Approximate Controllability and Application to Data Assimilation Problem for a Linear Population Dynamics Model

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Abstract—**In this paper, we consider a linear population dynamics model, in which the birth process is described by a nonlocal term and the initial distribution is unknown. We want here, to use an approximate controllability result for retrieving this unknowm datum. The method uses an approximate controllability result for the adjoint system. This result is proved using a global Carleman inequality for the direct problem. More precisely, we prove here that one can expect to compute the initial distribution using observations on a small part of the boundary.**

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1 Introduction

We consider in this paper an age and space structured population living in a bounded domain $\Omega \subset R^N$, $N \geq 1$ with a regular boundary $\partial\Omega$. In what follows, we will denote by $\hat{y}(t, a, x)$ the distribution of individuals of age $0 < a < A$ at time $t > 0$ and location x. Let $\hat{\mu}(a)$ and $\hat{\beta}(a)$ denote the natural death rate and the natural fertility rate of individuals of age a. Let T be a positive real and $A > 0$, the maximal life expectancy. When the diffusion of individuals in the domain follows the Fick's law, namely $\nabla\hat{y}(t, a, x))$ with ∇ the gradient with respect to the spatial variable, then $\hat{y}$ solves the following equation:

$$\frac{\partial \hat{y}}{\partial t} + \frac{\partial \hat{y}}{\partial a} - \Delta\hat{y} + \hat{\mu}(a)\hat{y} = 0 \quad \text{in} \quad (0, T) \times (0, A) \times \Omega \quad (1)$$

where $\frac{\partial \hat{y}}{\partial t}$ and $\frac{\partial \hat{y}}{\partial a}$ are partial derivatives with respect to the variables t and a in the sense of $D'\left((0, T) \times (0, A)\, ; \left(H^1(\Omega)\right)'\right)$. See [10] or [8].

Equation (1) is completed by:

$$\hat{y}(t, 0, x) = \int_0^A \hat{\beta}(a)\hat{y}(t, a, x)da \quad \text{in} \quad (0, T) \times \Omega; \quad (2)$$

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$$\hat{y}(0, a, x) = \hat{y}_0(a, x) \quad \text{in} \quad (0, A) \times \Omega \quad (3)$$

and by

$$\frac{\partial \hat{y}}{\partial \nu}(t, a, \sigma) = 0 \quad \text{in} \quad (0, T) \times (0, A) \times \partial\Omega. \quad (4)$$

Biologically, (2) gives the birth process. Indeed, this gives the distribution of newborn individuals at time t and location x. Equation (3) describes the initial distribution of individuals of age a at location x and relation (4) means that there is no flux of individuals through the boundary of Ω.
From [2] and [8], one can define the trace at $t = t_0$ or $a = a_0$ in the spaces $L^2(Q_A)$ and in $L^2(Q_T)$ respectively, for any solution of (1). This makes conditions (2) and (3) meaningfull.

In this paper, we assume that the function $\hat{y}_0$ is unknown and our goal is to establish a new technique for its determination using some "measurement" on a part of the boundary. More precisely, let us denote by Γ_0 and Γ_1 a partition of $\partial\Omega$, we present a method for recovering $\hat{y}_0$ using the values of $y(t, a, \sigma)$ when $(t, a, \sigma) \in (0, T) \times (0, A) \times \Gamma_0$.
This problem has a great importance in practice, since in the study of a population one cannot get directly the distribution of the individuals on all the domain. But, one can always get information on a part of the boundary or on a small open subset of the domain Ω. Note that, the method works well also in the case of distributed observation.
The problem under our consideration is in fact an inverse problem. Many papers deal with this topic. An introduction to the subject was given in [3] and in [11].The Inverse problems are generally ill- posed and a traditional way to solve them, is the use of the so called Tikhonov regularisation and the minimization of a quadratic functional [9]. Although thousands papers has been devoted to inverse problems and despite its great importance in practice, a little addresses population dynamic. Let us describe briefly what was done in the litterature on population dynamics.
In [15], the author performed a technique of recovering

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the natural death rate in a Mc Kendrick model. The method there, used an overdeterminated data $y(T,a) = \psi(a)$ and the explicit form of the solution. In [7], the goal is different from the previous. Indeed, the authors proposed in [7] a method for determining the individual survival and the reproduction function from data on the population size and the cumulative number of birth in a linear population model of a Mc Kendrick type.
These studies are quite different from the subject we consider here. Though our method uses essentially an approximate controllability result for the adjoint equation of (1-4), it is different from the method of [12]. In [12], the authors consider a Laplace equation and a generalized Stokes system for which the boundary value is unknown. They tried to retrieve an approximation of the boundary value from some "measurement" of the solutions on a given internal surface. The idea in [12] is to take the unknown value as a control function in a approximate controllability problem. In [16] the method used a null controllability result with distributed control and follows the idea of JP Puel [13] that consists to compute the state $\hat{y}$ at time $T > 0$ in order to compute the state at time $t > T$. This idea is also different, since presently, one want to compute the initial distribution.
As one will see, our method works well too for many evolutions equations.
We recall that, the first approximate controllability result for a linear population dynamics model was proved by B. Ainseba [1]. In [1], a linear population dynamics model with Dirichlet boundary conditions is considered and, an approximate controllability result with internal control was studied. The main ingredient there, is a Carleman inequality for the adjoint problem. Here, we give a Carleman inequality with homogenous Neuman boundary condition. This Carleman inequality is more difficult to prove and is in fact the main difficulty of this paper. Afterwards, we study an approximate controllability problem with boundary control.

The remainder of this paper is as follows; in the next section, we will state assumptions and we prove an approximate controllability results. In section 3, we describe a method for recovering the initial distribution.

2 Assumptions and approximate controllability results

Throughout this paper, we set: $Q = (0,T)\times(0,A)\times\Omega$; $Q_A = (0,A)\times\Omega$; $Q_T = (0,T)\times\Omega$; $\Sigma = (0,T)\times(0,A)\times\partial\Omega$ and $\Sigma_0 = (0,T)\times(0,A)\times\Gamma_0$ where Γ_0 is a nonempty open subset of $\partial\Omega$.
We assume that the following assumptions are fulfilled:

$A_1)$ $\hat{\beta} \in L^\infty(0,A)$; and $\hat{\beta} \geq 0\, a.e\, in\, (0,A)$;

$A_2)$ $\hat{\mu} = \mu + \mu_0$; $\mu \in L^\infty(0,A)$; $\mu_0 \in L^1_{loc}(0,A)$; $\mu \geq 0$ $a.e\,(0,A)$ and $\lim_{a\to A}\int_0^a \mu_0(s)ds = +\infty$;

$A_3)$ $\hat{y}_0$ is unknown but, $\hat{y}_0 \neq 0$ and belongs to $L^2_{\pi_0}((0,A)\times\Omega)$ with $\pi_0(a) = exp(\int_0^a \mu_0(s)ds)$;

$A_4)$ $y_{|\Sigma_0}\,(t,a,\sigma)$ is known a.e on Σ_0.

Let us consider for $v \in L^2(\Sigma)$ the following controlled system :

$$\begin{cases} -\frac{\partial z}{\partial t} - \frac{\partial z}{\partial a} - \Delta z + \mu p = \beta z\,(t,0,x) \text{ in } (0,T)\times(0,A)\times\Omega \\ \frac{\partial z}{\partial \nu}(t,a,\sigma) = v(t,a,\sigma)1_{\Sigma 0}(t,a\sigma) \text{ on } \Sigma \\ z(T,a,x) = 0 \text{ in } Q_A \\ z(t,A,x) = 0 \text{ in } Q_T \end{cases} \tag{5}$$

where $\beta = \pi(a)^{-1}\hat{\beta}$.
The system (5) is said to be approximately controllable at time T if the set $R^0 = \{z(0,.,.), z \text{ solves (5); } v \in L^2(\Sigma)\}$ is dense in $L^2(Q_A)$.
The main result of this section is :

Theorem 2.1 *Assume that assumptions $A_1 - A_2$ are fulfilled, then the system (5) is approximately controllable.*

The main ingredient of the proof is an unique continuation result. So, let ω be a nonempty open subset of Ω and ω_0 an open subset of ω such that $\overline{\omega_0} \subset \omega$. There exists a function $\Psi \in C^2(\overline{\Omega})$ such that $\Psi(x) = 0$, for all $x \in \partial\Omega$; $\nabla\Psi(x) \neq 0$, for $x \in \overline{\Omega - \omega_0}$ and $\Psi(x) > 0$, for all $x \in \Omega$. See [6].
In the sequel C will denote several positive constants.
We define for $\lambda > 0$: $\eta(t,a,x) = \frac{e^{2\lambda\|\Psi\|} - e^{\lambda\Psi(x)}}{a(A-a)t(T-t)}$ and $\varphi(t,a,x) = \frac{e^{\lambda\Psi(x)}}{a(A-a)t(T-t)}$.

