LMMSE Estimation Based on Counting Observations

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Abstract— **The problem of estimating the intensity process of a doubly stochastic Poisson process is analyzed. Using the knowledge of the first and second-order moments of the intensity process, a recursive linear minimum mean-square error estimate is designed. Moreover, an efficient procedure for the computation of its associated error covariance is shown. The proposed solution becomes an alternative approach to the Kalman filter which is applicable under the only structural assumption that the intensity process to be estimated has a finite-dimensional covariance function.**

Keywords: doubly stochastic Poisson processes, linear minimum mean-square error estimate

1 Introduction

This paper is focused on the problem of estimating the intensity process from counting observations of doubly stochastic Poisson processes (DSPP). These processes, introduced in [1], are Poisson processes whose rate is modulated by a second stochastic process, known as the intensity process. In the recent engineering literature, this problem has been of great interest since estimates of the intensity process are required in expressions for the counting and time statistics for DSPP which arise naturally in many practical situations of such diverse areas as optical communication systems [2], quantitative financial [3], network theory [4], among others [5, 6].

Thus, suppose that $\{N(t), t \geq t_0\}$ is a DSPP with a stochastic intensity process $\{\lambda(t), t \geq t_0\}$ whose mean $E[\lambda(t)]$ and covariance function $R_\lambda(t,s)$ are known. We consider that the observation interval $[t_0, t_f)$ is partitioned into m disjoint intervals according to the times $t_0 < t_1 < t_2 < \ldots < t_m = t_f$, and the number of points occurring in each subinterval is observed. Denote $\{N_1, N_2, \ldots, N_m\}$, with $N_i = N(t_i) - N(t_{i-1})$[1], these counting observations.

Observe that, the mean function $E[N_i]$ and the covariance function $R_N(t_i, t_j)$ associated with the observations N_i are given by the expressions [2]

$$\begin{aligned} E[N_i] &= \int_{t_{i-1}}^{t_i} E[\lambda(\sigma)]\, d\sigma \\ R_N(t_i,t_j) &= \int_{t_{j-1}}^{t_j}\int_{t_{i-1}}^{t_i} R_\lambda(\sigma,\tau) d\sigma d\tau + E[N_i]\delta_{ij} \end{aligned} \tag{1}$$

where δ_{ij} is the Kronecker delta function.

Moreover, the cross-covariance function between the intensity process $\lambda(t)$ and the observation N_i, $R_{\lambda N}(t, t_i)$, is of the form

$$R_{\lambda N}(t,t_i) = \int_{t_{i-1}}^{t_i} R_\lambda(t,\sigma) d\sigma \tag{2}$$

Next, our purpose is to derive a linear estimate $\hat{\lambda}(t)$ of the intensity process $\lambda(t)$ from the set of counting observations $\{N_1, N_2, \ldots, N_m\}$, with $t \geq t_m$. Specifically, we seek estimators which are optimal in the sense of minimizing the mean-square error

$$P(t) = E\left[\left\{\lambda(t) - \hat{\lambda}(t)\right\}^2\right] \tag{3}$$

Under this error criterion it is well known that the best solution, the linear minimum mean-square error (LMMSE) estimate, can be expressed as a linear functional of the data of the form [2]

$$\hat{\lambda}(t) = E[\lambda(t)] + \sum_{i=1}^{m} h(t,t_i)\{N_i - E[N_i]\}, \quad t \geq t_m \tag{4}$$

where the impulse-response function $h(t,\cdot)$, must satisfy the equation

$$R_{\lambda N}(t,t_j) = \sum_{i=1}^{m} h(t,t_i) R_N(t_i,t_j) \tag{5}$$

for $t_1 \leq t_j \leq t_m$ and $t \geq t_m$.

