Moment–type Estimation for Positive Stable Laws with Applications

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Abstract—**Strictly positive stable distributions are frequently encountered in such diverse areas as finance, engineering and survival analysis. Due to the non–existence of closed–form expression for the corresponding densities, standard procedures for estimation of the parameters of positive stable distributions appear to be computationally expensive. In this note we show that the first two moments of negative order provide a straightforward estimation procedure. The uniqueness of the estimators as well as their asymptotic distribution are shown. Simulations and application on real data are also included.** ***Keywords: stable distribution, moment estimation, heavy tails, skewness***

1 Introduction

Stable distributions arise as the only possible limit laws for normalized sums of independent and identically distributed random variables. Alternatively, a random variable X is said to have a stable distribution if, for any $a, b > 0$, there exists a positive number c and a real number d, such that

$$aX_1 + bX_2 \sim cX + d, \tag{1}$$

where X_1 and X_2 are independent copies of X and where $\sim$ denotes equality in law. If X follows a stable distribution then, there exists an $\alpha \in (0, 2]$ such that the number c in (1) satisfies, $c^\alpha = a^\alpha + b^\alpha$. The number α is called the index of stability or characteristic exponent. For $\alpha = 2$, the normal distribution results, which is the only member of the stable class having a finite variance.

Typically, the support of stable distributions is the entire real line. There exists however a subclass of stricly positive stable (SPS) laws. This subclass results by restricting the value of the characteristic exponent to the interval $(0, 1)$, and by setting the skewness parameter equal to its upper bound, $+1$ (or -1 depending upon the parametrization). Hence SPS laws are parameterized by (α, c), where $\alpha \in (0, 1)$ (resp. $c > 0$) denotes shape (resp. scale). Members of the SPS class will be denoted by $\mathcal{PS}_\alpha(c)$. SPS laws are important in their own right as building blocks of all stable distributions with index $\alpha \in (0, 1)$. In particular, each stable random variable with $\alpha \in (0, 1)$ can be written as a linear combination of two independent variables both following the same SPS law. Moreover, since as with all sub–Gaussian stable laws there is a considerable amount of mass in the tails of the distribution, SPS laws may be good models for positive heavy–tailed phenomena. See for example, [3], [7] and [5]. In [3] for instance it is shown that in exchange–rate markets, data on the so–called *intrinsic time* process, $T(t) = \{\text{Numbers of transactions up to time } t\}$, are satisfactorily fitted to a SPS distribution.

In this note, we first compute the theoretical moments of negative order via an entirely elementary argument involving the Laplace transform of the SPS law. Subsequently, the first two negative–integer moments are used to construct simple moment estimators of (α, c). It will be seen that the calculation of these highly intuitive estimates involves minimal computational effort, which leads to a unique solution. An illustration with simulated data is followed by application of these estimates to real data from the stock market.

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2 Derivation and computation of estimators

A most convenient definition of SPS laws is via the Laplace transform $L(t) = \mathrm{E}[\exp(-tX)]$. Specifically if $X \sim \mathcal{PS}_\alpha(c)$, it follows that

$$L(t) = \exp(-c^\alpha t^\alpha),\ t > 0. \tag{2}$$

With the aid of the Laplace transform we can prove the following lemma. For a proof refer to [4].

Lemma 2.1 Let E denote a unit exponential random variable, and $X \sim \mathcal{PS}_\alpha(c)$ be an independent SPS random variable with density $f(\cdot)$. Then,

$$W = \frac{E}{X},$$

follows a Weibull distribution with shape parameter equal to α, and scale equal to c^{-1}.