Let $p \in C^2([0,T]\times[0,A]\times\overline{\Omega})$ be such that:

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} - \Delta p + \mu p = f \text{ in } Q \tag{6}$$

and

$$\frac{\partial p}{\partial \nu} = 0 \text{ on } \Sigma, \tag{7}$$

then we have

Proposition 2.2 *There exist positive constants $s_0 > 1$; $\lambda_0 > 1$ and $C > 0$ such that for $s > s_0$ and $\lambda > \lambda_0$ the following inequality hold:*

$$L(p) \leq C\int_q s^3\lambda^4\varphi^3 e^{-2s\eta}p^2\,dadxdt + C\int_Q e^{-2s\eta}f^2\,dadxdt. \tag{8}$$

where p solves (6-7); $q = (0,T)\times(0,A)\times\omega$ and

$$L(p) = \int_Q \left(\frac{e^{-2s\eta}}{s\varphi}\left(\left|\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a}\right|^2 + |\Delta p|^2\right)\right) dtdadx +$$

$$\int_Q e^{-2s\eta}\left(s\lambda\varphi\left|\nabla p\right|^2+s^3\lambda^4\varphi^3\left|p\right|^2\right)dadtdx.$$

We adapt the proof of the Carleman inequality established in [6] for the heat equation .

Proof of the Proposition 2.2

Let $\overline{\eta}(t,a,x) = \frac{e^{2\lambda\|\Psi\|}-e^{-\lambda\Psi(x)}}{a(A-a)t(T-t)}$ and $\overline{\varphi}(t,a,x) = \frac{e^{-\lambda\Psi(x)}}{a(A-a)t(T-t)}$. We set $\overline{w}=e^{-s\overline{\eta}}p$ and $w=e^{-s\eta}p$, then we have:

$$\frac{\partial w}{\partial t}+\frac{\partial w}{\partial a}=e^{-s\eta}\left(\frac{\partial p}{\partial t}+\frac{\partial p}{\partial a}\right)-s\left(\frac{\partial \eta}{\partial t}+\frac{\partial \eta}{\partial a}\right)w \quad (9)$$

and

$$\nabla w=-s\nabla\eta w+e^{-s\eta}\nabla p=s\lambda\varphi w\nabla\Psi+e^{-s\eta}\nabla p. \quad (10)$$

Therefore:

$$\Delta w=s\lambda^2\varphi\left|\nabla\Psi\right|^2w+s\lambda\varphi w\Delta\Psi-s^2\lambda^2\varphi^2\left|\nabla\Psi\right|^2w+ 2s\lambda\varphi\nabla\Psi.\nabla w+e^{-s\eta}\Delta p. \quad (11)$$

So that, we get:

$$\frac{\partial w}{\partial t}+\frac{\partial w}{\partial a}-\Delta w+\mu w=e^{-s\eta}\left(\frac{\partial p}{\partial t}+\frac{\partial p}{\partial a}-\Delta p-\mu p\right)- s\lambda^2\varphi\left|\nabla\Psi\right|^2w-s\lambda\varphi\Delta\Psi w+s^2\lambda^2\varphi^2\left|\nabla\Psi\right|^2w-2s\lambda\varphi\nabla\Psi.\nabla w- s\left(\frac{\partial \eta}{\partial t}+\frac{\partial \eta}{\partial a}\right)w. \quad (12)$$

This gives:

$$\frac{\partial w}{\partial t}+\frac{\partial w}{\partial a}-\Delta w+2s\lambda^2\varphi\left|\nabla\Psi\right|^2w+2s\lambda\varphi\nabla\Psi.\nabla w- s^2\lambda^2\varphi^2\left|\nabla\Psi\right|^2w+s\left(\frac{\partial \eta}{\partial t}+\frac{\partial \eta}{\partial a}\right)w= e^{-s\eta}f-s\lambda\varphi\Delta\Psi w+s\lambda^2\varphi\left|\nabla\Psi\right|^2w-\mu w. \quad (13)$$

Note that:

$$\nabla\overline{w}=-s\nabla\overline{\eta}\ \overline{w}+e^{-s\overline{\eta}}\nabla p=-s\lambda\overline{\varphi}w\nabla\Psi+e^{-s\overline{\eta}}\nabla p. \quad (14)$$

In the same way, as for the variable w one can get:

$$\frac{\partial \overline{w}}{\partial t}+\frac{\partial \overline{w}}{\partial a}-\Delta\overline{w}+2s\lambda^2\overline{\varphi}\left|\nabla\Psi\right|^2\overline{w}-2s\lambda\overline{\varphi}\nabla\Psi.\nabla\overline{w} -s^2\lambda^2\overline{\varphi}^2\left|\nabla\Psi\right|^2\overline{w}+s\left(\frac{\partial \overline{\eta}}{\partial t}+\frac{\partial \overline{\eta}}{\partial a}\right)\overline{w}=e^{-s\overline{\eta}}f+s\lambda\overline{\varphi}\Delta\Psi\overline{w} +s\lambda^2\overline{\varphi}\left|\nabla\Psi\right|^2\overline{w}-\mu\overline{w}. \quad (15)$$

One can rewrite respectively (13) and (15) as:

$$P_1w+P_2w=g_s \quad (16)$$

and

$$\overline{P}_1\overline{w}+\overline{P}_2\overline{w}=\overline{g}_s \quad (17)$$

where

$$P_1w=\frac{\partial w}{\partial t}+\frac{\partial w}{\partial a}+2s\lambda\varphi\nabla\Psi.\nabla w+2s\lambda^2\varphi\left|\nabla\Psi\right|^2w; \quad (18)$$

$$P_2w=-\Delta w+s\left(\frac{\partial \eta}{\partial t}+\frac{\partial \eta}{\partial a}\right)w-s^2\lambda^2\varphi^2\left|\nabla\Psi\right|^2w; \quad (19)$$

$$g_s=e^{-s\eta}f-s\lambda\varphi\Delta\Psi w+s\lambda^2\varphi\left|\nabla\Psi\right|^2w-\mu w; \quad (20)$$

$$\overline{P}_1\overline{w}=\frac{\partial \overline{w}}{\partial t}+\frac{\partial \overline{w}}{\partial a}-2s\lambda\overline{\varphi}\nabla\Psi.\nabla\overline{w}+2s\lambda^2\overline{\varphi}\left|\nabla\Psi\right|^2\overline{w}; \quad (21)$$

$$\overline{P}_2\overline{w}=-\Delta w+s\left(\frac{\partial \overline{\eta}}{\partial t}+\frac{\partial \overline{\eta}}{\partial a}\right)w-s^2\lambda^2\overline{\varphi}^2\left|\nabla\Psi\right|^2\overline{w} \quad (22)$$

and

$$\overline{g}_s=e^{-s\overline{\eta}}f+s\lambda\overline{\varphi}\Delta\Psi\overline{w}+s\lambda^2\overline{\varphi}\left|\nabla\Psi\right|^2\overline{w}-\mu\overline{w}. \quad (23)$$

Now, let us take the square of (16) and integrate the result over Q, we get:

$$\int_Q\left|P_1w\right|^2dtdadx+\int_Q\left|P_2w\right|^2dtdadx+2\int_Q P_1wP_2wdtdadx =\int_Q\left|g_s\right|^2dtdadx. \quad (24)$$

Integrating the square of (17) over Q yields:

$$\int_Q\left|\overline{P}_1\overline{w}\right|^2dtdadx+\int_Q\left|\overline{P}_2\overline{w}\right|^2dtdadx+2\int_Q \overline{P}_1\overline{w}\overline{P}_2\overline{w}dtdadx =\int_Q\left|\overline{g}_s\right|^2dtdadx. \quad (25)$$

Let us compute first,

$$K=\int_Q P_1wP_2wdtdadx.$$

Note before this computation, that from the definition of Ψ one has

$$\nabla\Psi(\sigma)=\frac{\partial\Psi}{\partial\nu}(\sigma)\nu(\sigma),\ \ \forall\sigma\in\partial\Omega \quad (26)$$

and using (10); (26) and (7) we get clearly

$$\frac{\partial w}{\partial\nu}(t,a,\sigma)=s\lambda\varphi\frac{\partial\Psi}{\partial\nu}w(t,a\sigma)\ \ \text{a.e in}\ \ \Sigma. \quad (27)$$

Note also, that from the definition φ and η we have: $\left|\frac{\partial\eta}{\partial t}\right|\leq C\varphi^2$; $\left|\frac{\partial\eta}{\partial a}\right|\leq C\varphi^2$; $\left|\frac{\partial^2\eta}{\partial t^2}\right|\leq C\varphi^3$; $\left|\frac{\partial^2\eta}{\partial a^2}\right|\leq C\varphi^3$; $\left|\frac{\partial^2\eta}{\partial a^2}\right|\leq C\varphi^3$;

and

$\left|\frac{\partial\varphi}{\partial t}\right| \leq C\varphi^2$; $\left|\frac{\partial\varphi}{\partial a}\right| \leq C\varphi^2$; $\left|\frac{\partial^2\varphi}{\partial t^2}\right| \leq C\varphi^3$; $\left|\frac{\partial^2\varphi}{\partial a^2}\right| \leq C\varphi^3$; $\left|\frac{\partial^2\varphi}{\partial a^2}\right| \leq C\varphi^3$.

