On the convergence of finite steps iterative sequences with mean errors for asymptotically quasi-nonexpansive mappings

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Abstract— **The purpose of this paper is to study the convergence of a new finite steps iterative sequence with mean errors to a common fixed point for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces. The results presented in this paper extend and generalize some results in the literature.**

Keywords: finite steps iterative sequences with mean errors, common fixed point, asymptotically quasi-nonexpansive mappings, convergence, Banach space.

1 Introduction and Preliminaries

Let C be a nonempty subset of a real Banach space E, and let T be a self-mapping of C. T is called asymptotically quasi-nonexpansive if there exists $k_n \in [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$, such that $\|T^n x - p\| \leq k_n \|x-p\|$, $\forall x \in C$, $\forall p \in F(T)$ ($F(T)$ denotes the set of fixed points of T).

T is called asymptotically nonexpansive if $\|T^n x - T^n y\| \leq k_n \|x-y\|$, $\forall x, y \in C$. T is called quasi-nonexpansive if $\|Tx - p\| \leq \|x-p\|$, $\forall x \in C$, $\forall p \in F(T)$. T is called nonexpansive if $\|Tx - Ty\| \leq \|x-y\|$, $\forall x, y \in C$.

From the above definitions, it follows that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, and an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive. But the converse does not hold.

Petryshyn and Williamson [1], in 1973, proved a sufficient and necessary condition for the Picard iterative sequences and mann iterative sequences to converge to a fixed point of quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [2] extended the results of [1] and gave the sufficient and necessary condition for Ishikawa iterative sequences to converge to fixed points for quasi-nonexpansive mappings. Recently, Liu [3-5] extended the above results and proved some sufficiency and necessary conditions for Ishikawa iterative sequences and Ishikawa iterative sequences with errors of asymptotically quasi-nonexpansive mappings to converge to fixed points. Xu and Noor [6] studied some necessary conditions for three-step iterative sequences of asymptotically nonexpansive mappings to converge to fixed points. Cho, Zhou and Guo [7] researched some necessary conditions for three-step iterative sequences with errors of asymptotically nonexpansive mappings to converge to fixed points. Quan et. al. [8] studied the weak and strong convergence of finite steps iterative sequences with mean errors to a common fixed point for a finite family of asymptotically nonexpansive mappings.

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we now introduce a new iterative sequence as follows:

Definition 1.1. Let $T_1, T_2, ..., T_N : C \to C$ be any N mappings, and $x_1 \in C$ be a given point. Then sequence$\{x_n\}$ generated by

$$\begin{cases} x_{n+1} = \lambda_0 y_{n0} + \lambda_1 y_{n1} + \lambda_2 y_{n2} + ... + \lambda_{N-1} y_{nN-1}, \\ y_{n0} = (1 - a_{n1} - b_{n1})x_n + a_{n1} T_1^n y_{n1} + b_{n1} u_{n1}, \\ y_{n1} = (1 - a_{n2} - b_{n2})x_n + a_{n2} T_2^n y_{n2} + b_{n2} u_{n2}, \\ ... \\ y_{nN-2} = (1 - a_{nN-1} - b_{nN-1})x_n + a_{nN-1} T_{N-1}^n y_{nN-1} \\ \qquad + b_{nN-1} u_{nN-1}, \\ y_{nN-1} = (1 - a_{nN} - b_{nN})x_n + a_{nN} T_N^n x_n + b_{nN} u_{nN}. \end{cases} \tag{1.1}$$

is called the N-step iterative sequence with mean errors of $T_1, T_2, ..., T_N$, where $\{u_{ni}\}_{n=1}^{\infty}, i = 1, ..., N$ are N sequences in C, $\{a_{ni}\}_{n=1}^{\infty}, \{b_{ni}\}_{n=1}^{\infty}, i = 1, ..., N$ are N sequences in [0,1], $\lambda_i, i = 0, ..., N-1$ are N numbers in [0,1] satisfying the following conditions:

$$\begin{cases} \sum_{i=0}^{N-1} \lambda_i = 1, \\ a_{ni} + b_{ni} \leq 1, i = 1, ..., N \\ \sum_{n=1}^{\infty} b_{ni} \leq \infty, i = 1, ..., N \end{cases} \tag{1.2}$$

