Adding Relations in the Same Level of a Linking Pin Type Organization Structure

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Abstract—**This paper proposes two models of adding relations to a linking pin type organization structure where every pair of siblings in a complete K-ary tree of height H is adjacent: (i) a model of adding an edge between two nodes with the same depth N and (ii) a model of adding edges between every pair of nodes with the same depth N. For each of the two models, an optimal depth N^* is obtained by maximizing the total shortening path length which is the sum of shortening lengths of shortest paths between every pair of all nodes.**

Keywords: organization structure, linking pin, adding relation, complete K-ary tree, shortest path length

1 Introduction

The basic type of formal organization structure is a pyramid organization [6] which is a hierarchical structure based on the principle of unity of command [2] that every member except the top in the organization should have a single immediate superior. On the other hand an organization characterized by System 4 of Likert [3] has a structure in which relations between members of the same section are added to the pyramid organization structure. Members of middle layers of System 4 which are both members of the upper units and chiefs of the lower units are called linking pins, and this type of organization is called a linking pin type organization.

In the linking pin type organization there exist relations between each superior and his direct subordinates and those between members which have the same direct subordinate. However, it is desirable to have formed additional relations other than their relations in advance in case they need communication with other departments in the organization. In companies, the relations with other departments are built by meetings, group training, internal projects, and so on. Personal relations exceeding departments are also considered to be useful for the communication of information in the organization.

The linking pin type organization structure can be expressed as a structure where every pair of siblings which are nodes which have the same parent in a rooted tree is adjacent, if we let nodes and edges in the structure correspond to members and relations between members in the organization respectively. Then the linking pin type organization structure is characterized by the number of subordinates of each member, that is, the number of children of each node and the number of levels in the organization, that is, the height of the rooted tree, and so on [4, 5]. Moreover, the path between a pair of nodes in the structure is equivalent to the route of communication of information between a pair of members in the organization, and adding edges to the structure is equivalent to forming additional relations other than those between each superior and his direct subordinates and between members which have the same direct subordinate. The purpose of our study is to obtain an optimal set of additional relations to the linking pin type organization such that the communication of information between every member in the organization becomes the most efficient. This means that we obtain a set of additional edges to the structure minimizing the sum of lengths of shortest paths between every pair of all nodes.

This paper proposes two models of adding relations to a linking pin type organization structure which is a complete K-ary linking pin structure of height $H\,(H = 2, 3, \cdots)$ where every pair of siblings in a complete K-ary tree of height H is adjacent: (i) a model of adding an edge between two nodes with the same depth N and (ii) a model of adding edges between every pair of nodes with the same depth N. A complete K-ary tree is a rooted tree in which all leaves have the same depth and all internal nodes have $K\,(K = 2, 3, \cdots)$ children [1]. Figure 1 shows an example of a complete K-ary linking pin structure ($K = 2$, $H = 5$). In Figure 1 the value of N expresses the depth of each node.

The above model (i) corresponds to the formation of an additional relation between two members in the same level of an organization such as a personal communication. Model (ii) is equivalent to additional relations between every pair of all members in the same level such as section chief training.

If $l_{i,j}(= l_{j,i})$ denotes the path length, which is the number of edges in the shortest path from a node v_i to a node v_j $(i, j = 1, 2, \cdots, (K^{H+1} - 1)/(K - 1))$ in the complete K-ary linking pin structure of height H, then $\sum_{i<j} l_{i,j}$ is the total path length. Furthermore, if $l'_{i,j}$ denotes the

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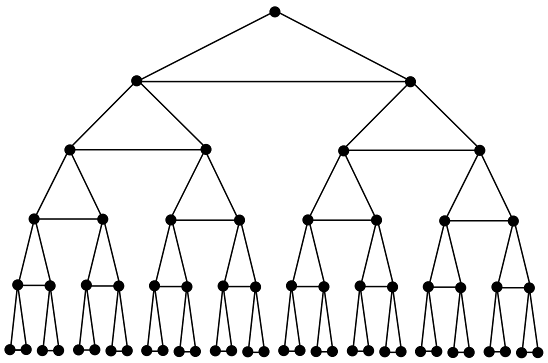

Figure 1: An example of a complete K-ary linking pin structure ($K = 2$, $H = 5$).

path length from v_i to v_j after adding edges in the above models, $l_{i,j} - l'_{i,j}$ is called the shortening path length between v_i and v_j, and $\sum_{i<j}(l_{i,j} - l'_{i,j})$ is called the *total shortening path length.*

In Section 2 and Section 3, for each of the above two models of adding edges respectively, we formulate the total shortening path length and obtain an optimal adding depth N^* which maximizes the total shortening path length.

