Differentiability of a 4-point ternary subdivision scheme and its application

Hongchan Zheng, Hongxing Zhao, Zhenglin Ye, and Min Zhou *

Abstract

Hassan *et al.* proposed a 4-point ternary interpolatory scheme with smaller sizes of the templates for the local averaging rules and with higher order smoothness property compared to most of the existing binary ones. It can be C^2-continuous when the subdivision parameter is chosen in a certain range. In this paper, we further investigate its differentiable properties to extend its application in the generating of smooth curves and surfaces with different continuity. Some important results about this scheme such as the conditions of C^0, C^1-continuous, Hölder exponent and the derivatives of the limit function are obtained and applied. A modified 4-point ternary interpolatory scheme for end points is also proposed to ameliorate the modelling ability of this scheme.

Keywords: ternary subdivision, interpolation, subdivision matrix, C^k-continuity

1 Introduction

Subdivision started as a tool for efficient computation of spline functions, and is now an important and independent subject with many applications in fields including Computer Aided Geometric Design, Computer Graphics, computer animation, surgical simulation and medical image processing. Especially it is used for developing new methods for curve and surface design.

A subdivision curve or surface is defined as the limit of a finer and finer control polygon or mesh by subdividing the polygon or mesh according to some refining rules recursively. In application, we always use a subdivided polygon or mesh at a certain refinement level to replace the limit curve or surface within the permitted accuracy. Subdivision schemes provide an efficient way to describe curves and surfaces for their convenience and flexibility.

Some binary interpolatory subdivision algorithms [1]-[7] were designed for the generation of interpolatory curves and surfaces, where the ratio of similarity between the edge of the initial regular mesh and the edge of the mesh after one subdivision step is 2 [8]. Recently ternary interpolatory subdivision algorithms received a lot of attention [9]-[15]. In [9] Hassan *et al.* proposed a 4-point ternary interpolatory subdivision scheme. It uses the following subdivision rules to refine the control polygon recursively:

$$\begin{aligned} P_{3j}^{k+1} &= P_j^k, \\ P_{3j+1}^{k+1} &= a_0 P_{j-1}^k + a_1 P_j^k + a_2 P_{j+1}^k + a_3 P_{j+2}^k, \\ P_{3j+2}^{k+1} &= a_3 P_{j-1}^k + a_2 P_j^k + a_1 P_{j+1}^k + a_0 P_{j+2}^k, \end{aligned} \tag{1}$$

where the weights a_i are given by $a_0 = -\frac{1}{18} - \frac{1}{6}\mu, a_1 = \frac{13}{18} + \frac{1}{2}\mu, a_2 = \frac{7}{18} - \frac{1}{2}\mu, a_3 = -\frac{1}{18} + \frac{1}{6}\mu$, and μ is the parameter of the scheme.

It has been proved that the 4-point ternary interpolatory subdivision scheme has a support of 5, which has the advantage over the 4-point binary one having a support of 6 [9]. Furthermore, for $\frac{1}{15} < \mu < \frac{1}{9}$, the limit curve of the 4-point ternary scheme is C^2-continuous. Since it is desirable to have subdivision methods which have small sizes of templates for the local averaging rules, smooth limit curves and parameters to control its shape, the 4-point ternary scheme compares favourably with most of the existing binary interpolatory subdivision schemes, which either have C^1-continuity, such as the 4-point binary scheme [1], or have a bigger template width, such as the 6-point binary scheme [2].

Furthermore, applying the 4-point ternary interpolatory subdivision scheme one can model interpolatory curves with different smoothness. To do this one must know more differentiable properties about the scheme.

In this paper we first perform a further differentia-

*This work is supported by grant 02JK192 from the Science Research Plan of Education Department of Shaanxi Province of China and Youth for NPU Teachers Scientific and Technological Innovation Foundation. Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, 710072, P.R. China. Email: zhenghc@nwpu.edu.cn

bility analysis on it, including its C^0, C^1-continuous conditions, Hölder exponent against μ and the expressions of the derivatives of the limit curve. Then we discuss its application in the modelling of smooth curves. Finally, we propose a modified 4-point ternary interpolatory scheme to deal with the problem of end points in the case of open polygon.

