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Saturation Effects and Cyclical Herd Behavior

Patrick L. Leoni*

Abstract— **We develop an original game-theoretical explanation of cyclical herd behaviors, where cyclicity occurs as a strategic equilibrium phenomena triggered by saturation effects associated with mass adoption of the same action. In our model, we show that herd behaviors occur almost surely, and so do cycles if saturation effects are present. Moreover, we show that the length of the transition phase between two consecutive herd behavior is at most the time needed for the saturation effect to disappear.**

Keywords: herd behavior, cycles, saturation effects.

1 Introduction

Many economic situations have in common that actions previously chosen by a large group of agents can have a direct influence on future decision-makers. Such situations are often described as *herd behavior*, where the information implicitly carried in previous choices about the value of some particular actions may overwhelm any private information or preferences of subsequent agents (see Banerjee [1] and Bikhchandani et al. [2] for a formalization).

Of particular interest are situations where herd behaviors are cyclical; i.e., a particular action is chosen for some time by a large group of agents in circumstances observationally equivalent to a herd behavior, then this action is abandoned for a new one to possibly reappear later. For instance in financial markets, periods of bull markets are usually followed by other periods of bear markets on basis that can be largely construed as speculative behavior driven by observations of previous trades.

Many explanations have been given to this phenomena. For instance, Kirman [4] argues that such patterns can be explained by private and noisy communication between agents about previously observed payoffs of some actions. As a positive message is believed to be reliable, agents tend to adopt a previously chosen action, whereas with noisy communication reliability declines and the action may be abandoned (see also Bikhchandani et al. [2] for other explanations).

In this paper, we develop an original game-theoretical explanation to the cyclical pattern of herd behaviors. Our approach is based on the idea that cyclicity of herd behavior is driven by publicly observable *saturation effects* caused by successive adoptions of the same action. We develop a game-theoretical model where herd behaviors occur almost surely, and repeated adoptions of the same action trigger future agents to abandon this action as long as the saturation is present. Cyclicity then appears as a strategic equilibrium phenomena, where an action triggering a herd behavior will disappear at some point because of saturation effects, to possibly reappear later. Moreover, we show that the length of the transition phase between two consecutive herd behaviors is at most the time needed for the saturation effect to disappear. Examples of saturation effects are well-known in the art industry or fashion industry.

2 The model

In this section, a formal description of the model is given. It is derived from that in Banerjee citeba. Our model generalizes this last reference in that we allow for a finite set of actions to have positive payoffs among a continuum, instead of one action only in this last reference. From a technical standpoint, we relax two of the three tie-breaking rules that Banerjee imposes. This framework is minimal to generate the results described in the Introduction.

Time is discrete and continues forever. In every period, a new player appears and selects an action from the set $[0,1]$. There are two types of actions: any action $a \in A = \{a_1, ..., a_n\}(\subset [0,1])$ has payoff $d_a > 0$, whereas any other action in $[0,1] \setminus A$ has a payoff of zero. We reorder A so that if $i > j$ then $d_i \leq d_j$. An player does not know the set A nor the payoffs, but she knows its cardinality n.

Every player receives a signal about A, which can take two forms. With probability $\alpha > 0$, the signal is informative and takes the form of an action chosen from A. When receiving an informative signal, the action a_i is received with probability $p_i > 0$ so that $\sum_i p_i = 1$. We assume that signals satisfy the following property.

Assumption 1. *The sequence $(p_i)_{i=1,...,n}$ is strictly decreasing.*

That is, when receiving a signal we assume that a player is more likely to know which action has the highest payoff.

*This project was supported by the grant "Complex Markets" at the GREQAM. Address: National University of Ireland at Maynooth, Department of Economics, Maynooth Co. Kildare, Ireland. E-mail: patrick.leoni@nuim.ie, Tel: +353 1708 6420.

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With probability $1-\alpha$, the signal is uninformative and takes the form of a variable $\xi \notin [0,1]$. The nature of the signal is private information.