2 Adding an edge between two nodes with the same depth

This section obtains an optimal depth N^* by maximizing the total shortening path length, when a new edge between two nodes with the same depth $N\,(N = 2, 3, \cdots, H\,)$ is added to a complete K-ary linking pin structure of height H.

2.1 Formulation of total shortening path length

We can add a new edge between two nodes with the same depth N in a complete K-ary linking pin structure in $N-1$ ways that lead to non-isomorphic graphs. Let $R_H(N, L)$ denote the total shortening path length by adding the new edge, where $L\,(L = 0, 1, 2, \cdots, N-2\,)$ is the depth of the deepest common ancestor of the two nodes on which the new edge is incident. For the case of $L = 0$, the total shortening path length is denoted by $S_{1,H}(N)$. Since addition of the new edge shortens path lengths only between pairs of descendants of the deepest common ancestor of the two nodes on which the new edge is incident, we obtain

$$R_H(N, L) = S_{1,H-L}(N - L)\,. \tag{1}$$

We formulate $S_{1,H}(N)$ in the following.

Let v_0^X and v_0^Y denote the two nodes on which the adding edge is incident and assume that $L = 0$. Let v_k^X and v_k^Y denote ancestors of v_0^X and v_0^Y, respectively, with depth $N-k$ for $k = 1, 2, \cdots, N-1$. The sets of descendants of v_0^X and v_0^Y are denoted by V_0^X and V_0^Y respectively. (Note that every node is a descendant of itself [1].) Let V_k^X denote the set obtained by removing v_k^X and the descendants of v_{k-1}^X from the set of descendants of v_k^X and let V_k^Y denote the set obtained by removing v_k^Y and the descendants of v_{k-1}^Y from the set of descendants of v_k^Y, where $k = 1, 2, \cdots, N-1$.

Since addition of the new edge doesn't shorten path lengths between pairs of nodes other than between pairs of $v_k^X\,(k = 1, 2, \cdots, N-1\,)$ or nodes in $V_k^X\,(k = 0, 1, 2, \cdots, N-1\,)$ and $v_k^Y\,(k = 1, 2, \cdots, N-1\,)$ or nodes in $V_k^Y\,(k = 0, 1, 2, \cdots, N-1\,)$, the total shortening path length can be formulated by adding up the following six sums of shortening path lengths: (i) the sum of shortening path lengths between every pair of nodes in V_0^X and nodes in V_0^Y, (ii) the sum of shortening path lengths between every pair of $v_k^X\,(k = 1, 2, \cdots, N-1\,)$ and nodes in V_0^Y and between every pair of $v_k^Y\,(k = 1, 2, \cdots, N-1\,)$ and nodes in V_0^X, (iii) the sum of shortening path lengths between every pair of nodes in V_0^X and nodes in $V_k^Y\,(k = 1, 2, \cdots, N-1\,)$ and between every pair of nodes in V_0^Y and nodes in $V_k^X\,(k = 1, 2, \cdots, N-1\,)$, (iv) the sum of shortening path lengths between every pair of $v_k^X\,(k = 1, 2, \cdots, N-1\,)$ and $v_k^Y\,(k = 1, 2, \cdots, N-1\,)$, (v) the sum of shortening path lengths between every pair of nodes in $V_k^X\,(k = 1, 2, \cdots, N-1\,)$ and nodes in $V_k^Y\,(k = 1, 2, \cdots, N-1\,)$ and (vi) the sum of shortening path lengths between every pair of $v_k^X\,(k = 1, 2, \cdots, N-1\,)$ and nodes in $V_k^Y\,(k = 1, 2, \cdots, N-1\,)$ and between every pair of $v_k^Y\,(k = 1, 2, \cdots, N-1\,)$ and nodes in $V_k^X\,(k = 1, 2, \cdots, N-1\,)$.