On the other hand, from the definition of η and w, we have:

$$w(0,a,x) = w(T,a,x) = 0 \ \text{ in } (0,A)\times\Omega \qquad (28)$$

and

$$w(t,0,x) = w(t,A,x) = 0 \ \text{ in } (0,T)\times\Omega. \qquad (29)$$

We compute now K. This computation gives twelve terms that we denote by $I_{i,j}$ for $i = 1,2,3,4; j = 1,2,3$. The notation $I_{i,j}$ denotes the product of the term number i of (18) and the term number j of (19). So that, we have:

$$I_{1,1} = -\int_Q \frac{\partial w}{\partial t}\Delta w dtdadx.$$

An integration by parts over Q with respect to the spatial variable gives:

$$I_{1,1} = -\int_\Sigma \frac{\partial w}{\partial t}\frac{\partial w}{\partial \nu}dtdad\sigma + \int_Q \frac{\partial \nabla w}{\partial t}.\nabla w dtdadx.$$

Using (27) we get

$$I_{1,1} = -\int_\Sigma \frac{\partial w}{\partial t}\left(s\lambda\varphi\frac{\partial\Psi}{\partial\nu}w\right)dtdad\sigma + \frac{1}{2}\int_Q \frac{\partial\left|\nabla w\right|^2}{\partial t}dtdadx.$$

Using now (28) we obtain after an integration by parts with respect to the variable t:

$$I_{1,1} = \frac{s\lambda}{2}\int_\Sigma \frac{\partial\varphi}{\partial t}\frac{\partial\Psi}{\partial\nu}\left|w\right|^2 dtdad\sigma. \qquad (30)$$

We have:

$$I_{1,2} = s\int_Q \frac{\partial w}{\partial t}\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)w dtdadx$$

$$= \frac{s}{2}\int_Q \frac{\partial\left|w\right|^2}{\partial t}\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)dtdadx.$$

An integration by parts and (28) yield

$$I_{1,2} = -\frac{s}{2}\int_Q \left|w\right|^2\left(\frac{\partial^2\eta}{\partial t^2}+\frac{\partial^2\eta}{\partial a\partial t}\right)dtdadx. \qquad (31)$$

We have,

$$I_{1,3} = -s^2\lambda^2\int_Q \frac{\partial w}{\partial t}\varphi^2\left|\nabla\Psi\right|^2 w dtdadx$$

$$= \frac{-s^2\lambda^2}{2}\int_Q \frac{\partial\left|w\right|^2}{\partial t}\varphi^2\left|\nabla\Psi\right|^2 dtdadx.$$

The above equality gives:

$$I_{1,3} = s^2\lambda^2\int_Q \frac{\partial\varphi}{\partial t}\varphi\left|\nabla\Psi\right|^2\left|w\right|^2 dtdadx \qquad (32)$$

Similarly one has:

$$I_{2,1} = \frac{s\lambda}{2}\int_\Sigma \frac{\partial\varphi}{\partial a}\frac{\partial\Psi}{\partial\nu}\left|w\right|^2 dtdad\sigma; \qquad (33)$$

$$I_{2,2} = -\frac{s}{2}\int_Q \left|w\right|^2\left(\frac{\partial^2\eta}{\partial a^2}+\frac{\partial^2\eta}{\partial a\partial t}\right)dtdadx \qquad (34)$$

and

$$I_{2,3} = s^2\lambda^2\int_Q \frac{\partial\varphi}{\partial a}\varphi\left|\nabla\Psi\right|^2\left|w\right|^2 dtdadx. \qquad (35)$$

Now, let us compute

$$I_{3,1} = -2s\lambda\int_Q \varphi\nabla\Psi.\nabla w\Delta w dtdadx.$$

Standard calculations, (26) and (27) give:

$$I_{3,1} = -2s^3\lambda^3\int_\Sigma \varphi^3\left(\frac{\partial\Psi}{\partial\nu}\right)^3\left|w\right|^2 dtdadx + 2s\lambda^2\int_Q \varphi\left|\nabla\Psi.\nabla w\right|^2 dtdadx$$

$$+s\lambda\int_\Sigma \varphi\frac{\partial\Psi}{\partial\nu}\left|\nabla w\right|^2 dtdadx - s\lambda\int_Q \varphi\left|\nabla w\right|^2\Delta\Psi dtdadx +$$

$$2s\lambda\sum_{i,j}\int_Q \varphi\frac{\partial^2\Psi}{\partial x_i\partial x_j}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}dtdadx - s\lambda^2\int_Q \varphi\left|\nabla\Psi\right|^2\left|\nabla w\right|^2 dtdadx. \qquad (36)$$

Let us compute now the term $I_{3,2}$

$$I_{3,2} = 2s^2\lambda\int_Q \varphi\nabla\Psi.\nabla w\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)w dtdadx$$

$$= s^2\lambda\int_Q \varphi\nabla\Psi.\nabla\left(\left|w\right|^2\right)\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)dtdadx$$

An integration by parts and (26) give

$$I_{3,2} = -s^2\lambda^2\int_Q \varphi\left|w\right|^2\left|\nabla\Psi\right|^2\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)dtdadx +$$

$$+s^2\lambda\int_\Sigma \varphi\frac{\partial\Psi}{\partial\nu}\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)\left|w\right|^2 dtdadx$$

$$-s^2\lambda\int_Q \left|w\right|^2\varphi\nabla.\left(\nabla\Psi\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)\right)dtdadx. \qquad (37)$$

We have:

$$I_{3,3} = -2s^3\lambda^3\int_Q \varphi^3\left|\nabla\Psi\right|^2 w\nabla\Psi.\nabla w dtdadx$$

$$= -s^3\lambda^3\int_Q \varphi^3\left|\nabla\Psi\right|^2 w\nabla\Psi.\nabla\left(\left|w\right|^2\right)dtdadx.$$

Then, this gives:

$$I_{3,3} = -s^3\lambda^3\int_\Sigma \varphi^3\left(\frac{\partial\Psi}{\partial\nu}\right)^3\left|w\right|^2 dtdadx$$

$$+3s^3\lambda^4\int_Q\varphi^3\left|\nabla\Psi\right|^4\left|w\right|^2dtdadx$$

$$+s^3\lambda^3\int_Q\varphi^3\nabla.\left(\left|\nabla\Psi\right|^2\nabla\Psi\right)\left|w\right|^2dtdadx \quad (38)$$

where $\nabla.\left(\left|\nabla\Psi\right|^2\Psi\right)=\sum_i\frac{\partial}{\partial x_i}\left(\left|\nabla\Psi\right|^2\nabla\Psi\right)$.
We want now to compute the three last terms $I_{4,j}$:

$$I_{4,1}=-2s\lambda^2\int_Q\varphi w\left|\nabla\Psi\right|^2\Delta wdtdadx$$

$$=-2s\lambda^2\int_\Sigma\varphi w\left|\nabla\Psi\right|^2\frac{\partial w}{\partial\nu}dtdad\sigma$$

$$+2s\lambda^2\int_Q\nabla\left(\varphi\left|\nabla\Psi\right|^2w\right)\nabla wdtdadx.$$

Therefore, one gets:

$$I_{4,1}=-2s^2\lambda^3\int_Q\varphi^2\left(\frac{\partial\Psi}{\partial\nu}\right)^3\left|w\right|^2dtdadx$$

$$+2s\lambda^3\int_Q\varphi\left|\nabla\Psi\right|^2w\nabla\Psi.\nabla wdtdadx+$$

$$2s\lambda^2\int_Q\varphi w\nabla\left(\left|\nabla\Psi\right|^2\right)\nabla wdtdadx$$

$$+2s\lambda^2\int_Q\varphi\left|\nabla\Psi\right|^2\left|\nabla w\right|^2dtdadx. \quad (39)$$

We get directly

$$I_{4,2}=2s^2\lambda^2\int_Q\varphi\left|\nabla\Psi\right|^2\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)\left|w\right|^2dtdadx. \quad (40)$$

$$I_{4,3}=-2s^3\lambda^4\int_Q\varphi\left|\nabla\Psi\right|^4\left|w\right|^2dtdadx. \quad (41)$$

Finally we obtain:

$$2K=s\lambda\int_\Sigma\left(\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial t}\right)\frac{\partial\Psi}{\partial\nu}\left|w\right|^2dtdad\sigma$$

$$-6s^3\lambda^3\int_\Sigma\varphi^3\left(\frac{\partial\Psi}{\partial\nu}\right)^3\left|w\right|^2dtdad\sigma+$$

$$2s\lambda\int_\Sigma\varphi\frac{\partial\Psi}{\partial\nu}\left|\nabla w\right|^2dtdad\sigma$$

$$+2s^2\lambda\int_\Sigma\varphi\frac{\partial\Psi}{\partial\nu}\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)\left|w\right|^2dtdad\sigma$$

$$-4s^2\lambda^3\int_\Sigma\varphi^2\left(\frac{\partial\Psi}{\partial\nu}\right)^3\left|w\right|^2dtdad\sigma$$

$$-s\int_Q\left|w\right|^2\left(\frac{\partial^2\eta}{\partial t^2}+2\frac{\partial^2\eta}{\partial a\partial t}+\frac{\partial^2\eta}{\partial a^2}\right)dtdadx$$

$$+2s^2\lambda^2\int_Q\left(\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial a}\right)\varphi\left|\nabla\Psi\right|^2\left|w\right|^2dtdadx+$$

$$4s\lambda^2\int_Q\varphi\left|\nabla\Psi.\nabla w\right|^2dtdadx-2s\lambda\int_Q\varphi\left|\nabla w\right|^2\Delta\Psi dtdadx$$

$$+4s\lambda\sum_{i,j}\int_Q\varphi\frac{\partial^2\Psi}{\partial x_i\partial x_j}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}dtdadx$$

$$-2s^2\lambda^2\int_Q\varphi\left|w\right|^2\left|\nabla\Psi\right|^2\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)dtdadx$$

$$-2s^2\lambda\int_Q\left|w\right|^2\varphi\nabla.\left(\nabla\Psi\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)\right)dtdadx+$$

$$2s^3\lambda^4\int_Q\varphi^3\left|\nabla\Psi\right|^4\left|w\right|^2dtdadx$$

$$+2s^3\lambda^3\int_Q\varphi^3\nabla.\left(\left|\nabla\Psi\right|^2\nabla\Psi\right)\left|w\right|^2dtdadx$$

$$-2s^2\lambda^3\int_\Sigma\varphi^2\left(\frac{\partial\Psi}{\partial\nu}\right)^3\left|w\right|^2dtdadx+4s\lambda^3\int_Q\varphi\left|\nabla\Psi\right|^2w\nabla\Psi.\nabla wdtdadx$$

$$+4s\lambda^2\int_Q\varphi w\nabla\left(\left|\nabla\Psi\right|^2\right)\nabla wdtdadx$$

$$+2s\lambda^2\int_Q\varphi\left|\nabla\Psi\right|^2\left|\nabla w\right|^2dtdadx$$

$$+4s^2\lambda^2\int_Q\varphi\left|\nabla\Psi\right|^2\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)\left|w\right|^2dtdadx. \quad (42)$$