2 Further convergence analysis of Hassan's scheme

2.1 C^0 and C^1 convergence analysis—necessary conditions

Hassan *et al.* has obtained the necessary condition for this scheme to be C^2 based on the eigenvalues of the mid-point and vertex subdivision matrices. Now we continue the analysis of this scheme to be C^0 and C^1.

The eigenvalues of the mid-point subdivision matrix [9] are

$$1, \frac{1}{3}, \frac{1}{9}, \mu, -\frac{1}{18}+\frac{1}{6}\mu, -\frac{1}{18}+\frac{1}{6}\mu.$$

The eigenvalues of the vertex subdivision matrix are

$$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{18}-\frac{1}{2}\mu, \frac{1}{6}-\frac{5}{6}\mu.$$

From [16], we can know that the necessary condition for this scheme to be C^0 is

$$|\mu|<1, |-\frac{1}{18}+\frac{1}{6}\mu|<1, |\frac{1}{18}-\frac{1}{2}\mu|<1, |\frac{1}{6}-\frac{5}{6}\mu|<1,$$

namely

$$-1<\mu<1.$$

Similarly the necessary condition to be C^1 is

$$|\mu|<\frac{1}{3}, |-\frac{1}{18}+\frac{1}{6}\mu|<\frac{1}{3}, |\frac{1}{18}-\frac{1}{2}\mu|<\frac{1}{3}, |\frac{1}{6}-\frac{5}{6}\mu|<\frac{1}{3},$$

namely

$$-\frac{1}{5}<\mu<\frac{1}{3}.$$

2.2 C^0 and C^1 convergence analysis—sufficient conditions

In this subsection we derive the sufficient conditions for the scheme to be C^0 and C^1.

Let S be the 4-point ternary interpolatory subdivision scheme and $S_i, i=1,2,3$ be its divided difference subdivision schemes, then by applying Dyn's method [17] to the case of ternary subdivision we obtain

$$\begin{aligned}\|(\frac{1}{3}S_1)^2\|_\infty = \frac{1}{324}max\{&72|\mu+3\mu^2|+36|1+2\mu+6\mu^2|,\\ &2|-1-18\mu+27\mu^2|+36|-1-2\mu+3\mu^2|\\ &+2|1-18\mu+27\mu^2|,\\ &|3\mu-1|^2+6|5\mu-9\mu^2|+|41-42\mu+81\mu^2|\\ &+6|-1+3\mu-6\mu^2|,\\ &6|\mu-3\mu^2|+6|1-\mu+18\mu^2|+6|-6+\mu+27\mu^2|\\ &+6|-1+\mu+12\mu^2|,\\ &|-1+9\mu^2|+6|-2-5\mu+9\mu^2|+|31-60\mu+81\mu^2|\\ &+6|1-5\mu+6\mu^2|\},\end{aligned}$$

and

$$\begin{aligned}\|(\frac{1}{3}S_2)^2\|_\infty = \frac{1}{36}max\{&2|1-81\mu^2|+2|1+81\mu^2|,\\ &6|1+6\mu-27\mu^2|+12|-2\mu+9\mu^2|,\\ &10|1-6\mu+9\mu^2|+2|7-30\mu+45\mu^2|,\\ &|-1+12\mu-27\mu^2|+|5-24\mu+135\mu^2|+12|\mu-9\mu^2|,\\ &2|2-27\mu+45\mu^2|+|-1+60\mu-99\mu^2|+|1-6\mu+9\mu^2|\}.\end{aligned}$$

Since for $-1<\mu<\frac{2}{3}$, $\|(\frac{1}{3}S_1)^2\|_\infty<1$, we know that for this improved range, the scheme is C^0. Similarly for $\frac{1}{3}-\frac{2}{15}\sqrt{10}<\mu<\frac{1}{3}$, $\|(\frac{1}{3}S_2)^2\|_\infty<1$, the scheme is C^1.

In conclusion, we have the following theorem.