Every player can observe all the previous choices of actions. Consider the player living in period $t > 1$, for every sequence of observed actions $(a^1, ..., a^{t-1})$ we associate the *information set* $\{(a^1, s^1, ..., a^{t-1}, s^{t-1})|$ $s^i \in [0,1] \cup \{\xi\} \forall i\}$. In other words, a player knows which actions have been chosen, but she is uncertain about the signals previously received. The information set of the first player is defined to be the null set. A *strategy* for the player living in period t assigns to every information set in period t and to every received sinal a (possibly randomized) action.

Every player has a common prior belief about the signals previously received at every information set. This prior belief is such that ξ (the uninformative signal) is received with probability $1-\alpha$, and with probability α the informative signal is drawn from a uniform distribution on $[0,1]$.

We say that an action $a \in [0,1]$ has a *saturation effect* if there exists $N_a > 1$ such that when $t + N_a$ players have consecutively chosen this action after any period t then the payoff of a is 0 with probability $\pi_{t'} > 0$, for every $t' \in \{t + N_a, ..., t + S\}$ and for some integer $S > N_a$. We assume that the sequence $(\pi_{t'})_{t'}$ is strictly decreasing and sums up to 1. After period $t + S$, the action regains its original payoff with probability $\phi_{t'} > 0$, for every $t' \in \{t + S + 1, ..., t + V\}$ and for some integer $V > S$, if it is not chosen in any such period t' and remains 0 otherwise. We assume that the finite sequence $(\phi_{t'})_{t'}$ is strictly increasing and sums up to 1.

The motivation for the notion of saturation effect is that, when choosing repeatedly too often the same action, subsequent agents may find it worthless with decreasing probability over time. This assumption can be justified as negative externalities occurring when the action is chosen too often, as in the example given in the Introduction. We keep the possibility of a decrease in payoff exogenous to simplify the exposition, our basic insight remaining the same when endogenous. The second aspect of the definition, namely that the action recovers its original payoff with probability that increases with the number of times the action is not chosen, captures the idea that the negative externality caused by the repeated use of the same action disappears over time as it gets temporarily abandoned.

We assume that every player is risk-neutral and maximizes the expected payoff of her action, where the expectation is based on observed actions and the signal received. Risk-neutrality is not central to our analysis. Similar qualitative results obtain with risk-aversion instead, this issue is omitted to simplify the analysis.

We must add a tie-breaking decision rule inherited from Banerjee [1] to carry out our analysis.

Assumption 2. *If a player does not have a signal, and if all the previous players (if any) have chosen $a = 0$, then this player will choose $a = 0$.*

The action $a = 0$ can be construed as an exit option, chosen by a player who has no information whatsoever about A. This assumption is consistent with the prior belief of the player, who is indifferent between any action in $[0,1]$ by assumption. Instead of randomizing, we impose this rational choice as the only outcome in this case.

Every player updates her belief in a Bayesian manner according to available information. The structure of the game is common knowledge to every player. The remainder of the paper is devoted to analyzing the Bayesian Nash Equilibria of this game. It is easy to see that there exists a continuum of such equilibria in this game; for instance, when a player believes that some actions are equally likely to yield the highest payoff then any randomization among those actions can be justified as an equilibrium strategy. This multiplicity of equilibria will not affect our qualitative analysis.

3 Cyclical Herd Behavior

In this section, we present our results on herd behavior. We first describe our notion of cyclical herd behavior triggered by a particular action. We then show that, in every equilibrium and for almost every equilibrium play path, there exists action in A will trigger a cyclical herd behavior. Finally, we analyze the length of the transition phases between two consecutive and distinct herd behavior.

We say that the action *a triggers a herd behavior* at the information set h if, given available information at this information set, the action a is chosen after receiving every possible signal. We also say that *a triggers a cyclical herd behavior* along the infinite play path s if there exist a sequence of information sets $(h_t)_{t \geq 0}$ along s such that none of those information sets are consecutive, and a triggers a herd behavior at every h_t.