The sum of shortening path lengths between every pair of nodes in V_0^X and nodes in V_0^Y is given by

$$A_{1,H}(N) = \{M(H-N)\}^2(2N-2)\,, \tag{2}$$

where $M(h)$ denotes the number of nodes of a complete K-ary tree of height $h\,(h = 0, 1, 2, \cdots)$. The sum of shortening path lengths between every pair of $v_k^X\,(k = 1, 2, \cdots, N-1\,)$ and nodes in V_0^Y and between every pair of $v_k^Y\,(k = 1, 2, \cdots, N-1\,)$ and nodes in V_0^X is given by

$$B_{1,H}(N) = 2M(H-N)\sum_{i=1}^{N-2} 2i\,, \tag{3}$$

and the sum of shortening path lengths between every pair of nodes in V_0^X and nodes in $V_k^Y\,(k = 1, 2, \cdots, N-1\,)$ and between every pair of nodes in V_0^Y and nodes in $V_k^X\,(k = 1, 2, \cdots, N-1\,)$ is given by

$$C_{1,H}(N) = 2M(H-N)\sum_{i=1}^{N-1}(K-1)M(H-i-1)(2i-1)\,. \tag{4}$$

Furthermore, the sum of shortening path lengths between every pair of v_k^X $(k=1,2,\cdots,N-1)$ and v_k^Y $(k=1,2,\cdots,N-1)$ is given by

$$D_{1,H}(N)=\sum_{i=1}^{N-3}\sum_{j=1}^{i}2j\,, \tag{5}$$

and the sum of shortening path lengths between every pair of nodes in V_k^X $(k=1,2,\cdots,N-1)$ and nodes in V_k^Y $(k=1,2,\cdots,N-1)$ is given by

$$\begin{aligned} & E_{1,H}(N) \\ &= \sum_{i=1}^{N-2}(K-1)M(H-i-2) \\ &\quad \times\sum_{j=1}^{i}(K-1)M(H-N+i-j)\,2j\,, \end{aligned} \tag{6}$$

and the sum of shortening path lengths between every pair of v_k^X $(k=1,2,\cdots,N-1)$ and nodes in V_k^Y $(k=1,2,\cdots,N-1)$ and between every pair of v_k^Y $(k=1,2,\cdots,N-1)$ and nodes in V_k^X $(k=1,2,\cdots,N-1)$ is given by

$$\begin{aligned} & F_{1,H}(N) \\ &= 2\sum_{i=1}^{N-2}(K-1)M(H-i-2)\sum_{j=1}^{i}(2j-1)\,. \end{aligned} \tag{7}$$

In Equations (3), (5), (6) and (7) we define

$$\sum_{i=1}^{-1}\cdot=0\,, \tag{8}$$

$$\sum_{i=1}^{0}\cdot=0\,. \tag{9}$$

From the above equations, the total shortening path length $S_{1,H}(N)$ is given by

$$\begin{aligned} & S_{1,H}(N) \\ &= A_{1,H}(N)+B_{1,H}(N)+C_{1,H}(N)+D_{1,H}(N) \\ &\quad +E_{1,H}(N)+F_{1,H}(N) \\ &= \big\{M(H-N)\big\}^2(2N-2)+2M(H-N)\sum_{i=1}^{N-2}2i \\ &\quad +2M(H-N)\sum_{i=1}^{N-1}(K-1)M(H-i-1)(2i-1) \\ &\quad +\sum_{i=1}^{N-3}\sum_{j=1}^{i}2j\;+\sum_{i=1}^{N-2}(K-1)M(H-i-2) \\ &\quad \times\sum_{j=1}^{i}(K-1)M(H-N+i-j)\,2j \\ &\quad +2\sum_{i=1}^{N-2}(K-1)M(H-i-2)\sum_{j=1}^{i}(2j-1)\,. \end{aligned} \tag{10}$$

From Equations (1) and (10), we have the following theorem.