Theorem 1. Given initial control points $\{P_j^0\}$, let P_j^k defined by (1) be the values corresponding to $\frac{j}{3^k}$, and let $p(t)$ be the limit function of this process, then, $p(t)$ is C^0 only if $-1<\mu<1$, and $p(t)$ is C^1 only if $-\frac{1}{5}<\mu<\frac{1}{3}$. On the other hand, for $-1<\mu<\frac{2}{3}$, $p(t)$ is C^0 , and for $\frac{1}{3}-\frac{2}{15}\sqrt{10}<\mu<\frac{1}{3}$, $p(t)$ is C^1. $p(t)$ is C^2 if and only if $\frac{1}{15}<\mu<\frac{1}{9}$.

2.3 Hölder exponent of the 4-point ternary scheme scheme

From the above two subsections it is easily to see that the 4-point ternary scheme has different continuity depending on the subdivision parameter μ. Furthermore, we can derive its highest smoothness by using Rioul's method for the ternary case [18, 11].

It is easily known that the 4-point ternary scheme scheme S and its divided difference subdivision schemes $S_i, i=1,2$ satisfy the necessary condition of a convergent subdivision scheme. Considering this and the fact that for $\frac{1}{15}<\mu<\frac{1}{9}$, $\|\frac{1}{3}S_3\|_\infty<1$, we know that the 4-point ternary scheme fulfil Rioul's conditions for the ternary case. Based on generalized Rioul's method [11] we can conclude that the scheme has Hölder regularity $R_H = 2+\nu^k$ for all $k \geq 1$, where ν^k is given by

$$3^{-k\nu^k} = \|(\frac{1}{3}S_3)^k\|_\infty.$$

For the convenience of computation, we set $k = 1$. Since

$$\|\frac{1}{3}S_3\|_\infty = \begin{cases} \frac{3}{2} - \frac{15}{2}\mu, & \frac{1}{15} < \mu \leq \frac{1}{11}, \\ 9\mu, & \frac{1}{11} < \mu < \frac{1}{9}, \end{cases}$$

we obtain that Hölder regularity against μ of the 4-point ternary scheme scheme is

$$R(\mu) = \begin{cases} 2 - \log_3(\frac{3}{2} - \frac{15}{2}\mu), & \frac{1}{15} < \mu \leq \frac{1}{11}, \\ 2 - \log_3(9\mu), & \frac{1}{11} < \mu < \frac{1}{9}. \end{cases}$$

Therefore the highest smoothness of the 4-point ternary scheme is achieved at $\mu = \frac{1}{11}$, and its Hölder exponent is

$$R_H = R(\frac{1}{11}) = 2.1827.$$

3 The derivatives of the limit function

In this section we derive the exact expressions of the first and the second derivatives of the limit function of the 4-point ternary scheme scheme. Since the differentiability of $p(t)$ in Theorem 1 can be reduced to that of its components, for simpleness we only need to present the results in the case of initial data being a sequence of real numbers.

Theorem 2. Given initial real numbers $\{f_i^0\}$, let f_i^k defined by the 4-point ternary interpolatory subdivision scheme be the values corresponding to $\frac{i}{3^k}(i, k \in Z, k \geq 0)$ and $f \in C^2$ be the corresponding limit function with $\frac{1}{15} < \mu < \frac{1}{9}$, then for arbitrarily fixed $m, n_0 \in Z, m \geq 0$, the derivatives of the limit function f are

$$f'(\frac{n_0}{3^m}) = \frac{3^{m+k}}{2(5+9\mu)}[(-1+3\mu)(f^{m+k}_{3^k n_0+2} - f^{m+k}_{3^k n_0-2}) + (7+3\mu)(f^{m+k}_{3^k n_0+1} - f^{m+k}_{3^k n_0-1})], \quad (2)$$

$$f''(\frac{n_0}{3^m}) = \frac{3^{2(m+k)}}{-1+15\mu}[(-1+3\mu)(f^{m+k}_{3^k n_0+2} + f^{m+k}_{3^k n_0-2}) - 4(1+3\mu)f^{m+k}_{3^k n_0} + 3(1+\mu)(f^{m+k}_{3^k n_0+1} + f^{m+k}_{3^k n_0-1})]. \quad (3)$$