We next state our main result on cyclical herd behavior. The aim is to know how often cyclical herd behavior occur, and which set of actions can potentially trigger this phenomena. Central to the next result is that every action with a positive payoff has a saturation effect.

Theorem 3. *Assume that every action in A has a saturation effect. For every equilibrium and for almost every equilibrium play path, there exists an action in A that triggers a cyclical herd behavior.*

The above result states that, in every equilibrium and for all but a set of measure zero of equilibrium play paths, an action with positive payoff will trigger a cyclical herd behavior. Implicit in the above result is that two distinct actions can trigger cyclical herd behavior along the same equilibrium play path, this can occur during the periods where the first action triggering the cyclical herd behavior exhibits a saturation effect. However, the action with the highest payoff will not necessarily triggers a cyclical herd behavior. Indeed, one can easily see from the proof of Theorem 1 that the first action in A to trigger a herd behavior will also trigger a cyclical herd behavior. This action is chosen by the early players as a function of their received informative signals, which can correspond to any action in A. Thus, saturation effects cannot eliminate the social inefficiency that is often seen in herd behavior.

4 Conclusion

We have developed a game-theoretical framework where herd behaviors occur almost surely, and where cyclicity of such behaviors is driven by our notion of saturation effect. That is, our model has the property that agents behaving strategically condition their actions both on the reliability of private information and public signals, and also on a socially perceived saturation associated with large adoption of the same actions. Our notion of saturation effects allows for empirical testing in term of occurrence and time length, since may examples can be found in the art industry and fashion industry for instance.

A Technical proofs

We now prove the result stated earlier. We first present the well-known Glivenko-Cantelli' Theorem, which will be used throughout. The proof of this result is given in Fristedt and Gray [3] p. 192 and extensions. It mostly states that, when dealing with identical i.i.d. drawings, the empirical distribution function converges almost surely to the original distribution function.

Theorem 4. (Glivenko-Cantelli)
Let $(Y_n)_{n\geq 0}$ be an i.i.d. sequence of real-valued random variables with common distribution function F. For every $y \in \boldsymbol{R}$, let $\boldsymbol{1}_y$ be the indicator function of the interval $(-\infty, y]$, and define the random variables

$$F_n(y) = \frac{1}{n}\sum_{k=1}^{n} \boldsymbol{1}_y(Y_k). \tag{1}$$

Then for every y, $F_n(y) \to F(y)$ almost everywhere.

When applied to our setting, the above result roughly states that, for almost every play path, the frequency of signals received by the players will converge in distribution to the original way nature draws. This result will help us in proving that some pathological infinite play paths have measure 0 with respect to the choices of the nature.

A.1 Proof of Theorem 3

We start our proof by presenting two simple technical lemmas, central to determining optimal choices given available information.

Our first lemma states that receiving a signal is a better information than observing a sequence of 0.

Lemma 5. *Assume that, along a path, every player before period t has chosen 0. If Player t receives an informative signal, then she will follow it.*

Proof. From an ex-ante standpoint, Player t believes that the event $0 \in A$ has probability 0. Since all the previous players have chosen 0, by Assumption 2 Player t assigns probability 1 to the event that all previous players had no informative signal. Thus, Player t believes with probability 1 that she is the only one to have an informative signal, and it is thus optimal to follow it. The proof is now complete.

Our second lemma states that observing two identical actions is more informative about the best action than one signal alone.