Remark 1.1. (1) If $\lambda_i = 0$, $i = 1, ..., N-1$, then the finite-step iterative sequence generated by (1.1) and (1.2) becomes that introduced by Quan et. al. [8] and generated by

$$\begin{cases} x_{n+1} = (1 - a_{n1} - b_{n1})x_n + a_{n1} T_1^n y_{n1} + b_{n1} u_{n1}, \\ y_{n1} = (1 - a_{n2} - b_{n2})x_n + a_{n2} T_2^n y_{n2} + b_{n2} u_{n2}, \\ ... \\ y_{nN-2} = (1 - a_{nN-1} - b_{nN-1})x_n + a_{nN-1} T_{N-1}^n y_{nN-1} \\ \qquad + b_{nN-1} u_{nN-1}, \\ y_{nN-1} = (1 - a_{nN} - b_{nN})x_n + a_{nN} T_N^n x_n + b_{nN} u_{nN}. \end{cases} \tag{1.3}$$

Where $\{u_{ni}\}_{n=1}^{\infty}, i = 1, ..., N$ are N sequences in C,

$\{a_{ni}\}_{n=1}^{\infty}, \{b_{ni}\}_{n=1}^{\infty}, i = 1, ..., N$ are N sequences in [0,1] satisfying the following conditions:

$$\begin{cases} a_{ni} + b_{ni} \leq 1, i = 1, ..., N \\ \sum_{n=1}^{\infty} b_{ni} \leq \infty, i = 1, ..., N \end{cases} \tag{1.4}$$

Hence, the N-step iterative sequence with mean errors of $T_1, T_2, ..., T_N$ generated by (1.1) and (1.2) is more general than that generated by (1.3) and (1.4).

(2) Let $S, T, R : C \to C$ be three mappings, $\{u_n\}, \{v_n\}, \{w_n\}$ be three given sequences in C, and $x_1 \in C$ be a given point. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\eta_n\}, \{\xi_n\}$ be sequences in [0,1] and $\lambda_0, \lambda_1, \lambda_2$ be three numbers in [0,1] satisfying the following conditions:

$$\begin{cases} \sum_{i=0}^{2} \lambda_i = 1, \\ \alpha_n + \gamma_n \leq 1, \beta_n + \delta_n \leq 1, \eta_n + \xi_n \leq 1, n \geq 1 \\ \sum_{n=1}^{\infty} \gamma_n \leq \infty, \sum_{n=1}^{\infty} \delta_n \leq \infty, \sum_{n=1}^{\infty} \xi_n \leq \infty, \end{cases}$$

Then the sequence $\{x_n\}$ generated by

$$\begin{cases} x_{n+1} = \lambda_0 p_n + \lambda_1 y_n + \lambda_2 z_n, n \geq 1 \\ p_n = (1 - \alpha_n - \gamma_n)x_n + \alpha_n S^n y_n + \gamma_n u_n, n \geq 1 \\ y_n = (1 - \beta_n - \delta_n)x_n + \beta_n T^n z_n + \delta_n v_n, n \geq 1 \\ z_n = (1 - \eta_n - \xi_n)x_n + \eta_n R^n x_n + \xi_n w_n, n \geq 1 \end{cases} \tag{1.5}$$

is called the three step iterative sequence with mean errors of S, T, R.

If $\lambda_0 = 1$, then the sequence generated by (1.5) becomes the three-step iterative sequence in [7, 8]. Hence the iterative sequence generated by (1.5) also contains Picard, Mann and Ishikawa iterative sequences in [1-6] as special cases.