As a consequence, the LMMSE estimation problem is theoretically determined from the solution of the equation (5) which only involves the covariance functions (1) and (2), that is, only requires the knowledge of the first and second-order moments of the intensity process. However,

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[1] N_i represents the points occurred in the observed doubly stochastic Poisson process during the interval $[t_{i-1}, t_i)$.

from the practical point of view an efficient algorithm for its computation is desirable. In this framework, different techniques have been applied to obtain recursive LMMSE estimation procedures for the intensity process of an observed DSPP (see, for example, [2] and [7]). In particular, the most extensively applied algorithm is the popular Kalman filter which requires that the intensity process to be estimated satisfies a state-space model. Although this condition is valid for a wide class of processes, there is a great number of practical situations where no linear dynamic model for the intensity process of a DSPP is available.

Therefore, in this paper we propose an alternative approach which is applicable under less restrictive structural conditions on the intensity process and leads to an efficient algorithm for the LMMSE estimator of the intensity process of a DSPP. In fact, we only assume that the intensity process has a finite-dimensional covariance function of the form

$$R_\lambda(t,s) = \mathbf{a}'(t)\mathbf{b}(s) \tag{6}$$

where $\mathbf{a}(\cdot)$ and $\mathbf{b}(\cdot)$ are vector-valued functions of dimension q.

Note that, this is not a very restrictive hypothesis since the kernel form of covariance (6) is easy to obtain (e.g. via inverse Fourier) and is suitable for expressing both stationary and non-stationary processes. Then, this type of covariance appears naturally in many general applications [2].

Next, under the hypotheses established in this section, efficient procedures for computing the LMMSE estimator (4) and its associated minimum mean-square error (3) are developed in the next section.

2 LMMSE Estimation Algorithm

The main objective now is the design of an efficient algorithm for the LMMSE estimate $\hat{\lambda}(t)$ for the intensity process $\lambda(t)$ of a DSPP $N(t)$, based on the discrete time counting observations $\{N_1, N_2, \ldots, N_m\}$, with $t \geq t_m$. For that, we first seek the solution, the optimal impulse-response function $h(t,t_j)$, of the equation (5). In the following theorem, a feasible procedure for its computation is presented.

Theorem 1 *The optimal impulse response funtion $h(t,t_j)$ can be expressed in the form*

$$h(t,t_j) = \mathbf{a}'(t)\mathbf{g}(t_j,t_m), \quad t_1 \leq t_j \leq t_m, \ t \geq t_m \tag{7}$$

where the q-dimensional vector-valued function $\mathbf{g}(t_j,\cdot)$ is resursively computed as follows

$$\mathbf{g}(t_j,t_k) = \mathbf{g}(t_j,t_{k-1}) - \mathbf{g}(t_k,t_k)\boldsymbol{\gamma}'(t_k)\mathbf{g}(t_j,t_{k-1}) \tag{8}$$

for $t_j < t_k$, with

$$\mathbf{g}(t_k,t_k) = \{\boldsymbol{I} - \mathbf{Q}(t_{k-1})\}\,\boldsymbol{\psi}(t_k)\rho(t_k)^{-1} \tag{9}$$

where $\rho(t_k) = \{R_N(t_k,t_k) - \boldsymbol{\gamma}'(t_k)\mathbf{Q}(t_{k-1})\boldsymbol{\psi}(t_k)\}$, $\boldsymbol{I}$ is the identity matrix, $\boldsymbol{\gamma}(t_k) = \int_{t_{k-1}}^{t_k}\mathbf{a}(\sigma)d\sigma$, $\boldsymbol{\psi}(t_k) = \int_{t_{k-1}}^{t_k}\mathbf{b}(\sigma)d\sigma$, and the $q\times q$-dimensional matrix $\mathbf{Q}(t_k)$, $k=1,\ldots,m$, satisfies the recursive equation

$$\begin{aligned}&\mathbf{Q}(t_k) = \mathbf{Q}(t_{k-1}) + \mathbf{g}(t_k,t_k)\boldsymbol{\gamma}'(t_k)\{\boldsymbol{I} - \mathbf{Q}(t_{k-1})\}\\ &\mathbf{Q}(t_0) = \mathbf{0}_{q\times q}\end{aligned} \tag{10}$$

with $\mathbf{0}_{q\times q}$ the $q\times q$-dimensional matrix whose elements are all zero.