Proof. Let us denote by

$$\mathbf{F}^k = (f^{m+k}_{3^k n_0-2}, f^{m+k}_{3^k n_0-1}, f^{m+k}_{3^k n_0}, f^{m+k}_{3^k n_0+1}, f^{m+k}_{3^k n_0+2})^T,$$

we get $f^{m+k}_{3^k n_0} = f^m_{n_0}$ and $\mathbf{F}^{k+1} = \mathbf{AF}^k$, where

$$\mathbf{A} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ 0 & a_3 & a_2 & a_1 & a_0 \end{pmatrix}$$

is the vertex subdivision matrix. For $\frac{1}{15} < \mu < \frac{1}{9}$, $\mathbf{A}$ has five different eigenvalues $\lambda_1 = 1, \lambda_2 = \frac{1}{3}, \lambda_3 = \frac{1}{9}, \lambda_4 = \frac{1}{8} - \frac{1}{2}\mu, \lambda_5 = \frac{1}{6} - \frac{5}{6}\mu$, and has five orthogonal eigenvectors. Let $\mathbf{r}_i, \mathbf{l}_i$ be the right and left eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues $\lambda_i, i = 1, ..., 5$, then direct computation leads to

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{r}_2 = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \mathbf{r}_3 = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \\ 4 \end{pmatrix},$$

$$\mathbf{r}_4 = \begin{pmatrix} -1 \\ \frac{-1+3\mu}{7+3\mu} \\ 0 \\ -\frac{-1+3\mu}{7+3\mu} \\ 1 \end{pmatrix}, \mathbf{r}_5 = \begin{pmatrix} 1 \\ -\frac{-1+3\mu}{3(1+\mu)} \\ 0 \\ -\frac{-1+3\mu}{3(1+\mu)} \\ 1 \end{pmatrix},$$

$$\mathbf{l}_2 = (-\frac{-1+3\mu}{2(5+9\mu)}, -\frac{7+3\mu}{2(5+9\mu)}, 0, \frac{7+3\mu}{2(5+9\mu)}, \frac{-1+3\mu}{2(5+9\mu)}),$$

$$\mathbf{l}_3 = (\frac{-1+3\mu}{2(-1+15\mu)}, \frac{3(1+\mu)}{2(-1+15\mu)}, -\frac{2(1+3\mu)}{-1+15\mu}, \frac{3(1+\mu)}{2(-1+15\mu)}, \frac{-1+3\mu}{2(-1+15\mu)}).$$

If the values generated by the 4-point ternary interpolatory subdivision process define a C^2-continuous function f for $\frac{1}{15} < \mu < \frac{1}{9}$, then necessarily

$$\lim_{k\to\infty} \mathbf{F}^k = f^m_{n_0}\mathbf{r}_1 = f(\frac{n_0}{3^m})\mathbf{r}_1, \quad (4)$$

$$\lim_{k\to\infty} \frac{f^{m+k}_{3^k n_0+j} - f^m_{n_0}}{\frac{j}{3^{m+k}}} = \lim_{k\to\infty} \frac{f(\frac{3^k n_0+j}{3^{m+k}}) - f(\frac{n_0}{3^m})}{\frac{j}{3^{m+k}}} = f'(\frac{n_0}{3^m}), j = \pm 1, \pm 2, \quad (5)$$

$$\lim_{k\to\infty} \frac{f^{m+k}_{3^k n_0+j} + f^{m+k}_{3^k n_0-j} - 2f^m_{n_0}}{(\frac{j}{3^{m+k}})^2}$$

$$= \lim_{k\to\infty} \frac{f(\frac{3^k n_0+j}{3^{m+k}}) + f(\frac{3^k n_0-j}{3^{m+k}}) - 2f(\frac{n_0}{3^m})}{(\frac{j}{3^{m+k}})^2}$$

$$= f''(\frac{n_0}{3^m}), j = \pm 1, \pm 2. \quad (6)$$

Let $\hat{\mathbf{e}} = (1,1,0,1,1)^T$, $\mathbf{D} = \text{diag}\,(-\frac{1}{2}, -1, 1, 1, \frac{1}{2})$ and