From Lemma 2.1, it follows that if $X \sim \mathcal{PS}_\alpha(c)$ then,

$$\mathrm{E}\left(\frac{1}{X}\right) = \frac{\Gamma(1 + \alpha^{-1})}{c}, \tag{3}$$

$$\mathrm{E}\left(\frac{1}{X^2}\right) = \frac{\Gamma(1+2\alpha^{-1})}{2c^2}, \tag{4}$$

where $\Gamma(\cdot)$ denotes the Gamma function. The moment equations result by replacing in (4), $\mathrm{E}(X^{-1})$ by

$$\overline{x_n^{(1)}} = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{x_j},$$

and in (5), $\mathrm{E}(X^{-2})$ by

$$\overline{x_n^{(2)}} = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{x_j^2},$$

where $x_1, x_2, ..., x_n$, denote specific independent realizations of X. Then the moment estimator $(\hat{\alpha}_n, \hat{c}_n)$, of (α, c), satisfies the system of equations,

$$\hat{c}_n = \frac{\Gamma(\hat{\alpha}_n^{-1})}{\hat{\alpha}_n \overline{x_n^{(1)}}}, \qquad F_n(\hat{\alpha}_n) = 0, \tag{5}$$

where

$$F_n(\alpha) = \frac{\alpha}{\mathrm{B}(\alpha^{-1}, \alpha^{-1})} - \frac{\overline{x_n^{(2)}}}{\left(\overline{x_n^{(1)}}\right)^2}, \tag{6}$$

and $\mathrm{B}(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$ denotes the Beta function.

The existence and uniqueness of $(\hat{\alpha}_n, \hat{c}_n)$, are consequences of the following lemma.

Lemma 2.2 The function $F_n(\cdot)$ defined by (6) is continuous for $\alpha \in (0, 1)$, and satisfies
1. $\lim_{\alpha\to 0^+} F_n(\alpha) = \infty$
2. $F_n(1) < 0$
3. $F_n'(\alpha) < 0, \ \forall\alpha \in (0, 1)$.

Proof. The continuity of $F_n(\cdot)$ follows directly from its definition, and the continuity of $\mathrm{B}(a, a)$ for $a > 0$.

Proof of 1. From the definition of $\mathrm{B}(a, a)$ and by noticing that $x(1-x) \le (1/4)$, we have,

$$\frac{\alpha}{\mathrm{B}(\alpha^{-1}, \alpha^{-1})} \ge \frac{1}{4}\alpha 4^{1/\alpha}. \tag{7}$$

Then the proof follows by taking the limit as $\alpha \to 0^+$ in (7), and applying L' Hospital's rule.

Proof of 2. Since $\mathrm{B}(1, 1) = 1$, we must show that

$$1 - \frac{\overline{x_n^{(2)}}}{\left(\overline{x_n^{(1)}}\right)^2} < 0 \Leftrightarrow \overline{x_n^{(2)}} - \left(\overline{x_n^{(1)}}\right)^2 > 0. \tag{8}$$

However the last inequality in (8) is true since its left–hand side defines the sample variance of $(1/x_j), \ j = 1, 2, ..., n$.

Proof of 3. By a straightforward calculation we have,

$$F_n'(\alpha) = \frac{1}{\mathrm{B}(1/\alpha, 1/\alpha)}\left(1 - \frac{2}{\alpha}[\Psi(2/\alpha) - \Psi(1/\alpha)]\right), \tag{9}$$

where,

$$\Psi(x) = \frac{d\log\Gamma(x)}{dx},$$

denotes the digamma function. In turn, from $\Psi(x) - \Psi(y) = \sum_{k=0}^{\infty}(y+k)^{-1} - (x+k)^{-1}$ (Gradshteyn and Ryzhik 1994, §8.363), it follows that $\Psi(x) - \Psi(y) > (1/y) - (1/x), \ x > y$. Hence $\Psi(2/\alpha) - \Psi(1/\alpha) > (\alpha/2)$, and consequently, one has from (9) that, $F_n'(\alpha) < 0$.

Lemma 2.2 implies that the equation $F_n(\alpha) = 0$ has a root in $(0, 1)$, which is unique. This root, say $\hat{\alpha}_n$, which may be found by a simple search procedure, is the estimate of the index parameter α. Subsequently, $\hat{\alpha}_n$ is used in the first equation in (5) in order to calculate the estimate $\hat{c}_n$ of the scale parameter c.