Proof

First of all, it should be observe that from (6) the covariance function $R_N(t_i,t_j)$ associated with the counting observations N_i defined in (1) can be written as

$$R_N(t_i,t_j) = \boldsymbol{\gamma}'(t_i)\boldsymbol{\psi}(t_j) + E[N_i]\delta_{ij} \tag{11}$$

with $\boldsymbol{\gamma}(t_k) = \int_{t_{k-1}}^{t_k}\mathbf{a}(\sigma)d\sigma$ and $\boldsymbol{\psi}(t_k) = \int_{t_{k-1}}^{t_k}\mathbf{b}(\sigma)d\sigma$. Moreover, the cross-covariance function $R_{\lambda N}(t,t_i)$ between the intensity process $\lambda(t)$ and the observation N_i given in (2) becomes

$$R_{\lambda N}(t,t_i) = \mathbf{a}'(t)\boldsymbol{\psi}(t_i) \tag{12}$$

Thus, substituting (11) and (12) in (5) we have

$$h(t,t_j)E[N_j] = \mathbf{a}'(t)\boldsymbol{\psi}(t_j) - \sum_{i=1}^{m} h(t,t_i)\boldsymbol{\gamma}'(t_i)\boldsymbol{\psi}(t_j)$$

where $t_1 \leq t_j \leq t_m$ and $t \geq t_m$.

Then, introducing a function $\mathbf{g}(t_j,t_k)$ such that

$$\mathbf{g}(t_j,t_k)E[N_j] = \boldsymbol{\psi}(t_j) - \sum_{i=1}^{k}\mathbf{g}(t_i,t_k)\boldsymbol{\gamma}'(t_i)\boldsymbol{\psi}(t_j) \tag{13}$$

for $t_1 \leq t_j \leq t_k$, the equation (7) for the optimal impulse-response $h(t,t_j)$ holds.

On the other hand, from (13) it follows that

$$\begin{aligned}\{\mathbf{g}(t_j,t_k) - \mathbf{g}(t_j,t_{k-1})\}\,E[N_j] &= -\mathbf{g}(t_k,t_k)\boldsymbol{\gamma}'(t_k)\boldsymbol{\psi}(t_j)\\ &\quad - \sum_{i=1}^{k-1}\{\mathbf{g}(t_i,t_k) - \mathbf{g}(t_i,t_{k-1})\}\,\boldsymbol{\gamma}'(t_i)\boldsymbol{\psi}(t_j)\end{aligned}$$

Therefore, taking (13) into account, the recursive formula (8) is derived.

Furthermore, for $j = k$, the equation (13) becomes

$$\mathbf{g}(t_k,t_k)E[N_k] = \boldsymbol{\psi}(t_k) - \sum_{i=1}^{k}\mathbf{g}(t_i,t_k)\boldsymbol{\gamma}'(t_i)\boldsymbol{\psi}(t_k)$$
$$= \boldsymbol{\psi}(t_k) - \sum_{i=1}^{k-1}\mathbf{g}(t_i,t_k)\boldsymbol{\gamma}'(t_i)\boldsymbol{\psi}(t_k) - \mathbf{g}(t_k,t_k)\boldsymbol{\gamma}'(t_k)\boldsymbol{\psi}(t_k) \quad (14)$$

Now, from (11), the equation (14) can be written as follows:

$$\mathbf{g}(t_k,t_k)R_N(t_k,t_k) = \boldsymbol{\psi}(t_k) - \sum_{i=1}^{k-1}\mathbf{g}(t_i,t_k)\boldsymbol{\gamma}'(t_i)\boldsymbol{\psi}(t_k)$$