The purpose of this paper is to study the weak and strong convergence of finite-step iterative sequences with mean errors $\{x_n\}$ generated by (1.1) and (1.2) to a common fixed point for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces. The results presented in this paper extend and improve some results in the literature.

In order to prove the main results of this paper, we need the following lemmas:

Lemma 1.1 [9]. Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{\delta_n\}_{n=1}^{\infty}$ be non-negative real sequences satisfying the following inequality

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 1.2 [8]. Let $T_1, T_2, ..., T_N : C \to C$ be N asymptotically quasi-nonexpansive mappings with $F(\Gamma) = \cap_{i=1}^{N} F(T_i) \neq \emptyset$, then there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $k_n \to 1$ such that for $i = 1, 2, ..., N$,

$$\|T_i^n x - p\| \leq k_n \|x - p\|, \forall x \in C, \forall p \in F(\Gamma), n \geq 1. \tag{1.6}$$

Proof. It follows from asymptotically quasi-nonexpansiveness of $T_1, T_2, ..., T_N : C \to C$ that for each $i = 1, 2, ...N$, there exists $\{k_i^n\} \subset [1, +\infty)$, $k_i^n \to 1$ and

$$\|T_i^n x - p\| \leq k_i^n \|x - p\|, \forall x \in C, \forall p \in F(T_i).$$

Since $F(\Gamma) = \cap_{i=1}^{N} F(T_i) \neq \emptyset$, we get that for each $i = 1, 2, ...N$,

$$\|T_i^n x - p\| \leq k_i^n \|x - p\|, \forall x \in C, \forall p \in F(\Gamma).$$

Taking $k_n = max\{k_1^n, k_2^n, ..., k_N^n\}$, then $\{k_n\} \subset [1, +\infty)$, $k_n \to 1$ and

$$\|T_i^n x - p\| \leq k_n \|x - p\|, \forall x \in C, \forall p \in F(\Gamma).$$

This completes the proof.

2 The Main Results

Theorem 2.1. Let E be a normed linear space and C be a nonempty bounded convex subset of E. Let $T_1, T_2, ..., T_N : C \to C$ be N asymptotically quasi-nonexpansive mappings with $F(\Gamma) = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{k_n\} \subset [1, \infty)$ be the sequence defined by (1.6) and the sequence $\{x_n\}$ be defined by (1.1) and (1.2). If $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, then

(i) there exists $M \geq 0$, such that

$$\|x_{n+1} - p\| \leq k_n^N \|x_n - p\| + Q_n, \forall p \in F(\Gamma), n \geq 1, \tag{2.1}$$

where $Q_n = \sum_{j=0}^{N-1} 2Mk_n^{N-1-j} b_{nN-j}$.

(ii)$\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F(\Gamma)$.

(iii) $\lim_{n\to\infty} d(x_n, F(\Gamma))$ exists, where $d(x, F(\Gamma))$ denotes the distance x to the set $F(\Gamma)$.

(iv) there exists $L = e^{N \sum_{j=1}^{\infty}(k_j - 1)} > 0$, such that

$$\|x_{n+m} - p\| \leq L\|x_n - p\| + L \sum_{j=n}^{n+m-1} Q_j, \forall p \in F(\Gamma), \forall m, n \geq 1. \tag{2.2}$$

Proof. (i) Since C is bounded, let $M = \sup_{x\in C} \|x\|$. It follows from (1.1) and (1.2) that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\lambda_0 y_{n0} + \lambda_1 y_{n1} + \lambda_2 y_{n2} + ... + \lambda_{N-1} y_{nN-1} - p\| \\ &\leq \lambda_0 \|y_{n0} - p\| + \lambda_1 \|y_{n0} - p\| + ... + \lambda_{N-1} \|y_{nN-1} - p\| \\ &\leq \max\{\|y_{n0} - p\|, \|y_{n1} - p\|, ..., \|y_{nN-1} - p\|\} \end{aligned} \tag{2.3}$$

$$\begin{aligned} \|y_{nN-1} - p\| &= \|(1 - a_{nN} - b_{nN})x_n + a_{nN} T_N^n x_n + b_{nN} u_{nN} - p\| \\ &\leq (1 - a_{nN} - b_{nN})\|x_n - p\| + a_{nN}\|T_N^n x_n - p\| + b_{nN}\|u_{nN} - p\| \end{aligned}$$