In the same way, recalling that

$$\frac{\partial\overline{w}}{\partial\nu}=-s\lambda\frac{\partial\Psi}{\partial\nu}\overline{w}\left(\sigma\right)\quad\text{a.e in }\ \Sigma, \quad (43)$$

we can compute

$$\overline{K}=\int_Q\overline{P}_1\overline{w}\overline{P}_2\overline{w}dtdadx$$

to get also twelve terms $\overline{I}_{i,j}$. Finally, one obtains:

$$2\overline{K}=-s\lambda\int_\Sigma\left(\frac{\partial\overline{\varphi}}{\partial t}+\frac{\partial\overline{\varphi}}{\partial t}\right)\frac{\partial\Psi}{\partial\nu}\left|\overline{w}\right|^2dtdad\sigma$$

$$+6s^3\lambda^3\int_\Sigma\overline{\varphi}^3\left(\frac{\partial\Psi}{\partial\nu}\right)^3\left|\overline{w}\right|^2dtdad\sigma$$

$$-2s\lambda\int_\Sigma\overline{\varphi}\frac{\partial\Psi}{\partial\nu}\left|\nabla\overline{w}\right|^2dtdad\sigma$$

$$-2s^2\lambda\int_\Sigma\overline{\varphi}\frac{\partial\Psi}{\partial\nu}\left(\frac{\partial\overline{\eta}}{\partial t}+\frac{\partial\overline{\eta}}{\partial a}\right)\left|\overline{w}\right|^2dtdad\sigma$$

$$+4s^2\lambda^3\int_\Sigma\overline{\varphi}^2\left(\frac{\partial\Psi}{\partial\nu}\right)^3\left|\overline{w}\right|^2dtdad\sigma$$

$$-s\int_Q\left|\overline{w}\right|^2\left(\frac{\partial^2\overline{\eta}}{\partial t^2}+2\frac{\partial^2\overline{\eta}}{\partial a\partial t}+\frac{\partial^2\overline{\eta}}{\partial a^2}\right)dtdadx$$

$$+2s^2\lambda^2\int_Q\left(\frac{\partial\overline{\varphi}}{\partial t}+\frac{\partial\overline{\varphi}}{\partial a}\right)\overline{\varphi}\left|\nabla\Psi\right|^2\left|\overline{w}\right|^2dtdadx$$

$$+4s\lambda^2\int_Q\overline{\varphi}\left|\nabla\Psi.\nabla\overline{w}\right|^2dtdadx$$

$$+2s\lambda\int_Q\overline{\varphi}\left|\nabla\overline{w}\right|^2\Delta\Psi dtdadx$$

$$-4s\lambda\sum_{i,j}\int_Q\overline{\varphi}\frac{\partial^2\Psi}{\partial x_i\partial x_j}\frac{\partial\overline{w}}{\partial x_i}\frac{\partial\overline{w}}{\partial x_j}dtdadx$$

$$-2s^2\lambda^2\int_Q\overline{\varphi}\left|\overline{w}\right|^2\left|\nabla\Psi\right|^2\left(\frac{\partial\overline{\eta}}{\partial t}+\frac{\partial\overline{\eta}}{\partial a}\right)dtdadx$$

$$+2s^2\lambda\int_Q\left|\overline{w}\right|^2\overline{\varphi}\nabla.\left(\nabla\Psi\frac{\partial\overline{\eta}}{\partial t}+\frac{\partial\overline{\eta}}{\partial a}\right)dtdadx+$$

$$2s^3\lambda^4\int_Q\overline{\varphi}^3\left|\nabla\Psi\right|^4\left|\overline{w}\right|^2dtdadx$$

$$-2s^3\lambda^3\int_Q\overline{\varphi}^3\nabla.\left(\left|\nabla\Psi\right|^2\nabla\Psi\right)\left|\overline{w}\right|^2dtdadx-$$

$$4s\lambda^3\int_Q\overline{\varphi}\left|\nabla\Psi\right|^2w\nabla\Psi.\nabla wdtdadx$$

$$+4s\lambda^2\int_Q\overline{\varphi}\;\overline{w}\nabla\left(\left|\nabla\Psi\right|^2\right)\nabla wdtdadx$$

$$+2s\lambda^2\int_Q\overline{\varphi}\left|\nabla\Psi\right|^2\left|\nabla\overline{w}\right|^2dtdadx$$

$$+4s^2\lambda^2\int_Q\overline{\varphi}\left|\nabla\Psi\right|^2\left(\frac{\partial\overline{\eta}}{\partial t}+\frac{\partial\overline{\eta}}{\partial a}\right)\left|\overline{w}\right|^2dtdadx. \qquad (44)$$

Note that

$$\varphi(t,a,\sigma)=\overline{\varphi}(t,a,\sigma)\quad\text{on}\quad\Sigma$$

and

$$\overline{\eta}(t,a,\sigma)=\eta(t,a,\sigma)\quad\text{on}\quad\Sigma.$$

So that,

$$\overline{w}(t,a,\sigma)=w(t,a,\sigma)\quad\text{on}\quad\Sigma.$$

Using these last identities and adding (42) and (44) one gets:

$$2K+2\overline{K}=A+B+D$$

$$+2s\lambda^2\int_Q\left(\varphi\left|\nabla w.\nabla\Psi\right|^2+\overline{\varphi}\left|\nabla\overline{w}.\nabla\Psi\right|^2\right)dtdadx$$

$$+2s\lambda^2\int_Q\varphi\left|\nabla\Psi\right|^2\left|\nabla w\right|^2dtdadx$$

$$+2s\lambda^2\int_Q\overline{\varphi}\left|\nabla\Psi\right|^2\left|\nabla\overline{w}\right|^2dtdadx+$$

$$2s^3\lambda^4\int_Q\varphi^3\left|\nabla\Psi\right|^4\left|w\right|^2dtdadx$$

$$+2s\lambda^2\int_Q\overline{\varphi}^3\left|\nabla\Psi\right|^4\left|\overline{w}\right|^2dtdadx \qquad (45)$$

where

$$A=4s\lambda\sum_{i,j}\int_Q\varphi\frac{\partial^2\Psi}{\partial x_i\partial x_j}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}dtdadx$$

$$+2s\lambda\int_Q\varphi\Delta\Psi\left|\nabla w\right|^2dtdadx$$

$$-4s\lambda\sum_{i,j}\int_Q\overline{\varphi}\frac{\partial^2\Psi}{\partial x_i\partial x_j}\frac{\partial\overline{w}}{\partial x_i}\frac{\partial\overline{w}}{\partial x_j}dtdadx$$

$$-2s\lambda\int_Q\overline{\varphi}\Delta\Psi\left|\nabla\overline{w}\right|^2dtdadx$$

$$B=-s\int_Q\left|w\right|^2\left(\frac{\partial^2\eta}{\partial t^2}+2\frac{\partial^2\eta}{\partial a\partial t}+\frac{\partial^2\eta}{\partial a^2}\right)dtdadx$$

$$+2s^2\lambda^2\int_Q\left(\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial a}\right)\varphi\left|\nabla\Psi\right|^2\left|w\right|^2dtdadx$$

$$-2s^2\lambda\int_Q\left|w\right|^2\varphi\nabla.\left(\nabla\Psi\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)\right)dtdadx$$

$$+2s^3\lambda^3\int_Q\varphi^3\nabla.\left(\left|\nabla\Psi\right|^2\nabla\Psi\right)\left|w\right|^2dtdadx$$

$$-s\int_Q\left|\overline{w}\right|^2\left(\frac{\partial^2\overline{\eta}}{\partial t^2}+2\frac{\partial^2\overline{\eta}}{\partial a\partial t}+\frac{\partial^2\overline{\eta}}{\partial a^2}\right)dtdadx$$

$$+2s^2\lambda^2\int_Q\left(\frac{\partial\overline{\varphi}}{\partial t}+\frac{\partial\overline{\varphi}}{\partial a}\right)\overline{\varphi}\left|\nabla\Psi\right|^2\left|\overline{w}\right|^2dtdadx$$

$$-2s^2\lambda\int_Q\left|\overline{w}\right|^2\overline{\varphi}\nabla.\left(\nabla\Psi\frac{\partial\overline{\eta}}{\partial t}+\frac{\partial\overline{\eta}}{\partial a}\right)dtdadx$$

$$-2s^3\lambda^3\int_Q\overline{\varphi}^3\nabla.\left(\left|\nabla\Psi\right|^2\nabla\Psi\right)\left|\overline{w}\right|^2dtdadx$$

and

$$D=4s\lambda^3\int_Q\varphi\left|\nabla\Psi\right|^2w\nabla\Psi.\nabla wdtdadx$$

$$+4s\lambda^2\int_Q\varphi w\nabla\left(\left|\nabla\Psi\right|^2\right)\nabla wdtdadx$$

$$-4s\lambda^3\int_Q\overline{\varphi}\left|\nabla\Psi\right|^2w\nabla\Psi.\nabla wdtdadx$$

$$+4s\lambda^2\int_Q\overline{\varphi}\overline{w}\nabla\left(\left|\nabla\Psi\right|^2\right)\nabla wdtdadx$$

