New Exact traveling wave solutions of the (3+1) dimensional Kadomtsev-Petviashvili (KP) equation

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Abstract—**The repeated homogeneous balance method is used to construct new exact traveling wave solutions of the (3+1) dimensional Kadomtsev-Petviashvili (KP) equation, in which the homogeneous balance method is applied to solve the Riccati equation and the reduced nonlinear ordinary differential equation, respectively. Many new exact traveling wave solutions are successfully obtained. This method is straightforward and concise, and it can be also applied to other nonlinear evolution equations.**

The investigation of the exact traveling wave solutions of nonlinear evolutions equations plays an important role in the study of nonlinear physical phenomena. For example, the wave phenomena observed in fluid dynamics, elastic media, optical fibers, etc.

In recent years Wong et al. presented a useful homogeneous balance (HB) method [1-3] for finding exact solutions of a given nonlinear partial differential equations. Fan [4]used HB method to search for Backlund transformation and similarity reduction of nonlinear partial differential equations. Also, he showed that there is a close connection among the HB method, Wiess, Tabor, Carnevale(WTC)method and Clarkson, Kruskal(CK)method.

In this paper, we use the HB method to solve the Riccati equation $\phi' = \alpha\phi^2 + \beta$ and the reduced nonlinear ordinary differential equation for the (3+1) KP equation, respectively. It makes the HB method use more extensively.

For the (3+1) KP equation [5-7]

$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} - 3u_{zz} = 0, \quad (1)$$

Let us consider the traveling wave solutions

$$u(x, y, t) = u(\zeta), \qquad \zeta = kx + ly + mz + nt + d, \quad (2)$$

where k, l, m, n and d are constants.

Substituting (2) into (1), then (1)is reduced to the following nonlinear ordinary differential equation

$$k^4 u'''' + (3(m^2 - l^2) + kn)u'' + 6k^2(uu')' = 0. \quad (3)$$

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We now seek the solutions of Eq.(3) in the form

$$u = \sum_{i=0}^{m} q_i \phi^i, \quad (4)$$

where $q_i$are constants to be determined later and ϕ satisfy the Riccati equation

$$\phi' = \alpha\phi^2 + \beta, \quad (5)$$

where α, β are constants. It is easy to show that $m = 2$ if Balancing u'' with uu'. Therefore use the ansatz

$$u = q_0 + q_1\phi + q_2\phi^2, \quad (6)$$

Substituting Eq.(5),and(6) into Eq.(3),and equating the coefficients of like powers of $\phi^i (i = 0, 1, 2, 3, 4, 5, 6)$ to zero yields the system of algebraic equations to q_0, q_1, q_2, k, l, m and n

$$\begin{gathered}
16k^4\alpha q_2\beta^3 + 2(3k^2q_1^2 - 3l^2q_2 - 3m^2q_2 + kn_2 + 6k^2q_0q_2)\beta^2 = 0,\\
4(9q_2 + 4k^2\alpha^2)k^2q_1\beta^2 - 2(3(l^2 + m^2) - kn - 6k^2q_0)q_1\alpha\beta = 0,\\
4(9q_2 + 34k^2\alpha^2)k^2q_2\beta^2 + 8\gamma\alpha\beta = 0,\\
4(27q_2 + 10k^2\alpha^2)k^2q_1\alpha\beta - 2(3(l^2 + m^2) - kn - 6k^2q_0)q_1\alpha^2 = 0,\\
48(2q_2 + 5k^2\alpha^2)k^2q_2\alpha\beta + 6\gamma\alpha^2 = 0,\\
24(3q_2 + k^2\alpha^2)k^2q_1\alpha^2 = 0,\\
60(q_2 + 2k^2\alpha^2)k^2q_2\alpha^2 = 0, \quad (7)
\end{gathered}$$

where $\gamma = 3k^2q_1^2 - 3l^2q_2 - 3m^2q_2 + knq_2 + 6k^2q_0q_2$. for which, with the aid of "Mathematica", we get the following solution

$$q_0 = \frac{3(l^2 + m^2) - k(n + 8k^3\alpha\beta)}{6k^2}, q_1 = 0, q_2 = -2k^2\alpha^2. \quad (8)$$

For the Riccati Eq.(5), we can solve it by using the HB method as follows

(I) Let $\phi = \Sigma_{i=0}^{m} b_i \tanh^i\zeta$. Balancing $\phi'$with ϕ^2leads to

$$\phi = b_0 + b_1 \tanh\zeta. \quad (9)$$

Substituting Eq.(9)into Eq.(5),we obtain the following solution of Eq.(5)

$$\phi = \beta\tanh\zeta = -\frac{1}{\alpha}\tanh\zeta, \qquad \alpha\beta = -1. \quad (10)$$

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From Eq.(6), (8)and (10), we get the following traveling wave solutions of (3+1) KP equation (1)

$$u(x,y,t) = \frac{3(l^2+m^2) - k(n+8k^3\alpha\beta)}{6k^2} - 2k^2\tanh^2\zeta. \quad (11)$$

Similarly, let $\phi = \Sigma_{i=0}^{m} b_i \coth^i\zeta$, then we obtain the following traveling wave solutions of (3+1) KP equation (1)

$$u(x,y,t) = \frac{3(l^2+m^2) - k(n+8k^3\alpha\beta)}{6k^2} - 2k^2\coth^2\zeta. \quad (12)$$

Where $\zeta = kx + ly + mz + nt + d$.