$\leq (1-a_{nN}-b_{nN})\|x_n-p\|+a_{nN}k_n\|x_n-p\|+b_{nN}\|u_{nN}-p\|$

$\leq k_n\|x_n-p\|+2Mb_{nN}$ (2.4)

$\|y_{nN-2}-p\| \leq (1-a_{nN-1}-b_{nN-1})\|x_n-p\|$
$+ a_{nN-1}\|T_{N-1}^n y_{nN-1}-p\| + b_{nN-1}\|u_{nN-1}-p\|$

$\leq (1-a_{nN-1}-b_{nN-1})\|x_n-p\| + a_{nN-1}k_n\|y_{nN-1}-p\|$
$+ b_{nN-1}\|u_{nN}-p\|$

$\leq (1-a_{nN-1}-b_{nN-1}+k_n^2 a_{nN-1})\|x_n-p\|+2Mb_{nN-1}+2Mb_{nN}k_n$

$\leq k_n^2\|x_n-p\|+2Mb_{nN-1}+2Mb_{nN}k_n$ (2.5)

...

$\|y_{n2}-p\| \leq k_n^{N-2}\|x_n-p\|+2Mb_{n3}+2Mk_nb_{n4}$
$+...+2Mk_n^{N-3}b_{nN}.$ (2.6)

$\|y_{n1}-p\| \leq (1-a_{n2}-b_{n2})\|x_n-p\| + a_{n2}k_n\|y_{n2}-p\|$
$+ b_{n2}\|u_{n2}-p\|$

$\leq (1-a_{n2}-b_{n2}+k_n^{N-1}a_{n2})\|x_n-p\|+2Mb_{n2}+2Mk_nb_{n3}$
$+2Mk_n^2b_{n4}+...+2Mk_n^{N-2}b_{nN}$

$\leq k_n^{N-1}\|x_n-p\|+2Mb_{n2}+2Mk_nb_{n3}+2Mk_n^2b_{n4}$
$+...+2Mk_n^{N-2}b_{nN}$ (2.7)

$\|y_{n0}-p\|$

$\leq (1-a_{n1}-b_{n1})\|x_n-p\|+a_{n1}k_n\|y_{n1}-p\|+b_{n1}\|u_{n1}-p\|$

$\leq (1-a_{n1}-b_{n1}+k_n^N a_{n1})\|x_n-p\|+2Mb_{n1}+2Mk_nb_{n2}$
$+2Mk_n^2b_{n3}+2Mk_n^3b_{n4}+...+2Mk_n^{N-1}b_{nN}$

$\leq k_n^N\|x_n-p\|+2Mb_{n1}+2Mk_nb_{n2}+2Mk_n^2b_{n3}+2Mk_n^3b_{n4}$
$+...+2Mk_n^{N-1}b_{nN}$ (2.8)

Substituting (2.4), (2.5), (2.6), (2.7) and (2.8) into (2.3), it can be obtained that

$\|x_{n+1}-p\|$

$\leq k_n^N\|x_n-p\|+2Mb_{n1}+2Mk_nb_{n2}+2Mk_n^2b_{n3}$
$+2Mk_n^3b_{n4}+...+2Mk_n^{N-1}b_{nN}$

$= k_n^N\|x_n-p\|+Q_n,$

where $Q_n=\sum_{j=0}^{N-1}2Mk_n^{N-1-j}b_{nN-j}$. This completes the proof of (i).

