x_n,\alpha_n) \times B_Y(y_n,\alpha_n) \times B_R(r_n,\alpha_n)).$$

Then we have

$$0 \leq h(x,y,r) + (1+\lambda)\eta_n$$

for all $(x,y,r) \in E$. Applying the Ekeland variational principle to h on E we have the existence of a sequence $(\tilde{x}_n, \tilde{y}_n, \tilde{s}_n) \in E$ satisfying

$$\| \tilde{x}_n - x_n \| + \| \tilde{y}_n - y_n \| + | \tilde{s}_n - s_n | \leq \sqrt{\eta_n}$$

and

$$h(\tilde{x}_n, \tilde{y}_n, \tilde{s}_n) \leq h(x,y,r) +$$

$$(1+\lambda)\sqrt{\eta_n}(\| \tilde{x}_n - x \| + \| \tilde{y}_n - y \| + | \tilde{s}_n - r |)$$

for all $(x,y,r) \in E$. Then by Proposition 2.4.3 in Clarke there exist $\gamma_n \in \left]0, \min(\frac{1}{n}, \frac{\alpha_n}{3}\right[$ and $b > 0$ such that

$$h(\tilde{x}_n, \tilde{y}_n, \tilde{s}_n) \leq h(x,y,r) +$$

$$(1+\lambda)\sqrt{\eta_n}(\| \tilde{x}_n - x \| + \| \tilde{y}_n - y \| + | \tilde{s}_n - r |) +$$

$$bd(x,y,r;E),$$

for all $(x,y,r) \in B_X(\tilde{x}_n,\gamma_n) \times B_Y(\tilde{y}_n,\gamma_n) \times B_R(\tilde{s}_n,\gamma_n))$. But for $(x,y,r) \in B_X(\tilde{x}_n,\gamma_n) \times B_Y(\tilde{y}_n,\gamma_n) \times B_R(\tilde{s}_n,\gamma_n))$ one has

$$d(x,y,r;E) = d(x,y,r;C_1 \cap C_2)$$

and for n large enough

$$\| x - \tilde{x}_n \| \leq \frac{\alpha}{2}, \| y - \tilde{y}_n \| \leq \frac{\alpha}{2} \quad \text{and} \quad | r - m(\bar{x}) | \leq \frac{\alpha}{2}.$$

Therefore, we can use the hypothesis (H) to get that $(\tilde{x}_n, \tilde{y}_n, \tilde{s}_n)$ is a local minimum of the function

$$(x,y,r) \longrightarrow h(x,y,r) + (1+\lambda)\sqrt{\eta_n}(\| \tilde{x}_n - x \| + \| \tilde{y}_n - y \| + | \tilde{s}_n - r |) +$$

$$kb\left[d(x,y,r;C_1) + d(x,y,r;C_2)\right].$$

So by subdifferential calculus rules we have

$$(x_n^*, 0, -\lambda_n) \in kb\partial_F d(\tilde{x}_n, \tilde{y}_n, \tilde{s}_n; epi f) + kb\partial_F d(\tilde{x}_n, \tilde{y}_n; gph\ G) \times \{0\} +$$

$$((1+\lambda)\sqrt{\eta_n} + \epsilon_n)B_{X^*} \times B_{Y^*} \times B_{\Re}.$$

Using the lower semicontinuity of m we get

$m(\bar{x}) \leq \liminf_n m(\tilde{x}_n) \leq \liminf_n f(\tilde{x}_n, \tilde{y}_n) \leq \limsup_n f(\tilde{x}_n, \tilde{y}_n) \leq \lim_n \tilde{s}_n = m(\bar{x})$.

and hence $\lim_n f(\tilde{x}_n, \tilde{y}_n) = m(\bar{x})$.

3.2 Theorem. Under the assumptions of Theorem 3.1 one has the following inclusion

$$\partial_F m(\bar{x}) \subset \bigcup_{(x^*,y^*) \in \hat{\partial}_F f(\bar{x})} \{x^* + \hat{D}_F^* G(\bar{x}) y^*\},$$

where (with $P :=]0, +\infty[$)

$\hat{\partial}_F f(\bar{x}) := \{(x^*, y^*) \in X^* \times Y^* :$

$$(x^*, y^*, -1) \in P \limsup_{x \to \bar{x}, f(x,y) \to m(\bar{x}), r \to m(\bar{x}), (x,y,r) \in C_1 \cap C_2} \partial_F d(x,y,r; epi\ f)\}$$

and

$$\hat{D}_F^* G(\bar{x}) y^* :=$$

$$\{x^* \in X^* : (x^*, -y^*) \in P \limsup_{x \to \bar{x}, f(x,y) \to m(\bar{x}), y \in G(x)} \partial_F d(x,y; gph\ G)$$

Proof. Fix $u^* \in \partial_F m(\bar{x})$ and apply the conclusion of Lemma 3.1. We get

$$(x_n^*, 0, -\lambda_n) \in kb\partial_F d(\tilde{x}_n, \tilde{y}_n, \tilde{s}_n; epi\ f) + kb\partial_F d(\tilde{x}_n, \tilde{y}_n; gph\ G) \times \{0\} +$$

$$((1+\lambda)\sqrt{\eta_n} + \epsilon_n)B_{X^*} \times B_{Y^*} \times B_{\Re}.$$

Since $\partial_F d(\tilde{x}_n, \tilde{y}_n; gph\ G) \subset B_{X^*} \times B_{Y^*}$ and since the closed balls of the dual of any Asplund space are weak-star sequentially compact, see [?], we obtain (after extraction of subsequences)

$\lambda(u^*, 0, -1) \in$

$kb \limsup_{x \to \bar{x}, f(x,y) \to m(\bar{x}), r \to m(\bar{x}), (x,y,r) \in C_1 \cap C_2} \partial_F d(x,y,r; epi\ f) +$

$$kb \limsup_{x\to\bar{x}, f(x,y)\to m(\bar{x}), y\in G(x)} \partial_F d(x,y; gph\ G)\times\{0\}.$$

Thus, there exist
$(z^*, y^*, -1) \in$
$\Re^+ \limsup_{x\to\bar{x}, f(x,y)\to m(\bar{x}), r\to m(\bar{x}), (x,y,r)\in C_1\cap C_2}$
$\partial_F d(x,y,r; epi\ f)$
and
$(v^*, w^*) \in \Re^+ \limsup_{x\to\bar{x}, f(x,y)\to m(\bar{x}), y\in G(x)}$
$\partial_F d(x,y; gph\ G)$
such that

$$(u^*, 0, -1) = (z^*, y^*, -1) + (v^*, w^*, 0).$$

So we may conclude that $u^* \in z^* + \hat{D}_F G(\bar{x}) y^*$ with $(z^*, y^*) \in \hat{\partial}_F f(\bar{x})$.

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