As a consequence, using (8) in this last equation, it can be checked that

$$\mathbf{g}(t_k,t_k)R_N(t_k,t_k) = \boldsymbol{\psi}(t_k) - \sum_{i=1}^{k-1}\mathbf{g}(t_i,t_{k-1})\boldsymbol{\gamma}'(t_i)\boldsymbol{\psi}(t_k) + \mathbf{g}(t_k,t_k)\boldsymbol{\gamma}'(t_k)\sum_{i=1}^{k-1}\mathbf{g}(t_i,t_{k-1})\boldsymbol{\gamma}'(t_i)\boldsymbol{\psi}(t_k) \quad (15)$$

Therefore, if we introduce the auxiliary function

$$\mathbf{Q}(t_k) = \sum_{i=1}^{k}\mathbf{g}(t_i,t_k)\boldsymbol{\gamma}'(t_i) \quad (16)$$

the equation (15) becomes

$$\mathbf{g}(t_k,t_k)\left[R_N(t_k,t_k) - \boldsymbol{\gamma}'(t_k)\mathbf{Q}(t_{k-1})\boldsymbol{\psi}(t_k)\right] = \boldsymbol{\psi}(t_k) - \mathbf{Q}(t_{k-1})\boldsymbol{\psi}(t_k)$$

and hence, the expression (9) is obtained.

Finally, using (8) and (16), we can write

$$\mathbf{Q}(t_k) - \mathbf{Q}(t_{k-1})$$
$$= \mathbf{g}(t_k,t_k)\boldsymbol{\gamma}'(t_k) + \sum_{i=1}^{k-1}\left\{\mathbf{g}(t_i,t_k) - \mathbf{g}(t_i,t_{k-1})\right\}\boldsymbol{\gamma}'(t_i)$$
$$= \mathbf{g}(t_k,t_k)\boldsymbol{\gamma}'(t_k) - \mathbf{g}(t_k,t_k)\boldsymbol{\gamma}'(t_k)\sum_{i=1}^{k-1}\mathbf{g}(t_i,t_{k-1})\boldsymbol{\gamma}'(t_i)$$
$$= \mathbf{g}(t_k,t_k)\boldsymbol{\gamma}'(t_k)\left\{\boldsymbol{I} - \mathbf{Q}(t_{k-1})\right\}$$

and thus, it is obvious that $\mathbf{Q}(t_k)$ obeys the equation (10) with the initialization at $k = 0$, $\mathbf{Q}(t_0) = \mathbf{0}_{q\times q}$ and the theorem is proven.

Next, from Theorem 1, a recursive algorithm for the LMMSE estimator of the intensity process is provided in the following result.

Theorem 2 *The LMMSE estimate for the intensity process $\lambda(t)$, $\hat{\lambda}(t)$, based on the observations $\{N_1, N_2, \ldots, N_m\}$, with $t \geq t_m$, can be computed through the equation*

$$\hat{\lambda}(t) = E\left[\lambda(t)\right] + \mathbf{a}'(t)\mathbf{e}(t_m), \quad t \geq t_m \quad (17)$$

where the q-dimensional vector $\mathbf{e}(t_k)$, $k = 1, \ldots, m$, obeys the recursive expression

$$\mathbf{e}(t_k) = \mathbf{e}(t_{k-1}) + \mathbf{g}(t_k,t_k)\left\{N_k - E\left[N_k\right] - \boldsymbol{\gamma}'(t_k)\mathbf{e}(t_{k-1})\right\}$$
$$\mathbf{e}(t_0) = \mathbf{0}_q \quad (18)$$

with $\mathbf{0}_q$ the q-dimensional vector whose elements are all zero and the function $\mathbf{g}(t_k,t_k)$ given by the equation (9).

Proof

Substituting (7) in (4) we get

$$\hat{\lambda}(t) = E\left[\lambda(t)\right] + \mathbf{a}'(t)\sum_{i=1}^{m}\mathbf{g}(t_i,t_m)\left\{N_i - E\left[N_i\right]\right\}, \quad t \geq t_m$$

Then, introducing the auxiliary function

$$\mathbf{e}(t_k) = \sum_{i=1}^{k}\mathbf{g}(t_i,t_k)\left\{N_i - E\left[N_i\right]\right\} \quad (19)$$

the expression (17) for the LMMSE estimate $\hat{\lambda}(t)$ holds.