(ii) By (2.1), we know that

$$\|x_{n+1}-p\| \leq [1+(k_n^N-1)]\|x_n-p\|+Q_n.$$

Notice that $\sum_{n=1}^{\infty}(k_n-1)<\infty$ implies that $\sum_{n=1}^{\infty}(k_{n-1}^i-1)<\infty$, $i=1,2,...,N$. By (1.2), we also know that $\sum_{n=1}^{\infty}Q_n<\infty$. Thus, it follows from Lemma 1.1 that $\lim_{n\to\infty}\|x_n-p\|$ exists. This completes the proof of (ii).

(iii) Also by (2.1), we have

$$d(x_{n+1},F(\Gamma)) \leq [1+(k_n^N-1)]d(x_n,F(\Gamma))+Q_n.$$

By Lemma 1.1, we get $\lim_{n\to\infty}d(x_n,F(\Gamma))$ exists. This completes the proof of (iii).

(iv) From (2.1), it can be obtained that

$\|x_{n+m}-p\| \leq k_{n+m-1}^N\|x_{n+m-1}-p\|+Q_{n+m-1}$

$\leq e^{N(k_{n+m-1}-1)}\|x_{n+m-1}-p\|+Q_{n+m-1}$

$\leq e^{N(k_{n+m-1}-1)+N(k_{n+m-2}-1)}\|x_{n+m-2}-p\|$
$+Q_{n+m-1}+e^{N(k_{n+m-1}-1)}Q_{n+m-2}$

$\leq e^{N(k_{n+m-1}-1)+N(k_{n+m-2}-1)}\|x_{n+m-2}-p\|$
$+e^{N(k_{n+m-1}-1)}(Q_{n+m-2}+Q_{n+m-1})$

$\leq ...$

$\leq e^{N\sum_{j=n}^{n+m-1}(k_j-1)}\|x_n-p\|+e^{N\sum_{j=n}^{n+m-1}(k_j-1)}\sum_{j=n}^{n+m-1}Q_j$

By $\sum_{j=1}^{\infty}(k_j-1)<\infty$, it can be obtained that $L=e^{N\sum_{j=1}^{\infty}(k_j-1)}<\infty$. Therefore,

$$\|x_{n+m}-p\| \leq L\|x_n-p\|+L\sum_{j=n}^{n+m-1}Q_j.$$

This completes the proof of (iv).

Remark 2.1 Theorem 2.1 unifies and extends Lemma 1 in [4], Lemma 1 in [5], Lemma 1.2 in [7] and Lemma 5 in [8].

Theorem 2.2. Let E be a Banach space and C be a nonempty bounded convex subset of E. Let $T_1,T_2,...,T_N : C \to C$ be N asymptotically quasi-nonexpansive mappings with $F(\Gamma)=\cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{k_n\} \subset [1,\infty)$ be the sequence defined by (1.6) satisfies $\sum_{n=1}^{\infty}(k_n-1)<\infty$. Then the iterative sequence $\{x_n\}$ defined by (1.1) and (1.2) converges to a common fixed point if and only if $\lim_{n\to\infty}\inf d(x_n,F(\Gamma))=0$, where $d(y,A)$ denotes the distance of y to set A, i.e., $d(y,A)=\inf_{x\in A}\|y-x\|$.

Proof. The necessity is obvious. It is only need to prove the sufficiency.

By $\lim_{n\to\infty}\inf d(x_n,F(\Gamma))=0$ and Theorem 2.1 (iii), it can be obtained that

$$\lim_{n\to\infty} d(x_n,F(\Gamma))=0. \quad (2.9)$$

From the proof of Theorem 2.1 (ii), we know that

$$\sum_{n=1}^{\infty} Q_n < \infty. \tag{2.10}$$

By (2.9) and (2.10), we know that for all $\varepsilon > 0$, there is $N_0 > 0$ such that

$$d(x_n, F(\Gamma)) < \frac{\varepsilon}{2}, \forall n \geq N_0. \tag{2.11}$$

$$\sum_{n=N_0}^{\infty} Q_n < \varepsilon, \forall n \geq N_0. \tag{2.12}$$

From (2.11), it is easy to know that there exists $p_0 \in F(\Gamma)$ such that

$$\|x_{N_0} - p_0\| < \varepsilon. \tag{2.13}$$

From (2.2), we have

$$\|x_n - p_0\| \leq L\|x_{N_0} - p_0\| + L\sum_{j=N_0}^{n-1} Q_j, \forall n > N_0. \tag{2.14}$$