Lemma 6. *Fix any equilibrium, and assume that the same action (different from 0) has been chosen by the first and second player. Then this action triggers a herd behavior next period.*

Proof. Consider any equilibrium, and let H be the event that the first two players have chosen the same action a and player 3 has received the signal a^3. Clearly, if $a^3 = a$ then Player 3 will choose a. Otherwise, we compute the probability of the event $[a = a_i]$ for every i, conditional on H, to derive our result. We have that

$$P([a = a_i]|H) = \alpha^3 p_i^2(1-p_i) + \alpha^2(1-\alpha)p_i(1-p_i). \tag{2}$$

Moreover, we have that

$$P([a^3 = a_i]|H) = \alpha^2 p_i(1-p_i)(1-\alpha). \tag{3}$$

It is easy to see that, from (2) and (3) together with Assumption 1, the expected payoff of choosing a is greater than that of a^3, and thus Player 3 will choose to ignore her signal. The proof is complete.

The previous lemma states that observing two identical actions will offset any private information. The intuition of the result is central to our analysis. The first player to choose this action must have received the corresponding signal, which as good as any signal received by Player 3 ex-ante. Moreover, with strictly positive probability the second player has chosen this action because he also received the corresponding signal. Thus, receiving twice the same signal with strictly positive probability makes the corresponding action more likely to have a higher payoff than any private signal Player 3 can receive.

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With all the previous results, we can prove Theorem 3. We start our proof by showing that, for almost every equilibrium play path, some action in A will trigger a herd behavior. The remainder of the proof is based on the method used to derive this property. From now on, we will refer to Player i as i (for every i) to simplify the exposition.

Consider 1, if she has the uninformative signal she chooses 0 by Assumption 2, and otherwise she chooses her own signal. Consider now 2, if she has the uninformative signal she follows the same choice as 1, and otherwise one must distinguish two cases. If 2's signal matches 1's action, then 2 follows her signal. If 2's signal is different from 1's action then she randomizes between 1's action and her own signal (2 does not randomize if 1's action is 0, she chooses her own signal instead by Lemma 5).

We now analyze the decision problem of 3. This player can observe four different class of past actions: case a) both previous players have chosen 0; case b) 1 has chosen 0 and 2 has chosen $a^2 > 0$; case c) 1 and 2 have chosen the same action; case d) two distinct actions have been chosen that are not 0. We next examine those four cases.

In case a), if 3 has the uninformative signal she chooses 0 by Assumption 2, and otherwise she chooses her own signal by Lemma 5. In case b), if 3 has the uninformative signal then she will choose a^2, and otherwise she randomizes between her own signal and a^2. In case c), 3 always chooses the action chosen twice by Lemma 6. In case d), if 3 has the uninformative signal then she randomizes between the two previously chosen actions, and otherwise one must distinguish two subcases. If 3's signal is identical to one of the previously chosen actions then she follows her own signal (the idea is the same as in Lemma 6), and otherwise she randomizes between the two actions and her own signal.

From the above analysis, it is easy to derive that the first action (different from 0) to be chosen twice triggers a herd behavior at the information set immediately following the second choice, unless the signal received in this period matches an already chosen and different action. By Theorem 4, for almost every equilibrium play path every element in A will be drawn by nature at least twice. Moreover, by Assumption 1 the set of play paths where two different signals are always sent one after the other also has probability 0. Thus, by an argument similar to that in Lemma 6, for almost every path an action in A will be chosen often enough to rule out the above case. Thus, for almost every equilibrium play path, there exists an action in A that will trigger a herd behavior.

By an argument similar to that in Lemma 6, once an action in A has trigger a herd behavior at a particular information set, it will also trigger a herd behavior at the following information set. Since every action in A has a saturation effect, it follows that once $a \in A$ has triggered a herd behavior for the first time, it will also trigger $N_a - 1$ consecutive herd behavior until it becomes common knowledge that its payoff may become 0 for the next $S + V$ consecutive periods with positive probability.

However, after those $S + V$ periods the information structure of every subsequent player about which action has the highest payoff is identical to that of the players who followed the previous herd behavior. Thus, any optimal action for those players must also be optimal for the subsequent players, and a new cycle of herd behavior start. This situation will be repeated infinitely often, proving the result. The proof is now complete.

References

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