3 Asymptotic properties

To obtain asymptotically linear representations of the estimators, let $X_1, X_2, ..., X_n$, be independent copies on the random variable X, denote by (α_0, c_0) the true parameter values and, without loss of generality, assume that $c_0 = 1$. Also let $Z_j = 1/X_j, \ j = 1, 2, ..., n$. A linear Taylor expansion of $F_n(\hat{\alpha}_n)$ around $F_n(\alpha_0)$ yields

$$\sqrt{n}(\hat{\alpha}_n - \alpha_0) = -\sqrt{n}\frac{F_n(\alpha_0)}{F_n'(\alpha_0)} + o_P(1)$$

$$= \sqrt{n}\left(\frac{S_n^2}{\bar{Z}_n^2} - \frac{\sigma_z^2}{\mu_z^2}\right) + o_P(1), \tag{10}$$

where μ_z and σ_z^2 denote the mean and the variance, respectively, of $Z_1 = 1/X_1$, $\bar{Z}_n = n^{-1}\sum_{j=1}^n Z_j$, and $S_n^2 = n^{-1}\sum_{j=1}^n (Z_j - \bar{Z}_n)^2$, and $o_P(1)$ denotes a term which converges to zero in probability.

However it is well known that (refer to [6], §3.4),

$$\sqrt{n}\left[(\bar{Z}_n, S_n^2) - (\mu_z, \sigma_z^2)\right] =$$

$$\sqrt{n}\left[\left(\frac{\sum_{j=1}^n Z_j}{n}, \frac{\sum_{j=1}^n (Z_j - \mu_z)^2}{n}\right) - (\mu_z, \sigma_z^2)\right] + o_P(1).$$

Hence let $g(u, v) = v/u^2$, and expand $g(\bar{Z}_n, S_n^2)$ around $g(\mu_z, \sigma_z^2)$ to get

$$\sqrt{n}\left(\frac{S_n^2}{\bar{Z}_n^2} - \frac{\sigma_z^2}{\mu_z^2}\right) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} W_{\alpha_0}(Z_j) + o_P(1), \tag{11}$$

where

$$W_\alpha(z) = \frac{1}{\mu_z^2}\left[(z - \mu_z)^2 - \sigma_z^2 - \frac{2\sigma_z^2}{\mu_z}(z - \mu_z)\right].$$

Then by inserting (11) in (10), we conclude that the asymptotic linear respresentation for the estimator of α is

$$\sqrt{n}(\hat{\alpha}_n - \alpha_0) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} A_{\alpha_0}(X_j) + \mathrm{o_P}(1), \qquad (12)$$

where $A_\alpha(x) = W_\alpha(1/x)/F'(\alpha)$, with $F'(\cdot) := F'_n(\cdot)$ given by (9).

Likewise, consider the first equation in (5) and let $g(u,v) = \Gamma(1+u^{-1})/v$. Expand then $g(\hat{\alpha}_n, \bar{Z}_n)$ around $g(\alpha_0, \mu_z)$ to get after some algebra

$$\sqrt{n}(\hat{c}_n - 1) =$$
$$-\frac{1}{\mu_z}\Gamma(1+\alpha_0^{-1})\big[\frac{\Psi(1+\alpha_0^{-1})}{\alpha_0^2}\sqrt{n}(\hat{\alpha}_n - \alpha_0)$$
$$+\frac{1}{\mu_z}\sqrt{n}(\bar{Z}_n - \mu_z)\big] + \mathrm{o_P}(1).$$

Consequently, the asymptotic linear respresentation of the estimator of c is

$$\sqrt{n}(\hat{c}_n - 1) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} C_{\alpha_0}(X_j) + \mathrm{o_P}(1), \qquad (13)$$

where $C_\alpha(X) = \Omega_\alpha(1/X)$ with

$$\Omega_\alpha(z) = -\frac{\Gamma(1+\frac{1}{\alpha})}{\mu_z}\left[\frac{\Psi(1+\frac{1}{\alpha})W_\alpha(z)}{[\alpha F'(\alpha)]^2} + \frac{z-\mu_z}{\mu_z}\right].$$