Classical algebra show that:

$$|A|\le C\left(s\lambda+\lambda^2\right)\int_Q\left(\varphi\left|\nabla w\right|^2+\overline{\varphi}\left|\nabla\overline{w}\right|^2\right)dtdadx;$$

$$|B|\le C\left(s^3\lambda^3+s^2\lambda^4\right)\int_Q\left(\varphi^3\left|w\right|^2+\overline{\varphi}^3\left|\overline{w}\right|^2\right)dtdadx$$

and

$$|D|\le C\left(s\lambda+\lambda^2\right)\int_Q\left(\varphi\left|\nabla w\right|^2+\overline{\varphi}\left|\nabla\overline{w}\right|^2\right)dtdadx+$$

$$C\left(s^3\lambda^3+s^2\lambda^4\right)\int_Q\left(\varphi^3\left|w\right|^2+\overline{\varphi}^3\left|\overline{w}\right|^2\right)dtdadx.$$

Note also that $\Psi\in C^2\left(\overline{\Omega}\right)$ and $|\nabla\Psi|\ne 0$ in $\overline{\Omega-\omega_0}$. Therefore, there exists δ such that $|\nabla\Psi|\ge\delta$ in $\overline{\Omega-\omega_0}$. Next, (45) gives:

$$2K+2\overline{K}+2s\lambda^2\delta^2\int_{q_0}\left(\overline{\varphi}\left|\nabla\overline{w}\right|^2+\varphi\left|\nabla w\right|^2\right)dtdadx$$

$$+2s^3\lambda^4\delta^4\int_{q_0}\left(\varphi^3|w|^2+\overline{\varphi}^3|\overline{w}|^2\right)dtdadx$$

$$\geq A+B+D+2s\lambda^2\delta^2\int_Q\left(\varphi|\nabla w|^2+\overline{\varphi}|\nabla\overline{w}|^2\right)dtdadx$$

$$+2s^3\lambda^4\delta^4\int_Q\left(\varphi^3|w|^2+\overline{\varphi}^3|\overline{w}|^2\right)dtdadx \qquad (46)$$

where $q_0=(0,A)\times(0,T)\times\omega_0$.
Furthermore,

$$\int_Q\left(g_s^2+\overline{g}_s^2\right)dtdadx\leq\int_Q\left(e^{-2s\eta}+e^{-2s\overline{\eta}}\right)f^2dtdadx$$

$$+C\left(s\lambda+\lambda^2\right)\int_Q\left(\varphi|\nabla w|^2+\overline{\varphi}|\nabla\overline{w}|^2\right)dtdadx+$$

$$C\left(s^3\lambda^3+s^2\lambda^4\right)\int_Q\left(\varphi^3|w|^2+\overline{\varphi}^3|\overline{w}|^2\right)dtdadx. \qquad (47)$$

Adding (24) and (25) we get, taking account of (47) and (46) that

$$\int_Q\left(|P_1w|^2+|P_2w|^2+\left|\overline{P}_1\overline{w}\right|^2+\left|\overline{P}_2\overline{w}\right|^2\right)dtdadx$$

$$+2s\lambda^2\delta^2\int_Q\left(\overline{\varphi}|\nabla\overline{w}|^2+\varphi|\nabla w|^2\right)dtdadx$$

$$+2s^3\lambda^4\delta^4\int_Q\left(\varphi^3|w|^2+\overline{\varphi}^3|\overline{w}|^2\right)dtdadx$$

$$\leq\int_Q\left(e^{-2s\eta}+e^{-2s\overline{\eta}}\right)f^2dtdadx$$

$$+C\left(s\lambda+\lambda^2\right)\int_Q\left(\varphi|\nabla w|^2+\overline{\varphi}|\nabla\overline{w}|^2\right)dtdadx$$

$$+C\left(s^3\lambda^3+s^2\lambda^4\right)\int_Q\left(\varphi^3|w|^2+\overline{\varphi}^3|\overline{w}|^2\right)dtdadx$$

$$2s\lambda^2\delta^2\int_{q_0}\left(\overline{\varphi}|\nabla\overline{w}|^2+\varphi|\nabla w|^2\right)dtdadx$$

$$+2s^3\lambda^4\delta^4\int_{q_0}\left(\varphi^3|w|^2+\overline{\varphi}^3|\overline{w}|^2\right)dtdadx. \qquad (48)$$

Let us choose s and λ large enough such that:

$$s\lambda^2\delta^2\int_Q\left(\overline{\varphi}|\nabla\overline{w}|^2+\varphi|\nabla w|^2\right)dtdadx$$

$$+s^3\lambda^4\delta^4\int_Q\left(\varphi^3|w|^2+\overline{\varphi}^3|\overline{w}|^2\right)dtdadx$$

$$\geq C\left(s\lambda+\lambda^2\right)\int_Q\left(\varphi|\nabla w|^2+\overline{\varphi}|\nabla\overline{w}|^2\right)dtdadx$$

$$+C\left(s^3\lambda^3+s^2\lambda^4\right)\int_Q\left(\varphi^3|w|^2+\overline{\varphi}^3|\overline{w}|^2\right)dtdadx.$$

Therefore, there exist $s_1>1$ and $\lambda_1>1$ such that for $s>s_1$ and $\lambda_1>\lambda_1$, (48) gives

$$\int_Q\left(|P_1w|^2+|P_2w|^2+\left|\overline{P}_1\overline{w}\right|^2+\left|\overline{P}_2\overline{w}\right|^2\right)dtdadx$$

$$+s\lambda^2\delta^2\int_Q\left(\overline{\varphi}|\nabla\overline{w}|^2+\varphi|\nabla w|^2\right)dtdadx$$

$$+s^3\lambda^4\delta^4\int_Q\left(\varphi^3|w|^2+\overline{\varphi}^3|\overline{w}|^2\right)dtdadx$$

$$\leq\int_Q\left(e^{-2s\eta}+e^{-2s\overline{\eta}}\right)f^2dtdadx$$

$$+2s\lambda^2\delta^2\int_{q_0}\left(\overline{\varphi}|\nabla\overline{w}|^2+\varphi|\nabla w|^2\right)dtdadx$$

$$+2s^3\lambda^4\delta^4\int_{q_0}\left(\varphi^3|w|^2+\overline{\varphi}^3|\overline{w}|^2\right)dtdadx. \qquad (49)$$

We want now to eliminate the term $2s\lambda^2\delta^2\int_{q_0}\left(\overline{\varphi}|\nabla\overline{w}|^2+\varphi|\nabla w|^2\right)dtdadx$.

For this aim, we introduce a cut-off function α such that: $\alpha\in C_0^\infty(\omega)$; $0\leq\alpha\leq1$; and $\alpha=1$ on ω_0.
Multiplying P_2w by $\varphi\alpha^2w$ and integrating the result over Q give:

$$\int_Q\varphi\alpha^2wP_2wdtdadx=-s\int_Q\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)w^2\varphi\alpha^2dtdadx$$

$$-\int_Q w\Delta w\varphi\alpha^2dtdadx-s^2\lambda^2\int_Q w^2\varphi^3\alpha^2|\Psi|^2dtdadx. \qquad (50)$$

Note that:

$$\int_Q w\Delta w\varphi\alpha^2dtdadx=-\int_Q|\nabla w|^2\varphi\alpha^2dtdadx$$

$$-\lambda\int_Q w\nabla w.\nabla\Psi\varphi\alpha^2dtdadx-2\int_Q w\nabla w.\nabla\alpha\varphi\alpha dtdadx. \qquad (51)$$

Therefore,

$$\int_Q\varphi\alpha^2wP_2wdtdadx=-s\int_Q\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)w^2\varphi\alpha^2dtdadx$$

$$+\int_Q|\nabla w|^2\varphi\alpha^2dtdadx-s^2\lambda^2\int_Q w^2\varphi^3\alpha^2|\Psi|^2dtdadx$$

$$+\lambda\int_Q w\nabla w.\nabla\Psi\varphi\alpha^2dtdadx+2\int_Q w\nabla w.\nabla\alpha\varphi\alpha dtdadx. \qquad (52)$$