Moreover, from (8) and (19), we have

$$\mathbf{e}(t_k) - \mathbf{e}(t_{k-1}) = \mathbf{g}(t_k,t_k)\left\{N_k - E\left[N_k\right]\right\} + \sum_{i=1}^{k-1}\left\{\mathbf{g}(t_i,t_k) - \mathbf{g}(t_i,t_{k-1})\right\}\left\{N_i - E\left[N_i\right]\right\}$$
$$= \mathbf{g}(t_k,t_k)\left\{N_k - E\left[N_k\right]\right\} - \mathbf{g}(t_k,t_k)\boldsymbol{\gamma}'(t_k)\sum_{i=1}^{k-1}\mathbf{g}(t_i,t_{k-1})\left\{N_i - E\left[N_i\right]\right\}$$
$$= \mathbf{g}(t_k,t_k)\left\{N_k - E\left[N_k\right] - \boldsymbol{\gamma}'(t_k)\mathbf{e}(t_{k-1})\right\}$$

and the equation (18) for $\mathbf{e}(t_k)$ is obtained with the initialization at $k = 0$, $\mathbf{e}(t_0) = \mathbf{0}_q$.

In the next theorem, a recursive procedure for computing $P(t)$, a measure of the estimation accuracy for the LMMSE estimate of the intensity process (17) is shown.

Theorem 3 *The LMMSE estimation error covariance $P(t)$ associated with (17) is*

$$\mathbf{P}(t) = R_\lambda(t,t) - \mathbf{a}'(t)\mathbf{Q}(t_m)\mathbf{b}(t), \quad t \geq t_m \quad (20)$$

where $\mathbf{Q}(t_m)$ satisfies the equation (10).

Proof

From (4), the minimum mean-square error (3) can be written as

$$\mathbf{P}(t) = R_\lambda(t,t) - \sum_{i=1}^{m} h(t,t_i) R_{N\lambda}(t_i,t), \quad t \geq t_m$$

Now, using (2) and (7) in the above equation, we get

$$P(t) = R_\lambda(t,t) - \mathbf{a}'(t) \sum_{i=1}^{m} \mathbf{g}(t_i,t_m) \int_{t_{i-1}}^{t_i} R_\lambda(\sigma,t) d\sigma$$

Then, applying that $R_\lambda(\sigma,t)$ is a finite-dimensional covariance function of the form (6), we have

$$P(t) = R_\lambda(t,t) - \mathbf{a}'(t) \sum_{i=1}^{m} \mathbf{g}(t_i,t_m) \boldsymbol{\gamma}'(t_i) \mathbf{b}(t)$$

Finally, taking (16) into account, the equation (20) is verified.

3 Numerical Example

In this section, the behavior of the proposed LLMSE estimate is numerically analyzed. For that, as an illustrative example, we consider a DSPP $\{N(t), t \geq 0\}$ whose intensity process $\{\lambda(t), t \geq 0\}$ is a gaussian random process with mean function

$$E[\lambda(t)] = 1 - e^{-2t}$$

and covariance function

$$R_\lambda(t,s) = \frac{1}{4}(1 - e^{-2t})(1 - e^{-2s}) \tag{21}$$

It should be noted that, processes with exponential covariance are common in different areas such as in telecommunications networks with the aim of modelling the call arrival intensity of any given traffic stream [8] or in quantitative financial for modelling the intensity rate in the study of the pricing of defaultable derivatives, such as bonds, bond options, and credit default swaps (see, e.g., [9] and [10]).

Moreover, remark that the covariance function of the intensity process $R_\lambda(t,s)$ given in (21) is a finite-dimensional covariance of the form (6) where $\mathbf{a}(t) = \frac{1}{4}(1 - e^{-2t})$ and $\mathbf{b}(t) = (1 - e^{-2t})$.