It follows from Lemma (2.2), (2.12), (2.13) and (2.14) that when $n \geq N_0$,

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p_0\| + \|x_n - p_0\|$$

$$\leq L\|x_n - p_0\| + L\sum_{j=n}^{n+m-1} Q_j + \|x_n - p_0\|$$

$$= (L+1)\|x_n - p_0\| + L\sum_{j=n}^{n+m-1} Q_j$$

$$= (L+1)\{L\|x_{N_0} - p_0\| + L\sum_{j=N_0}^{n-1} Q_j\} + L\sum_{j=n}^{n+m-1} Q_j$$

$$< (L+1)L\varepsilon + L\varepsilon = (L+2)L\varepsilon, \forall m \geq 1.$$

This implies that $\{x_n\}$ is a Cauchy sequence in C, hence $\lim_{n\to\infty} x_n$ exists. Let $x_n \to p \in C$. Then for any $\epsilon > 0$, there is a natural number N_1 such that

$$\|x_n - p\| < \epsilon, \forall n > N_1. \tag{2.15}$$

Since $\lim_{n\to\infty} \inf d(x_n, F(\Gamma)) = 0$, there exists $N_2 \geq N_1$ such that

$$d(x_n, F(\Gamma)) < \epsilon, \forall n \geq N_2. \tag{2.16}$$

And hence, there exists $p_1 \in F(\Gamma)$, such that

$$\|x_{N_2} - p_1\| \leq \epsilon. \tag{2.17}$$

It follows (2.15) and (2.17) that for $i = 1, 2, ..., N$,

$$\|T_i p - p\| \leq \|T_i p - p_1\| + \|p_1 - x_{N2}\| + \|x_{N2} - p\|$$

$$\leq k_1\|p - p_1\| + \|p_1 - x_{N2}\| + \|x_{N2} - p\|$$

$$\leq k_1\|p - x_{N2}\| + k_1\|x_{N2} - p_1\| + \|p_1 - x_{N2}\| + \|x_{N2} - p\|$$

$$\leq 2(k_1 + 1)\epsilon$$

By the arbitrariness of ϵ, it can be obtained that $T_i p = p$ for all $i = 1, 2, ..., N$. And hence p is a common fixed point of $T_1, T_2, ..., T_N$. This completes the proof.

Using same method, it is easy to obtain

Corollary 2.3. Let E be a Banach space and C be a nonempty bounded convex subset of E. Let $T_1, T_2, ..., T_N : C \to C$ be N quasi-nonexpansive mappings with $F(\Gamma) = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Then the iterative sequence $\{x_n\}$ defined by (1.1) and (1.2) converges to a common fixed point if and only if $\lim_{n\to\infty} \inf d(x_n, F(\Gamma)) = 0$.

By Theorem 2.2, it is easy to obtain

Corollary 2.4. Let E be a Banach space and C be a nonempty bounded convex subset of E. Let $T_1, T_2, ..., T_N : C \to C$ be N asymptotically quasi-nonexpansive mappings with $F(\Gamma) = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Then the iterative sequence $\{x_n\}$ defined by (1.1) and (1.2) converges to a common fixed point if and only if there exists some infinite subsequence of $\{x_n\}$ which converges to p.

Remark 2.2. (i) Theorem 2.2 extends and improves Theorem 6 in [8], Theorem 1 in [4], Theorem 1 and Corollary 2 in [3]. Corollary 2.3 generalizes Corollary 1 in [3] and Theorem 2 in [4]. And Corollary 2.4 extends Theorem 3 in [4].

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