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

then by (4)-(6), we derive

$$\lim_{k\to\infty} 3^{m+k}\mathbf{D}(\mathbf{F}^k - f^m_{n_0}\mathbf{r}_1) = f'(\frac{n_0}{3^m})\hat{\mathbf{e}}, \quad (7)$$

$$\lim_{k\to\infty} 3^{2(m+k)}\mathbf{D}^2(\mathbf{F}^k + \mathbf{J}\mathbf{F}^k - 2f^m_{n_0}\mathbf{r}_1) = f''(\frac{n_0}{3^m})\hat{\mathbf{e}}. \quad (8)$$

Since $\mathbf{A}$ has five linear independent eigenvectors, there exist $\alpha_1, \alpha_2, ..., \alpha_5$ such that $\mathbf{F}^0$ can be written as

$$\mathbf{F}^0 = \sum_{i=1}^{5} \alpha_i \mathbf{r}_i.$$

From the expressions of $\mathbf{r}_i, i = 1, ..., 5$, we get $\alpha_1 = f^m_{n_0}$. Therefore $\mathbf{F}^k$ can be written as

$$\mathbf{F}^k = \mathbf{A}^k\mathbf{F}^0 = \sum_{i=1}^{5} \alpha_i\lambda_i^k\mathbf{r}_i = f^m_{n_0}\mathbf{r}_1 + \alpha_2(\frac{1}{3})^k\mathbf{r}_2$$

$$+\alpha_3(\frac{1}{9})^k\mathbf{r}_3 + \sum_{i=4}^{5} \alpha_i\lambda_i^k\mathbf{r}_i. \quad (9)$$

By (7),(8) and (9), in view of

$$\mathbf{J}\mathbf{r}_i = \begin{cases} \mathbf{r}_i, i = 1, 3, 5, \\ -\mathbf{r}_i, i = 2, 4, \end{cases}$$

we can show

$$\lim_{k\to\infty} 3^m[\alpha_2\mathbf{D}\mathbf{r}_2 + (\frac{1}{3})^k\alpha_3\mathbf{D}\mathbf{r}_3 + \sum_{i=4}^{5}\alpha_i(3\lambda_i)^k\mathbf{D}\mathbf{r}_i]$$

$$= f'(\frac{n_0}{3^m})\hat{\mathbf{e}},$$

$$\lim_{k\to\infty} 3^{2m}[2\alpha_3\mathbf{D}^2\mathbf{r}_3 + 2\alpha_5(9\lambda_5)^k\mathbf{D}^2\mathbf{r}_5] = f''(\frac{n_0}{3^m})\hat{\mathbf{e}}.$$

It is easily seen that for $\frac{1}{15} < \mu < \frac{1}{9}$, the eigenvalues of matrix A satisfy

$$|\lambda_i| < \frac{1}{9} < \frac{1}{3}, i = 4, 5,$$

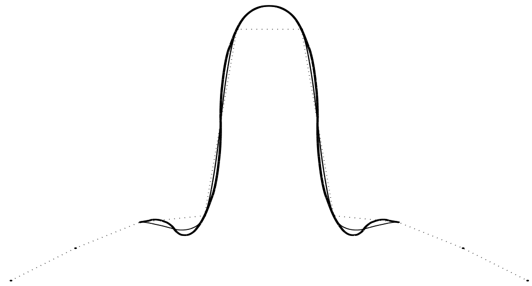

Figure 1: Open curves produced by the 4-point ternary interpolatory scheme. The thin solid line is produced by setting $\mu = \frac{1}{11}$ and the bold solid line is produced by setting $\mu = \frac{1}{4}$.

therefore by using $\mathbf{D}\mathbf{r}_2 = \mathbf{D}^2\mathbf{r}_3 = \hat{\mathbf{e}}$, we can get

$$f'(\frac{n_0}{3^m}) = 3^m\alpha_2, f''(\frac{n_0}{3^m}) = 3^{2m}2\alpha_3. \quad (10)$$

In order to derive the explicit formulae of $f'(\frac{n_0}{3^m})$ and $f''(\frac{n_0}{3^m})$, we multiply (9) by the left eigenvector $\mathbf{l}_2$ corresponding to the eigenvalue $\lambda_2 = \frac{1}{3}$ and the left eigenvector $\mathbf{l}_3$ corresponding to the eigenvalue $\lambda_3 = \frac{1}{9}$ respectively, in view of

$$\mathbf{l}_j\mathbf{r}_i = \begin{cases} 1, i = j, \\ 0, i \neq j, \end{cases}$$

we have

$$\mathbf{l}_2\mathbf{F}^k = \alpha_2(\frac{1}{3})^k, \mathbf{l}_3\mathbf{F}^k = \alpha_3(\frac{1}{9})^k,$$