(II) From [8], when $\alpha = 1$, the Riccati equation 5)has the following solutions

$$\phi = \begin{cases} -\sqrt{-\beta}\tanh(\sqrt{-\beta}\zeta), & \beta < 0, \\ -\frac{1}{\zeta}, & \beta = 0, \\ \sqrt{\beta}\tan(\sqrt{\beta}\zeta). & \beta > 0. \end{cases} \quad (13)$$

From (6),(8) and (13), we have the following traveling wave solutions of (3+1) KP equation (1).

When $\beta < 0$, **we have**

$$u(x,y,t) = \frac{3(l^2+m^2) - k(n+8k^3\beta)}{6k^2} + 2k^2\beta\tanh^2(\sqrt{-\beta}\zeta)). \quad (14)$$

When $\beta = 0$, **we have**

$$u(x,y,t) = \frac{3(l^2+m^2) - k(n+8k^3\beta)}{6k^2} - \frac{2k^2}{a\zeta^2}. \quad (15)$$

When $\beta > 0$, **we have**

$$u(x,y,t) = \frac{3(l^2+m^2) - k(n+8k^3\beta)}{6k^2} + 2k^2\beta\tan^2(\sqrt{-\beta}\zeta)). \quad (16)$$

Where $\zeta = kx + ly + mz + nt + d$.

(III) We suppose that the Riccati equation (5) have the following solutions of the form

$$\phi = A_0 + \sum_{i=1}^{m}(A_i f^i + B_i f^{i-1} g), \quad (17)$$

with

$$f = \frac{1}{\cosh\zeta + r}, \qquad g = \frac{\sinh\zeta}{\cosh\zeta + r},$$

which satisfy

$$f'(\zeta) = -f(\zeta)g(\zeta), \quad g'(\zeta) = 1 - g^2(\zeta) - rf(\zeta),$$
$$g^2(\zeta) = 1 - 2rf(\zeta) + (r^2-1)f^2(\zeta).$$

Balancing ϕ' with ϕ^2 leads to

$$\phi = A_0 + A_1 F + B_1 g. \quad (18)$$

Substituting Eq.(18)into (5), collecting the coefficient of the same power $f^i g^j$ $(i = 0,1,2; j = 0,1)$ and setting each of the obtained coefficients to zero yield the following set of algebra equations

$$\alpha A_1^2 + \alpha(r^2-1)B_1^2 + (r^2-1)B_1 = 0,$$
$$2\alpha A_1 B_1 + A_1 = 0,$$
$$2\alpha A_0 A_1 - 2\alpha r B_1^2 - r B_1 = 0,$$
$$2\alpha A_0 B_1 = 0,$$
$$\alpha A_0^2 + \alpha B_1^2 + \beta = 0, \quad (19)$$

which have solutions

$$A_0 = 0, \quad A_1 = \pm\sqrt{\frac{(r^2-1)}{4\alpha^2}}, \quad B_1 = -\frac{1}{2\alpha}. \quad (20)$$

where $4\alpha\beta = -1$. **From Eqs.(17-20), we have**

$$\phi = \frac{-1}{2\alpha}\left(\frac{\sinh\zeta \mp \sqrt{(r^2-1)}}{\cosh\zeta + r}\right) \quad (21)$$

From Eqs.(6),(8)and (21), we obtain

$$u(x,y,t) = \frac{1}{6k^2}(3(l^2+m^2) - k(n+8k^3\alpha\beta)$$
$$-3k^4\left(\frac{\pm\sqrt{r^2-1} - \sinh(\zeta)}{r + \cosh(\zeta)}\right)^2), \quad (22)$$

where

$$\zeta = kx + ly + mz + nt + d.$$

(IV) We take ϕ in the Riccati equation(5) being of the form

$$\phi = e^{p_1\zeta}\rho(z) + p_4(\zeta), \quad (23)$$

where

$$z = e^{p_2\zeta} + p_3, \quad (24)$$

where p_1, p_2 and p_3 are constants to be determined.