Theorem 1. *$L^*=0$ maximizes $R_H(N,L)$ for each N.*

Proof. For $N=2$, $L^*=0$ trivially. For $N=3,4,\cdots,H$, let

$$\Delta R_H(N,L)\equiv R_H(N,L+1)-R_H(N,L)\,. \tag{11}$$

We then have

$$\begin{aligned} & \Delta R_H(N,L) \\ &= S_{1,H-(L+1)}(N-(L+1))-S_{1,H-L}(N-L) \\ &= -2\big\{M(H-N)\big\}^2-4M(H-N)(N-L-2) \\ &\quad -2M(H-N)\sum_{i=1}^{N-L-2}(K-1)(2i-1) \\ &\quad \times\big\{M(H-L-i-1)-M(H-L-i-2)\big\} \\ &\quad -2(K-1)\big\{M(H-N)\big\}^2(2N-2L-3) \\ &\quad -\sum_{j=1}^{N-L-3}2j\;-\sum_{i=1}^{N-L-3}(K-1) \\ &\quad \times\big\{M(H-L-i-2)-M(H-L-i-3)\big\} \\ &\quad \times\sum_{j=1}^{i}(K-1)M(H-N+i-j)\,2j \\ &\quad -(K-1)M(H-N) \\ &\quad \times\sum_{j=1}^{N-L-2}(K-1)M(H-L-j-2)\,2j \\ &\quad -2\sum_{i=1}^{N-L-3}(K-1)\big\{M(H-L-i-2) \\ &\quad -M(H-L-i-3)\big\}\sum_{j=1}^{i}(2j-1) \\ &\quad -2(K-1)M(H-N)\sum_{j=1}^{N-L-2}(2j-1)\,, \end{aligned} \tag{12}$$

for $L=0,1,2,\cdots,N-3$. Since $M(h)$ increases with h, we obtain

$$\Delta R_H(N,L)<0\,. \tag{13}$$

Therefore, $R_H(N,L)$ takes its maximum at $L^*=0$ for each N. □

We next discuss the optimal adding depth $N=N^*$ which maximizes $S_{1,H}(N)=R_H(N,0)$. Since the number of nodes of a complete K-ary tree of height h is

$$M(h)=\frac{K^{h+1}-1}{K-1}\,, \tag{14}$$

$S_{1,H}(N)$ of Equation (10) becomes

$$S_{1,H}(N)$$

$$= \frac{2}{(K-1)^3}\Big\{(N-1)(K-1)K^{2H-N+1} + 2K^{H-N+2} - 2K^{H+1} + K(N-1)(K-1)\Big\}. \quad (15)$$

2.2 An optimal adding depth

In this subsection, we seek $N = N^*$ which maximizes $S_{1,H}(N)$ in Equation (15).

Let $\Delta S_{1,H}(N) \equiv S_{1,H}(N+1) - S_{1,H}(N)$, so that we have

$$\Delta S_{1,H}(N) = \frac{2}{(K-1)^2}\Big\{(K-NK+N)K^{2H-N} - 2K^{H-N+1} + K\Big\}, \quad (16)$$

for $N = 2, 3, \cdots, H-1$. Let us define a continuous variable x which depends on H as

$$x = K^H , \quad (17)$$

then $\Delta S_{1,H}(N)$ becomes

$$T_{1,N}(x) = \frac{2}{(K-1)^2}\Big\{(K-NK+N)K^{-N}x^2 - 2K^{-N+1}x + K\Big\}, \quad (18)$$

which is a quadratic function of x.

From the coefficient of x^2 in Equation (18), the following two cases can be discussed:
(i) When $K = 2$ and $N = 2$, then $2(K - NK + N)K^{-N}/(K-1)^2 = 0$ which indicates that $T_{1,N}(x)$ is a linear function.
(ii) When $K = 2$ and $N = 3, 4, \cdots, H-1$ or $K = 3, 4, \cdots$, then $2(K - NK + N)K^{-N}/(K-1)^2 < 0$ which means that $T_{1,N}(x)$ is convex upward.

In the case of (i), $T_{1,N}(x)$ becomes

$$T_{1,N}(x) = -2x + 4 . \quad (19)$$

Since the coefficient of x in Equation (19) is negative and

$$T_{1,N}(2^3) = -12 < 0 , \quad (20)$$

we have $T_{1,N}(x) < 0$ for $x \geq 2^3$. Therefore, when $K = 2$ and $N = 2$, then we have $\Delta S_{1,H}(N) < 0$ for $H = 3, 4, \cdots$.