Hence

$$\alpha_2 = 3^k\mathbf{l}_2\mathbf{F}^k, \alpha_3 = 9^k\mathbf{l}_3\mathbf{F}^k. \quad (11)$$

From (10), (11) and the expressions of $\mathbf{l}_2, \mathbf{l}_3$, we obtain (2) and (3).

4 The application of the C^k-continuity of the 4-point ternary interpolatory subdivision scheme

The subdivision scheme (1) can be used to define a curve interpolating initial control points $P_0, P_1, ..., P_m$. In the case of an open curve, we need to supply four extra and assistant control points P_{-2}, P_{-1}, P_{m+1} and P_{m+2}, which affect the behavior of the curve near its end points P_0 and P_m. In the case of a closed curve, we only need to let $P_{-2} = P_{m-1}, P_{-1} = P_m, P_{m+1} = P_0, P_{m+2} = P_1$.

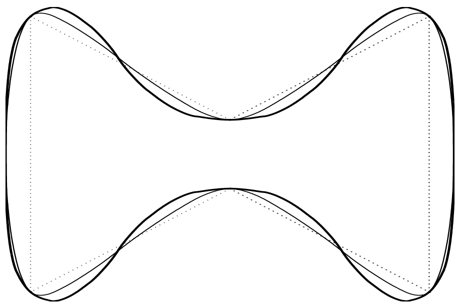

Figure 2: Closed curves produced by the 4-point ternary interpolatory scheme. The thin solid line is produced by setting $\mu = \frac{1}{10}$ and the bold solid line is produced by setting $\mu = \frac{1}{5}$.

Fig. 1 and Fig. 2 show the results applied (1) to the common initial control polygon after four subdivision steps respectively. In this two figures the control polygons are drawn by dotted lines, and the subdivision curves drawn by solid lines. Fig. 1 shows examples of open curves. In Fig. 1 the thin solid line is produced by setting $\mu = \frac{1}{11}$ (the limit curve is C^2) and the bold solid line is produced by setting $\mu = \frac{1}{4}$ (the limit curve is C^1). Fig. 2 shows examples of closed curves. In Fig. 2 the thin solid line is produced by setting $\mu = \frac{1}{10}$ (the limit curve is C^2) and the bold solid line is produced by setting $\mu = \frac{1}{5}$ (the limit curve is C^1).

From the above two examples we conclude that we can model interpolatory curves with different smoothness fast by using the 4-point ternary scheme and Theorem 1 and can adjust the shape of the subdivision curves to a certain extend by choosing the subdivision parameter μ appropriately.

But the 4-point ternary scheme still has the disadvantage that it can not handles the situation of end points conveniently in the case of open polygon. In view of this, we propose an ameliorated 4-point ternary interpolatory subdivision scheme interpolating end points directly in the next section.

5 A modified 4-point ternary interpolatory subdivision scheme

In this section we present a modified edition of the 4-point ternary scheme which interpolate every initial control points without supplying any extra control points in the case of open polygon as follows:

Algorithm 1. Given initial control points $P_j^0, j = 0, ..., m$, let P_j^k be the values corresponding to $\frac{j}{3^k}(0 \leq j \leq 3^k n, k \geq 0)$, P_j^k are defined recursively by