This gives:

$$\int_Q|\nabla w|^2\varphi\alpha^2dtdadx=\int_Q\varphi\alpha^2wP_2wdtdadx$$

$$+s^2\lambda^2\int_Q w^2\varphi^3\alpha^2|\Psi|^2dtdadx$$

$$s\int_Q\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)w^2\varphi\alpha^2dtdadx$$
$$-\lambda\int_Q w\nabla w.\nabla\Psi\varphi\alpha^2dtdadx-2\int_Q w\nabla w.\nabla\alpha\varphi\alpha dtdadx. \tag{53}$$

Note that:

$$-\lambda\int_Q w\nabla w.\nabla\Psi\varphi\alpha^2dtdadx\le C\lambda^2\int_Q|w|^2\varphi\alpha^2dtdadx$$
$$+\frac{1}{2}\int_Q|\nabla w|^2\varphi\alpha^2dtdadx$$

where C is a positive constant. As $\varphi\le C\varphi^3$ with C a positive constant, using now the properties of α and Ψ we deduce:

$$\int_{q_0}|\nabla w|^2\varphi\alpha^2dtdadx\le C\int_Q\varphi\alpha^2wP_2wdtdadx$$
$$+Cs^2\lambda^2\int_Q w^2\varphi^3\alpha^2dtdadx$$
$$+C\int_Q w\varphi^{1/2}|\nabla w|\varphi^{1/2}\alpha dtdadx. \tag{54}$$

From (54) we deduce that:

$$2s\lambda^2\delta^2\int_{q_0}|\nabla w|^2\varphi dtdadx\le\frac{1}{2}\int_Q|P_2w|^2dtdadx$$
$$+Cs^3\lambda^4\int_q w^2\varphi^3dtdadx \tag{55}$$

where C is a positive constant.
Analogous calculations yield

$$2s\lambda^2\delta^2\int_{q_0}|\nabla\overline{w}|^2\overline{\varphi}dtdadx\le\frac{1}{2}\int_Q|P_2\overline{w}|^2dtdadx$$
$$+Cs^3\lambda^4\int_q\overline{w}^2\overline{\varphi}^3dtdadx. \tag{56}$$

Combining (49),(55) and (56) one gets:

$$\int_Q\left(|P_1w|^2+|P_2w|^2+\left|\overline{P}_1\overline{w}\right|^2+\left|\overline{P}_2\overline{w}\right|^2\right)dtdadx$$
$$+s\lambda^2\delta^2\int_Q\left(\overline{\varphi}|\nabla\overline{w}|^2+\varphi|\nabla w|^2\right)dtdadx$$
$$+s^3\lambda^4\delta^4\int_Q\left(\varphi^3|w|^2+\overline{\varphi}^3|\overline{w}|^2\right)dtdadx$$
$$\le\int_Q\left(e^{-2s\eta}+e^{-2s\overline{\eta}}\right)f^2dtdadx+$$
$$Cs^3\lambda^4\delta^4\int_{q_0}\left(\varphi^3|w|^2+\overline{\varphi}^3|\overline{w}|^2\right)dtdadx.$$

This gives, as $\overline{\varphi}\le\varphi$ and $e^{-2s\overline{\eta}}\le e^{-2s\eta}$:

$$\int_Q\left(|P_1w|^2+|P_2w|^2\right)dtdadx+s\lambda^2\delta^2\int_Q\varphi|\nabla w|^2dtdadx$$
$$+s^3\lambda^4\delta^4\int_Q\varphi^3|w|^2dtdadx$$
$$\le C\int_Q e^{-2s\eta}f^2dtdadx+Cs^3\lambda^4\delta^4\int_q\varphi^3|w|^2dtdadx. \tag{57}$$

Note that $w=e^{-s\eta}p$, so, we get from (57)

$$s^3\lambda^4\int_Q\varphi^3e^{-2s\eta}|p|^2dtdadx\le C\int_Q e^{-2s\eta}f^2dtdadx+$$
$$Cs^3\lambda^4\int_q\varphi^3e^{-2s\eta}|p|^2dtdadx. \tag{58}$$

Recalling (10) and inequality $\varphi\le C\varphi^2$ one derives:

$$s\lambda^2e^{-2s\eta}|\nabla p|^2\le C\left(s\lambda^2|\nabla w|^2+s^2\lambda^2\varphi^3|w|^2\right).$$

Therefore, (57) and the inequality above yield:

$$s\lambda^2\int_Q\varphi e^{-2s\eta}|\nabla p|^2dtdadx$$
$$\le C\int_Q e^{-2s\eta}f^2dtdadx$$
$$+Cs^3\lambda^4\int_q\varphi^3e^{-2s\eta}|p|^2dtdadx.$$

Using the definition of P_1w we see that:

$$\frac{1}{s\varphi}\left|\frac{\partial w}{\partial t}+\frac{\partial w}{\partial a}\right|^2\le C\left(\frac{1}{s\varphi}|P_1w|^2+s\lambda^4\varphi|w|^2+s\lambda^2\varphi|\nabla w|^2\right). \tag{60}$$

Moreover, we have from (9):

$$e^{-s\eta}\left(\frac{\partial p}{\partial t}+\frac{\partial p}{\partial a}\right)=\frac{\partial w}{\partial t}+\frac{\partial w}{\partial a}+s\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)w.$$

This gives

$$e^{-2s\eta}\left|\frac{\partial p}{\partial t}+\frac{\partial p}{\partial a}\right|^2\le C\left(\left|\frac{\partial w}{\partial t}+\frac{\partial w}{\partial a}\right|^2+s^2|w|^2\right). \tag{61}$$

Now, (57); (60) and (61) gives:

$$\int_Q\frac{e^{-2s\eta}}{s\varphi}\left|\frac{\partial p}{\partial t}+\frac{\partial p}{\partial a}\right|^2dtdadx\le C\int_Q e^{-2s\eta}f^2dtdadx$$

$$+Cs^3\lambda^4 \int_q \varphi^3 e^{-2s\eta} |p|^2 \, dtdadx. \qquad (62)$$

We use now the definition of $P_2 w$ and the fact that $\left|\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial a}\right|^2 \leq C\varphi^4$ to find that on the one hand

$$\frac{1}{s\varphi} |\Delta w|^2 \leq C \left(\frac{1}{s\varphi} |P_2 w| + s^3 \lambda^4 \varphi^3 |w|^2 \right).$$

On the other hand we have from (11):

$$e^{-s\eta} |\Delta p|^2 \leq C \left(|\Delta w|^2 + s^2\lambda^2\varphi^2 |\nabla w|^2 + s^3\lambda^4\varphi^4 |w|^2 \right).$$

The two last inequalities together with (57) give:

$$\int_Q \frac{e^{-2s\eta}}{s\varphi} |\Delta p|^2 \, dtdadx \leq C \int_Q e^{-2s\eta} f^2 dtdadx$$

$$+Cs^3\lambda^4 \int_q \varphi^3 e^{-2s\eta} |p|^2 \, dtdadx. \qquad (63)$$

Finally adding (58), (59) (62) (63) one gets (8).

Our goal now, is to derive from the **Proposition 2.2** the following result.

Corollary 2.3 *Suppose that $f = 0$. Let p be a function that verifies (6-7) and*

$$p(t,a,\sigma) = 0 \ \ for \ \ (t,a,\sigma) \in \Sigma_0 \qquad (64)$$

then

$$p(t,a,x) = 0 \ \ a.e \ \ in \ \ Q. \qquad (65)$$

Proof of Corollary 2.3
We suppose that $f = 0$. Let $\sigma_0 \in \Gamma_0$ and $r > 0$ such that the ball $B(\sigma_0, 2r)$ verifies
$B(\sigma_0, 2r) \cap \partial\Omega \subset \Gamma_0$. Set $\hat{\Omega} = \Omega \cup B(\sigma_0, 2r)$ and
$\hat{p}(t,a,x) = \begin{cases} p(t,a,x) & (t,a,x) \in (0,T) \times (0,A) \times \Omega \\ 0 & (t,a,x) \in (0,T) \times (0,A) \times \ B(\sigma_0, 2r) \end{cases}$
Then standard device gives that $\hat{p}$ verifies:

$$\frac{\partial \hat{p}}{\partial t} + \frac{\partial \hat{p}}{\partial a} - \Delta\hat{p} + \mu\hat{p} = 0 \ \ in \ \ (0,T) \times (0,A) \times \hat{\Omega} \qquad (66)$$

and

$$\frac{\partial \hat{p}}{\partial \nu} = 0 \ \ on \ \ (0,T) \times (0,A) \times \partial\hat{\Omega}. \qquad (67)$$

Let $\sigma_1 \in \hat{\Omega} - \overline{\Omega}$. There exists r_0 such that $B(\sigma_1, r_0) \subset B(\sigma_0, 2r)$.
As $\hat{p}(t,a,x) = 0$ a.e in $(0,T) \times (0,A) \times B(\sigma_1, r_0)$ then, the **Proposition 2.1** with $\omega = B(\sigma_1, r_0)$ implies that $\hat{p}(t,a,x) = 0 \ \ in \ \ (0,T) \times (0,A) \times \hat{\Omega}$. Therefore, $p \equiv 0$.