In the case of (ii), by differentiating $T_{1,N}(x)$ in Equation (18) with respect to x, we obtain

$$T'_{1,N}(x) = \frac{4}{(K-1)^2}\Big\{(K-NK+N)K^{-N}x - K^{-N+1}\Big\}. \quad (21)$$

Since

$$T_{1,N}(K^{N+1}) = \frac{2}{(K-1)^2}\Big\{(K-NK+N)K^{N+2} - K(2K-1)\Big\} < 0 \quad (22)$$

and

$$T'_{1,N}(K^{N+1}) = \frac{4}{(K-1)^2}\Big\{(K-NK+N)K - K^{-N+1}\Big\} < 0 , \quad (23)$$

we have $T_{1,N}(x) < 0$ for $x \geq K^{N+1}$. Therefore, we have $\Delta S_{1,H}(N) < 0$ for $H \geq N+1$; that is, $N = 3, 4, \cdots, H-1$ when $K = 2$ and $N = 2, 3, \cdots, H-1$ when $K = 3, 4, \cdots$.

From the above results, the optimal adding depth N^* can be obtained and is given in Theorem 2.

Theorem 2. *The optimal adding depth is $N^* = 2$.*

Proof. If $H = 2$, then $N^* = 2$ trivially. If $H = 3, 4, \cdots$, then $N^* = 2$ from $\Delta S_{1,H}(N) < 0$, for $N = 2, 3, \cdots, H-1$. □

3 Adding edges between every pair of nodes with the same depth

This section obtains an optimal depth N^* by maximizing the total shortening path length, when new edges between every pair of nodes with the same depth N ($N = 2, 3, \cdots, H$) are added to a complete K-ary linking pin structure of height H.

3.1 Formulation of total shortening path length

Let $S_{2,H}(N)$ denote the total shortening path length, when we add edges between every pair of nodes with a depth of N.

The total shortening path length $S_{2,H}(N)$ can be formulated by adding up the following three sums of shortening path lengths: (i) the sum of shortening path lengths between every pair of nodes whose depths are equal to or more than N, (ii) the sum of shortening path lengths between every pair of nodes whose depths are less than N and those whose depths are equal to or more than N and (iii) the sum of shortening path lengths between every pair of nodes whose depths are less than N.

The sum of shortening path lengths between every pair of nodes whose depths are equal to or more than N is given by

$$A_{2,H}(N) = \big\{M(H-N)\big\}^2 K^N (K-1) \sum_{i=1}^{N-1} iK^i , \quad (24)$$

where $M(h)$ is as before. The sum of shortening path lengths between every pair of nodes whose depths are less than N and those whose depths are equal to or more than N is given by

$$B_{2,H}(N) = 2M(H-N)K^N(K-1)\sum_{i=1}^{N-2}\sum_{j=1}^{i} jK^j \,, \tag{25}$$

and the sum of shortening path lengths between every pair of nodes whose depths are less than N is given by

$$C_{2,H}(N) = K^N(K-1)\sum_{i=1}^{N-3}\sum_{j=1}^{i} j\,(i-j+1)\,K^j \,, \tag{26}$$

where Equations (8) and (9) apply.

From these equations, the total shortening path length $S_{2,H}(N)$ is given by

$$\begin{aligned} S_{2,H}(N) &= A_{2,H}(N) + B_{2,H}(N) + C_{2,H}(N) \\ &= \left\{M(H-N)\right\}^2 K^N(K-1)\sum_{i=1}^{N-1} iK^i \\ &\quad + 2M(H-N)K^N(K-1)\sum_{i=1}^{N-2}\sum_{j=1}^{i} jK^j \\ &\quad + K^N(K-1)\sum_{i=1}^{N-3}\sum_{j=1}^{i} j\,(i-j+1)\,K^j \,. \end{aligned} \tag{27}$$

From Equations (14) and (27), we have that

$$\begin{aligned} &S_{2,H}(N) \\ &= \frac{1}{2(K-1)^3}\Big\{2K^{2H-N+3} \\ &\quad + 2(NK-K-N)K^{2H+2} - 4K^{H+N+2} \\ &\quad + 4(NK-N+1)K^{H+2} \\ &\quad + N(N-1)(K-1)^2K^{N+1}\Big\} \,. \end{aligned} \tag{28}$$

3.2 An optimal adding depth

In this subsection, we seek $N = N^*$ which maximizes $S_{2,H}(N)$ in Equation (28).