$$\begin{aligned}
P_{3j}^{k+1} &= P_j^k, \quad 0 \leq j \leq 3^k m,\\
P_{3j+1}^{k+1} &= a_0 P_{j-1}^k + a_1 P_j^k + a_2 P_{j+1}^k + a_3 P_{j+2}^k,\\
&\qquad 1 \leq j \leq 3^k m - 2,\\
P_{3j+2}^{k+1} &= a_3 P_{j-1}^k + a_2 P_j^k + a_1 P_{j+1}^k + a_0 P_{j+2}^k,\\
&\qquad 1 \leq j \leq 3^k m - 2,\\
P_1^{k+1} &= (\tfrac{1}{3} - 5a_0 + 3a_3)P_0^k + (\tfrac{10}{3} - 5a_1 + 3a_2)P_1^k\\
&\quad + (-\tfrac{2}{3} - 5a_2 + 3a_1)P_2^k + (-5a_3 + 3a_0)P_3^k,\\
P_2^{k+1} &= (\tfrac{1}{15} - 3a_0 + \tfrac{8}{5}a_3)P_0^k + (\tfrac{8}{3} - 3a_1 + \tfrac{8}{5}a_2)P_1^k\\
&\quad + (-\tfrac{1}{3} - 3a_2 + \tfrac{8}{5}a_1)P_2^k + (-3a_3 + \tfrac{8}{5}a_0)P_3^k,\\
P_{3^{k+1}n-2}^{k+1} &= (-3a_3 + \tfrac{8}{5}a_0)P_{3^k n-3}^k\\
&\quad + (-\tfrac{1}{3} - 3a_2 + \tfrac{8}{5}a_1)P_{3^k n-2}^k\\
&\quad + (\tfrac{8}{3} - 3a_1 + \tfrac{8}{5}a_2)P_{3^k n-1}^k\\
&\quad + (\tfrac{1}{15} - 3a_0 + \tfrac{8}{5}a_3)P_{3^k n}^k,\\
P_{3^{k+1}n-1}^{k+1} &= (-5a_3 + 3a_0)P_{3^k n-3}^k\\
&\quad + (-\tfrac{2}{3} - 5a_2 + 3a_1)P_{3^k n-2}^k\\
&\quad + (\tfrac{10}{3} - 5a_1 + 3a_2)P_{3^k n-1}^k\\
&\quad + (\tfrac{1}{3} - 5a_0 + 3a_3)P_{3^k n}^k.
\end{aligned}$$

Remark. $\mathrm{P}_1^{k+1}, P_2^{k+1} (k \geq 0)$ are derived such that $P_0^{k+1}, P_1^{k+1}, P_2^{k+1}, P_3^{k+1}, P_4^{k+1}, P_5^{k+1}, P_6^{k+1}$ are on one quartic curve. So do $P_{3^{k+1}n-2}^{k+1}, P_{3^{k+1}n-1}^{k+1} (k \geq 0)$. Thus the C^1-continuity of the modified 4-point ternary interpolatory subdivision scheme for $\frac{1}{3} - \frac{2}{15}\sqrt{10} < \mu < \frac{1}{3}$ and the C^2-continuity for $\frac{1}{15} < \mu < \frac{1}{9}$ can be guaranteed. Here we will not give the details owing to the limitation of space.

Fig. 3 depicts the difference between the results obtained by applying the modified scheme and the original scheme with the same $\mu = \frac{1}{11}$ (the limit curves are all C^2) after four subdivision steps given the same initial control points. In Fig. 3 the bold solid line is produced by applying the modified scheme and the thin solid line is produced by applying the original scheme with the same control polygon as in Fig. 1.

6 Conclusion

Further convergence analysis on the C^0 and C^1 continuity of the 4-point ternary scheme is presented. The expressions of the first and the second derivatives of the limit function are derived. A modified 4-point ternary interpolatory subdivision scheme is proposed which interpolates the endpoints of the initial control polygon directly in the case of open polygon. The results obtained in this paper can be extended to the case of the surface. By using the obtained results one can model smooth curves and surfaces with different smoothness efficiently.

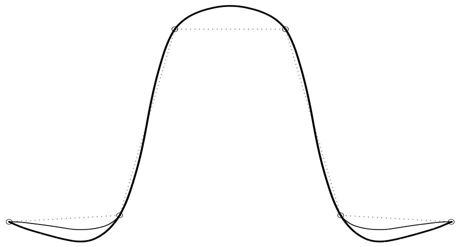

Figure 3: The examples of C^2 interpolatory curves by setting $\mu = \frac{1}{11}$. The bold solid line is produced by applying the modified scheme and the thin solid line is produced by applying the original scheme with the same control polygon as in Fig. 1.

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