Remark 2.4 *One can prove easily **theorem 2.1** using a version of the Hahn Banach theorem like in [1]. More precisely, let g^0 be a vector of the orthogonal set of $R^0 = \left\{ z(0,.,.), z \ \ solves \ \ (5); \ \ v \in L^2(\Sigma) \right\}$. Let us multiply (5) by $p(g^0)$ solution of (68) below with $p^0 = g^0$. Integrating the result over Q gives*

$$\int_{Q_A} g^0(a,x) z(0,a,x) dadx = \int_{\Sigma_0} v(t,a,\sigma) p(t,a,\sigma) dtdadx.$$

Therefore, as

$$\int_{Q_A} g^0(a,x) z(0,a,x) dadx = 0,$$

one obtains:

$$\int_{\Sigma_0} v(t,a,\sigma) p(t,a,\sigma) dtdadx = 0,$$

for all $v \in L^2((0,T) \times (0,A) \times \partial\Omega)$.
*This gives via the **Corollary 2.3** that $p \equiv 0$. So, it follows that $g^0 = 0$. Therefore, the Hahn Banach theorem yields that R^0 is dense in $L^2((0,T) \times (0,A) \times \Omega)$.*

We prefer the proof below, since it is very usefull in practice.

Proof of theorem 2.1 Let $g \in L^2((0,A) \times \Omega)$ and $\epsilon > 0$. We have to prove that there exists a control $\tilde{v}$ such that the associated solution $\tilde{z}$ of (5) verifies $\|\tilde{z}(0,.,.) - g\| \leq \epsilon$.
If $\|g\| \leq \epsilon$ one can take $\tilde{v} = 0$ to get that $\tilde{z} = 0$.
Therefore, it follows that

$$\|z(0,.,) - g\| = \|g\| \leq \epsilon.$$

Suppose now that $\|g\| > \epsilon$. For a given $p^0 \in L^2((0,A) \times \Omega)$, we consider the following system

$$\begin{cases} \frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} - \Delta p + \mu p = 0 \text{ in } (0,T) \times (0,A) \times \Omega \\ \frac{\partial p}{\partial \nu}(t,a,\sigma) = 0 \text{ on } (0,T) \times (0,A) \times \partial\Omega \\ p(0,a,x) = p^0(a,x) \text{ in } (0,A) \times \Omega \\ p(t,0,x) = \int_0^A \beta p da \text{ in } (0,T) \times \Omega \end{cases} \qquad (68)$$

we take $v = p_{|\Sigma}$ and consider the new controlled system:

$$\begin{cases} -\frac{\partial z}{\partial t} - \frac{\partial z}{\partial a} - \Delta z + \mu_0 p = \beta z(t,0,x) \text{ in } (0,T) \times (0,A) \times \Omega \\ \frac{\partial z}{\partial \nu}(t,a,\sigma) = p(t,a,\sigma) 1_{\Sigma 0}(\sigma) \text{ on } (0,T) \times (0,A) \times \partial\Omega \\ z(T,a,x) = 0 \text{ in } (0,A) \times \Omega \\ z(t,A,x) = 0 \text{ in } (0,T) \times \Omega \end{cases}. \qquad (69)$$

We recall that the solution of (69) is taken in the following sense: $z \in L^2((0,T) \times (0,A); H^1(\Omega))$ and $\forall \theta \in L^2((0,T) \times (0,A); H^1(\Omega))$ we have

$$-\int_0^A \int_0^T \left\langle \frac{\partial z}{\partial t} + \frac{\partial z}{\partial a}, \theta \right\rangle_{(H^1)', H^1} dtda + \int_Q (\nabla z . \nabla \theta + \mu z \theta) \, dtdadx$$

$$-\int_{\Sigma_0} p\theta dtdad\sigma = \int_Q \beta z(t,0,x) \theta dtdadx$$

and

$$\left\{ \begin{array}{l} z(T,a,x)=0 \text{ in } (0,A)\times\Omega \\ z(t,A,x)=0 \text{ in } (0,T)\times\Omega \end{array} \right. .$$

One can use the method of [14] to prove that (69) admits a unique solution.

Let us denote by $p(p_0)$ and $z(p)$ respectively, the solutions of (68) and (69). We consider now the functional

$$J_g(p^0)=\frac{1}{2}\int_{\Sigma_0} p^2 dadtdx+\epsilon \left\|p^0\right\|_{L^2(Q_A)}-\int_{Q_A} gp^0 dadx. \quad (70)$$

Note that, the function $p^0 \mapsto p$ is continuous from $L^2(Q_A)$ into $L^2\left((0,T)\times(0,A);H^1(\Omega)\right)$. So that, the function $p^0 \mapsto p_{|\Sigma_0}$ is continuous from $L^2(Q_A)$ into $L^2(\Sigma)$. Therefore, J_g is continuous. In addition, the functional J_g is trivially convex. Moreover, J_g is coercive. More precisely, we have:

$$\liminf_{\|p^0\|_{L^2(Q_A)}\to+\infty} \frac{J(p^0)}{\|p^0\|_{L^2(Q_A)}} \geq \epsilon. \quad (71)$$

The proof of (71) follows what was proposed in [5] for the heat equation. Let us consider the sequence p_n^0 in $L^2((0,A)\times\Omega)$ such that $\|p_n^0\|_{L^2(Q_A)}\to\infty$ as $n\to\infty$. We note $\hat{p}_n^0=\frac{p_n^0}{\|p_n^0\|_{L^2(Q_A)}}$ and $\hat{p}$ the associated solution of (68).
Then:

$$\frac{J_g(p_n^0)}{\|p_n^0\|_{L^2(Q_A)}}=\frac{\|p_n^0\|_{L^2(Q_A)}}{2}\int_{\Sigma_0}\hat{p}_n^2 dadtd\sigma+\epsilon-\int_{Q_A} g\hat{p}_n^0 dadx. \quad (72)$$

Using the fact that

$$\int_{\Sigma_0}\hat{p}_n^2 dtdad\sigma \geq 0, \quad (73)$$

we obtain

$$\liminf_{n\to\infty}\int_{\Sigma_0}\hat{p}_n^2 dtdad\sigma>0 \ \ or \ \ \liminf_{n\to\infty}\int_{\Sigma_0}\hat{p}_n^2 dtdad\sigma=0. \quad (74)$$

In the first case, we get obviously

$$\liminf_{n\to\infty}\frac{J(p_n^0)}{\|p_n^0\|_{L^2(Q_A)}}=+\infty. \quad (75)$$

This gives obviously (71).
In the secund case, we extract a subsequence still denoted $\hat{p}_n^0$ such that

$$\int_{\Sigma_0}\hat{p}_n^2 dtdad\sigma \to 0 \quad \text{as} \quad n\to+\infty,$$

$$\hat{p}_n^0 \rightharpoonup \hat{p}^0 \ \text{ weakly in } \ L^2(0,A)\times\Omega)$$

and

$$\hat{p}_n \rightharpoonup \hat{p}=p(\hat{p}^0)\,\text{weakly in } \ L^2\left((0,T)\times(0,A)\times\Omega\right).$$

Therefore, we get that $\hat{p}=p(\hat{p}^0)$ verifies

$$\hat{p}_{|\Sigma_0}=0 \ \ a.e \ \ \Sigma_0.$$

Using now **Corollary 2.3** we obtain that

$$\hat{p}=0 \ \text{ a.e in } \ Q.$$

This gives $\hat{p}^0=0$ a.e in Q_A. Then, equality (72) yields (71).

Recalling that J_g is continuous, convex and coercive, it follows that it admits a minimizer denoted $\tilde{p}^0$. More precisely, there exists $\tilde{p}^0 \in L^2((0,A)\times\Omega)$ such that $J_g(\tilde{p}^0))=\min_{p^0\in L^2(Q_A)} J_g(p^0)$
Using the fact that $J_g(0)=0$ and $\|g\|<\epsilon$, we infer that there exists $p^0\in L^2((0,A)\times\Omega)$ such that $J_g(p^0)<0$. This implies that $\tilde{p}^0\neq 0$.

Next, for $p^0\in L^2((0,A)\times\Omega)$ setting $p=p(p^0)$ and $\tilde{p}=p(\tilde{p}^0)$ one has:

$$\lim_{\tau\to 0}\frac{J_g(\tilde{p}^0+\tau p^0)-J_g(\tilde{p}^0)}{\tau}=\int_{\Sigma_0} p\tilde{p}dtdad\sigma+\frac{\epsilon}{\|\tilde{p}^0\|}\int_{Q_A}\tilde{p}^0p^0 dadx$$

$$-\int_{Q_A} gp^0 dadx=0.$$

This gives,

$$\int_{\Sigma_0} p\tilde{p}dtdad\sigma=-\frac{\epsilon}{\|\tilde{p}^0\|}\int_{Q_A}\tilde{p}^0p^0 dadx+\int_{Q_A} gp^0 dadx. \quad (76)$$

Let us multiply now equation (69) with $\tilde{p}$ instead of p by $p=p(p^0)$. An integration by parts of the result over Q yields:

$$\int_{Q_A}\tilde{z}(0,a,x)p^0(a,x)\,dadx-\int_{\Sigma_0} p\tilde{p}dtda\sigma=0. \quad (77)$$

This equality and (76) yield for all p^0

$$\int_{Q_A}\left(\tilde{z}(0,a,x)-g(a,x)+\frac{\epsilon}{\|\tilde{p}^0\|}\tilde{p}^0(a,x)\right)p^0 dadx=0 \quad (78)$$

Therefore, we get:

$$\tilde{z}(0,a,x)-g(a,x)=\frac{\epsilon}{\|\tilde{p}^0\|}\tilde{p}^0 \ \ a.e \ \ in \ \ (0,A)\times\Omega. \quad (79)$$