Substituting (23),(24) into (5), we have

$$p_2 e^{(p_1+p_2)\zeta}\rho' - \alpha e^{2p_1\zeta}\rho^2 + (p_1 - 2\alpha p_4)e^{p_1\zeta}\rho + p_4' - \alpha p_4^2 - \beta = 0. \quad (25)$$

Setting $p_1 + p_2 = 2p_1$, we get $p_1 = p_2$, if we assume that $p_4 = \frac{p_1}{2\alpha}$ and $\beta = -\frac{p_1^2}{4\alpha}$, then Eq.(25) becomes

$$p_2\rho' - \alpha\rho^2 = 0. \quad (26)$$

By solving Eq.(26), we have

$$\rho = -\frac{p_1}{\alpha z} = -\frac{p_1}{\alpha e^{p_1\zeta} + p_3}. \quad (27)$$

Substituting (27) and $p_4 = \frac{p_1}{2a}$ into (23), we have

$$\phi = -\frac{p_1 e^{p_1\zeta}}{\alpha(e^{p_1\zeta} + p_3)} + \frac{p_1}{2\alpha}. \quad (28)$$

If $p_3 = 1$ in (28), we get

$$\phi = -\frac{p_1}{2\alpha}\tanh(\frac{1}{2}p_1\zeta). \quad (29)$$

If $p_3 = -1$ in (28), we get

$$\phi = -\frac{p_1}{2\alpha}\coth(\frac{1}{2}p_1\zeta). \quad (30)$$

From (6),(8) and (28), we obtain the following traveling wave solutions of (3+1) KP equation (1)

$$u(x,y,t) = \frac{1}{6k^2}(3(l^2+m^2) - k(n+8k^3\alpha\beta) - 3p_1^2k^4\left(\frac{2e^{p_1\zeta} - 1}{e^{p_1\zeta} + p_3}\right)^2). \quad (31)$$

When $p_3 = 1$, we have from (29)

$$u(x,y,t) = \frac{1}{6k^2}(3(l^2+m^2)-k(n+8k^3\alpha\beta)-3p_1^2k^4\tanh^2(\frac{p_1}{2}\zeta)). \quad (32)$$

Clearly, (11) is the special case of (32) with $p_1 = 2$. When $p_3 = -1$, we have from (30)

$$u(x,y,t) = \frac{1}{6k^2}(3(l^2+m^2)-k(n+8k^3\alpha\beta)-3p_1^2k^4\coth^2(\frac{p_1}{2}\zeta)). \quad (33)$$

Where $\zeta = kx + ly + mz + nt + d$. Clearly, (12) is the special case of (33) with $p_1 = 2$.

(V) We suppose that the Riccati equation (5) have the following solutions of the form

$$\phi = A_0 + \sum_{i=1}^{m}\sinh^{i-1}(A_i\sinh\omega + B_i\cosh\omega),$$

where $d\omega/d\zeta = \sinh\omega$ or $d\omega/d\zeta = \cosh\omega$. It is easy to find that $m = 1$ by balancing ϕ' and ϕ^2. So we choose

$$\phi = A_0 + A_1\sinh\omega + B_1\cosh\omega, \quad (34)$$

when $d\omega/d\zeta = \sinh\omega$, we substitute (34)and $d\omega/d\zeta = \sinh\omega$, into (5) and set the coefficient of $\sinh^i\omega\cosh^j\omega(i = 0,1,2; j = 0,1)$ to zero. A set of algebraic equations is obtained as follows

$$\alpha A_0^2 + \alpha B_1^2 + \beta = 0,$$

$$2\alpha A_0A_1 = 0,$$
$$\alpha A_1^2 + \alpha B_1^2 = B_1$$
$$2\alpha A_0B_1 = 0,$$
$$2\alpha A_1B_1 = A_1, \quad (35)$$

for which, we have the following solutions

$$A_0 = 0, \quad A_1 = 0, \quad B_1 = \frac{1}{\alpha}, \quad (36)$$

where $\beta = \frac{-1}{\alpha}$, and

$$A_0 = 0, \quad A_1 = \pm\frac{1}{2\alpha}, \quad B_1 = \frac{1}{2\alpha}, \quad (37)$$

where $\beta = -\frac{1}{4\alpha}$.

To $d\omega/d\zeta = \sinh\omega$, we have

$$\sinh\omega = -csch\zeta, \quad \cosh\omega = -\coth\zeta. \quad (38)$$

From (35)-(38), we obtain

$$\phi = -\frac{coth\zeta}{\alpha}, \quad (39)$$

where $\beta = -\frac{1}{\alpha}$, and

$$\phi = \frac{\coth\zeta \pm csch\zeta}{2\alpha}, \quad (40)$$

where $\beta = -\frac{1}{4\alpha}$.

Clearly, (39)is the special case of (31)with $p_1 = 2$.

From (6),(8),(39) and (40), we get the exact traveling wave solutions of (3+1) KP equation (1) in the following form

$$u(x,y,t) = \frac{3(l^2+m^2)-k(n+8k^3\alpha\beta)}{6k^2} - 2k^2\coth^2\zeta. \quad (41)$$

Which is identical with (12).

$$u(x,y,t) = \frac{1}{6k^2}(3(l^2+m^2)-k(n+8k^3\alpha\beta)-3k^4(coth\zeta \pm csch\zeta)^2). \quad (42)$$

Where $\zeta = kx + ly + mz + nt + d$.

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