Let $\Delta S_{2,H}(N) \equiv S_{2,H}(N+1) - S_{2,H}(N)$, so that we have

$$\begin{aligned} &\Delta S_{2,H}(N) \\ &= \frac{1}{2(K-1)^2}\Big\{\left(2K^2 - 2K^{-N+2}\right)K^{2H} \\ &\quad + \left(-4K^{N+2} + 4K^2\right)K^H \\ &\quad + N(NK+K-N+1)(K-1)K^{N+1}\Big\} \,, \end{aligned} \tag{29}$$

for $N = 2, 3, \cdots, H-1$. Let us define a continuous variable x which depends on H as in Equation (17), so that $\Delta S_{2,H}(N)$ becomes

$$\begin{aligned} &T_{2,N}(x) \\ &= \frac{1}{2(K-1)^2}\Big\{\left(2K^2 - 2K^{-N+2}\right)x^2 \\ &\quad + \left(-4K^{N+2} + 4K^2\right)x \\ &\quad + N(NK+K-N+1)(K-1)K^{N+1}\Big\} \,, \end{aligned} \tag{30}$$

which is a quadratic function of x.

By differentiating $T_{2,N}(x)$ with respect to x, we obtain

$$\begin{aligned} T'_{2,N}(x) &= \frac{1}{(K-1)^2}\Big\{\left(2K^2 - 2K^{-N+2}\right)x \\ &\quad - 2K^{N+2} + 2K^2\Big\} \,. \end{aligned} \tag{31}$$

Since $T_{2,N}(x)$ is convex downward from $(K^2 - K^{-N+2})/(K-1)^2 > 0$, and

$$\begin{aligned} &T_{2,N}(K^{N+1}) \\ &= \frac{1}{2(K-1)^2}\Big[(2K-4)K^{N+3}(K^N-1) \\ &\quad + N\left\{N(K-1)+K+1\right\}(K-1)K^{N+1}\Big] \\ &> 0 \end{aligned} \tag{32}$$

and

$$T'_{2,N}(K^{N+1}) = \frac{2K^2(K^N-1)}{K-1} > 0 \,, \tag{33}$$

we have $T_{2,N}(x) > 0$ for $x \geq K^{N+1}$. Hence, we have $\Delta S_{2,H}(N) > 0$ for $H = N+1, N+2, \cdots$; that is, $N = 2, 3, \cdots, H-1$.

From the above results, the optimal adding depth N^* can be obtained and is given in Theorem 3.

Theorem 3. *The optimal adding depth is $N^* = H$.*

Proof. If $H = 2$, then $N^* = 2$ trivially; that is $N^* = H$. If $H = 3, 4, \cdots$, then $N^* = H$ since $\Delta S_{2,H}(N) > 0$, for $N = 2, 3, \cdots, H-1$. □

4 Conclusions

This study considered the addition of relations to an organization structure such that the communication of information between every member in the organization becomes the most efficient. For each of two models of adding edges between nodes of the same depth N to a complete K-ary linking pin structure of height H where every pair of siblings in a complete K-ary tree of height H is adjacent, we obtained an optimal adding depth N^* which maximizes the total shortening path length.

Theorem 1 and Theorem 2 on adding an edge between two nodes with the same depth in Section 2 show that the most efficient manner of adding a single relation between two members in the same level such as a personal communication is to add the relation between two members which doesn't have common superiors except the top at the second level, irrespective of the number of subordinates and the number of levels in the organization structure. Theorem 3 on adding edges between every pair of nodes with the same depth in Section 3 shows that the most efficient way to add relations between every pair of all members at the same level such as section chief training is to use the lowest level, irrespective of the number of subordinates. These two results reveal optimal sets of additional relations to a linking pin type organization which is a complete K-ary linking pin structure such that the communication of information between every member in the organization becomes the most efficient.

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