Consequently:

$$\|\tilde{z}(0,.,.)-g\|\leq\epsilon. \quad (80)$$

3 The method for recovering the initial distribution

Let us consider the function $\hat{y}$ solution of system (1-4). In order to work with bounded coefficients we make the following change of variables : $y = \pi(a)\hat{y}$; $\beta = \pi^{-1}(a)\hat{\beta}$; $y_0 = \pi(a)\hat{y}_0$.
As $\hat{y}_0$ is unknown, y_0 is also unknown, but y solves the system:

$$\begin{cases} \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - \Delta y + \mu p = 0 \text{ in } (0,T)\times(0,A)\times\Omega \\ \frac{\partial y}{\partial \nu}(t,a,\sigma) = 0 \text{ on } (0,T)\times(0,A)\times\partial\Omega \\ y(0,a,x) = y_0(a,x) \text{ in } (0,A)\times\Omega \\ y(t,0,x) = \int_0^A \beta y da \text{ in } (0,T)\times\Omega \end{cases} \quad (81)$$

Let $\epsilon > 0$ be small enough and consider $N > 1$, an integer. Let (g_k) be a orthonormal basis of $L^2((0,A)\times\Omega)$. We set $y_{0,N} = \sum_{j=1}^N \theta_k g_k$ with $\theta_k = \int_{Q_A} y_0(a,x) g_k(a,x) dadx$. Clearly $y_{0,N}$ converges strongly to y_0.
For all $k > 0$, there exists a function $\tilde{p}_k^0 \in L^2(Q_A)$ and consequently v_k such that the associated solution z_k of (5) verifies:

$$z_k(0,a,x) = g_k(a,x) + \frac{\epsilon}{N\left\|\tilde{p_k^0}\right\|}\tilde{p}_k^0(a,x)$$

Let

$$\hat{\theta}_k = \int_{\Sigma_0} y_{|\Sigma_0} v_k dtdad\sigma \text{ and } \hat{y}_{0,N,\epsilon} = \sum_{j=1}^N \hat{\theta}_k g_k.$$

As one can compute $\hat{\theta}_k$ then we may consider $\hat{y}_{0,N,\epsilon}$ as a known datum . Then the main result is:

Theorem 3.1 *Assume that assumptions $A_1 - A_4$ are fulfilled. Then we have:*
(i) the following estimate:

$$\|\hat{y}_{0,N,\epsilon} - y_{0,N}\| \le \epsilon \|y_0\| \quad (82)$$

(ii) $\hat{y}_{0,N,\epsilon}$ converges strongly to $y_{0,N}$ in $L^2((0,A)\times\Omega)$ as ϵ tends to 0.

(iii) Let y^i be the solution of (81) associated to the initial distribution y_0^i, $i = 1,2$. Then, $y^1_{|\Sigma_0} \le y^2_{|\Sigma_0}$ a.e on $\Sigma_0 \Rightarrow y_0^1 \le y_0^2$ a.e in Q

Remark 3.2 *Proposition iii) seems to be natural. It means that the greater the value on Σ_0, the higher the initial distribution. We note that one cannot derive the above property by the use of the traditional Tykonov regularisation method. In fact, our method can give some properties of the unknown datum.*

Proof of Theorem 3.1 (i) For all k, multiplying (5) by z_k associated solution of (69) to v_k, integrating the result by parts over Q and using the definition of $\hat{\theta}_k$ lead to

$$\hat{\theta}_k = \int_{Q_A} y_0(a,x) z_k(0,a,x) dadx.$$

Therefore,

$$\hat{\theta}_k = \int_{Q_A} g_k(a,x) y_0(a,x) dadx + \frac{\epsilon}{N\left\|\tilde{p_k^0}\right\|}\int_{Q_A} \tilde{p}_k^0(a,x) y_0(a,x) dadx \quad (83)$$

Consequently

$$\hat{\theta}_k - \theta_k = \frac{\epsilon}{N\left\|\tilde{p_k^0}\right\|}\int_{Q_A} \tilde{p}_k^0(a,x) y_0(a,x) dadx. \quad (84)$$

This implies that:

$$\left|\hat{\theta}_k - \theta_k\right| \le \|y_0\| \frac{\epsilon}{N}. \quad (85)$$

On the other hand we have:

$$\hat{y}_{0,N,\epsilon} - y_{0,N} = \sum_{k=1}^N \left(\hat{\theta}_k - \theta_k\right) g_k.$$

Owing to (85) we get:

$$\|\hat{y}_{0,N,\epsilon} - y_{0,N}\| \le \epsilon \|y_0\|. \quad (86)$$

Let us prove ii).
As ϵ is independent of $\|y_0\|$ the second proposition follows easily.

We now perform the proof of (iii).
Using the fact that $y^1_{|\Sigma_0} \le y^2_{|\Sigma_0}$ a.e on Σ_0, setting $\hat{\theta}_k^j = \int_{\Sigma_0} y^j_{|\Sigma_0} v_k dtdad\sigma$ it follows that:

$$\left|\hat{\theta}_k^1\right| \le \left|\hat{\theta}_k^2\right|.$$

Therefore, we derive that

$$\left\|\hat{y}^1_{0,N,\epsilon}\right\| \le \left\|\hat{y}^2_{0,N,\epsilon}\right\|. \quad (86)$$

Using (ii) we get now that:

$$\left\|y^1_{0,N}\right\| \le \left\|y^2_{0,N}\right\|.$$

As $y^j_{0,N}$ converges strongly to y_0^j in $L^2(Q)$ the last inequality implies

$$\left\|y_0^1\right\| \le \left\|y_0^2\right\|.$$

Now, since y_0^j is non negative we infer that: $y_0^1 \le y_0^2$ a.e in Q_A. This achieves the proof.

4 Concluding Remark

In this paper, we have studied first, an approximate controllability problem by means a new Carleman inequality. Afterwards, our goal was to show, how one can use this approximate controllability result in the study of an inverse problem. The method outlined in this paper, gives in fact an approximation of the unknown datum, y_0. This approximation is somewhat incomplete in order, it becomes more precise when $\|y_0\|$ is known.
The traditional way, that consists to minimize the functional
$J(z) = \frac{1}{2}\int_{\Sigma_0} |y_z - y_{obs}|^2 \, dtdad\sigma + \alpha \int_{Q_A} z^2(a,x)dadx$
where, y_z is the solution of (1-4) with z instead of y_0 and y_{obs} the observed value, gives in fact the function that is close to y_0 and moreover, minimizes the cost: $\alpha \int_{Q_A} z^2(a,x)dadx$. Therefore, this traditional method is not better than ours in term of precision.
Nevertheless, in practice, our method is not trivial, since it requires the minimization of many functionals. We are planning to check numerically our method and to do a comparative study with the traditional method elsewhere.

References

[1] B.E Ainseba,"Exact and Approximate Controllability of the Age and Space Population Structured Problem", *J. Math. Anal.App*,275 (2002) pp 562-574.

[2] S. Anita, *Analysis and Control of Age Dependent Population Dynamics*, Kluwer Acad. Publ 2000.

[3] V. Arsenin, A. Tikonov, *Solutions of Ill-posed problems* John Wiley and Sons, 1977.

[4] F.X. Le Dimet, O. Talagrand, *variational algorithms for analysis and assimilation of meteorological observations: theoritical aspects.* Tellus, 38A pp 97-110 1986

[5] C. Fabre, J-P Puel ,E. Zuazua, "Approximate Controllability of the semilinear heat equation", *Proc. of the royal Soc of Edinburg*, 125A, 1995 pp 31-61.

[6] A. Fursikov, O. Imanuvilov , *Controllability of Evolution equations*, Lectures Notes series 34, RIM, Seoul, National University Korea, 1996.

[7] M. Gyllenberg, A. Osipov, L. Paivarinta, "The Inverse Problem of Linear Age Structured Population Dynamics",*J.E.E*, 2 (2002) pp 223-239.

[8] M. Langlais, "A nonlinear problem in age-dependent population diffusion", *SIAM J. Math. anal*, 16,1985, pp 510-529.

[9] Z. Li, K. Zheng, "An Inverse problem in Parabolic Equation",*E.J.D.E,Conf 01*, 1997, pp 193-199.

[10] JL Lions, E. Magenes, *Problèmes aux limites non homogènes et application*, vol. 1, Dunod, Paris 1968

[11] D. Maclaughlin, *Inverse problems*, editor SIAM AMS proc, 14, AMS Providence,

[12] A. Osser,J.-P. Puel," Some extension of approximate controllability results to inverse problem" Esaim Proc, Vol.2, 1997, pp 133-143.

[13] J.P Puel, "A non standart Approach to Data Assimilation",*C. R. Math. Acad. Sci*, Paris, Ser I 335 (2) (2002) pp161-166.

[14] A. Ouedraogo, O. Traore," Sur un problème de dynamique des populations", *Imhotep, J. Afr Math P and Appl* vol (2003) pp 15-23

[15] W. Rundell, "Determining the Death Rate for Age-Structured Population from Census Data", *SIAM, j. Applied Mathematics*, Vol 53, No 6.(Dec., 1993),pp 1731-1746.

[16] O. Traore, "A non Standard Approach to Data Assimilation problem for a Linear Population Dynamics", (submitted)