On the other hand, we consider that the process $N(t)$ is observed in the interval $[0, 10)$ which is partitioned into $m = 100$ disjoint intervals according to the times $t_i = (i-1)/10$, for $i = 1, \ldots, 100$. Thus, we have the observations set $\{N_1, \ldots, N_{100}\}$, with $N_i = N(t_i) - N(t_{i-1})$.

Next, from the set of counting observation $\{N_1, \ldots, N_m\}$, the LMMSE filtering estimate (17) of the intensity process $\lambda(t)$, $\hat{\lambda}(t)$ with $t = t_m$, as well as its mean square error (20), have been computed.

Figure 1 illustrates the simulated values for the intensity process in comparison with their filtering estimations computed through Theorem 2. Notice that different simulations have been made and the one presented here is representative. Furthermore, the LMMSE filtering error (2) associated with the above estimate (20) is shown in Figure 2.

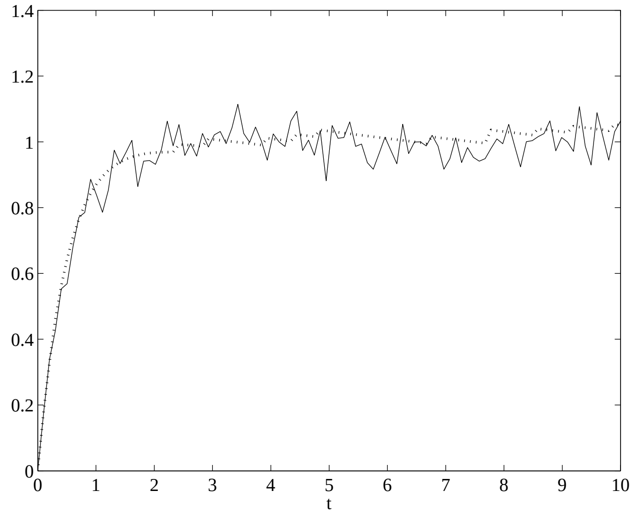

Figure 1: Simulated values for $\lambda(t)$ (solid line) and the filtering estimate $\hat{\lambda}(t)$.

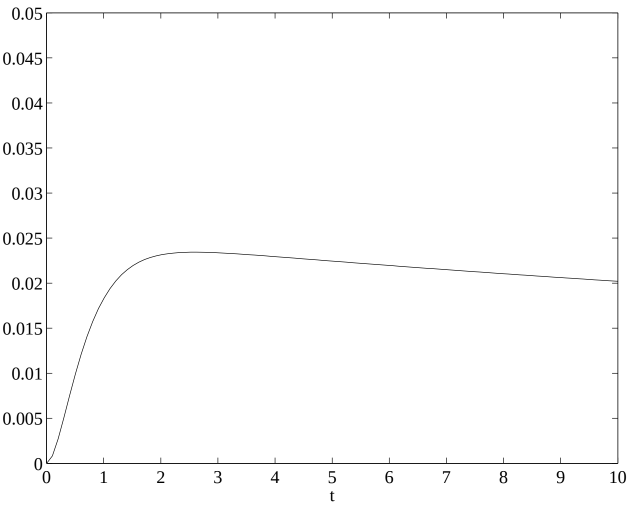

Figure 2: LLMSE filtering error $P(t)$.

4 Conclusions and Future Work

In this paper, a new LMMSE estimation algorithm has been developed for computing the intensity process of a DSPP under the only assumption that the intensity process has a finite-dimensional covariance function. This hypothesis is valid for general stationary and non-stationary processes and then, it can be widely applied. Hence, the proposed methodology is an alternative approach to the Kalman-Bucy filter for those situations in which a state-space model is not readily at hand.

In future work our efforts will be directed to developing a general LMMSE estimation algorithm valid for all types of estimators (smoothing, filtering and prediction estimates) of any linear or nonlinear operation of the intensity process and extend these results to those situations where more than one DSPP is observed simultaneously, that is, to include doubly stochastic multichannel Poisson processes.

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