From (12) and (13) and the Central Limit Theorem we obtain the following theorem:

Theorem 3.1 Let $X_1, X_2, ..., X_n$, be independent copies on the random variable X, with $X \sim \mathcal{PS}_{\alpha_0}(1)$. Then the estimators $(\hat{\alpha}_n, \hat{c}_n)$ satisfying equations (5) asymptotically follow a bivariate normal distribution. In particular

$$\sqrt{n}\left(\hat{\alpha}_n - \alpha_0, \hat{c}_n - 1\right) \to^D N\left(\mathbf{0}, \mathbf{V}\right),$$

where $\to^D$ denotes convergence in distribution and, in obvious notations,

$$\mathbf{V} = \begin{pmatrix} \sigma_\alpha^2 & \tau_{\alpha,c} \\ \tau_{\alpha,c} & \sigma_c^2 \end{pmatrix}.$$

Remark 3.2 The limit variance of $\hat{\alpha}_n$ may be obtained by tedious but straightforward calculations as $\sigma_\alpha^2 = \upsilon/[F'(\alpha_0)]^2$ where F' is given by (9) and

$$\upsilon = \frac{1}{\mu^4}\left(\mu_4 - \sigma^4 + \frac{4\sigma^6}{\mu^2} - 4\frac{\sigma^2}{\mu}\mu_3\right),$$

with $\mu = \mathbf{E}(Z)$, $\sigma^2 = Var(Z)$, and $\mu_r = \mathbf{E}[Z - \mathbf{E}(Z)]^r$, where $Z = 1/X$.

Likewise we obtain the limit variance of $\hat{c}_n$ as

$$\sigma_c^2 = \frac{\Gamma^2(1+\alpha_0^{-1})}{\mu^2}\big[\frac{\upsilon\,\Psi^2(1+\alpha_0^{-1})}{[\alpha_0 F'(\alpha_0)]^4} + \frac{\sigma^2}{\mu^2}$$
$$+\frac{2\Psi(1+\alpha_0^{-1})}{[\alpha_0 F'(\alpha_0)]^2}\left(\frac{\mu_3}{\mu^3} - \frac{2\sigma^4}{\mu^4}\right)\big].$$

4 Applications

In this section we illustrate the method of estimation by applying it first to pseudo–random numbers from SPS laws with scale parameter $c = 1$ and characteristic exponent α. The normalized ($\times n$) mean squared error (MSE) of the moment estimators (ME) is computed for sample size n. We also compute the corresponding MSE for the highly efficient generalized moment estimators (GME) in [1]. These estimators are computed as follows:

1. Compute the ME estimator $\hat{\alpha}_n$, from the second equation in (5).
2. Let $t = \hat{\alpha}_n(\hat{\alpha}_n + 4.2)/(10(1 - \hat{\alpha}_n))$.
3. With $\overline{x_n^{(t)}} = n^{-1}\sum_{j=1}^n (1/x_j^t)$, find the GME, say $\tilde{\alpha}_n$, of α as the solution of

$$F_n(\alpha) = \frac{\alpha \mathrm{B}(t,t)}{\mathrm{B}(t/\alpha, t/\alpha)} - \frac{\overline{x_n^{(2t)}}}{\left(\overline{x_n^{(t)}}\right)^2}.$$

4. Let $\tau = \tilde{\alpha}_n(3\tilde{\alpha}_n + 2.5)/(10(1 - \tilde{\alpha}_n))$.

5. Compute the GME, say $\tilde{c}_n$, of c as

$$\tilde{c}_n = \left[\frac{\Gamma(\tau/\tilde{\alpha}_n)}{\Gamma(\tau)\tilde{\alpha}_n \overline{x_n^{(\tau)}}}\right]^{1/\tau}.$$

In Table 1, the MSE of the ME and the GME computed from 10,000 replications is reported for sample size n = 20, 40, and n = 100. From these figures it may be observed that for small or moderate sample size (n = 20, 40), the ME of scale is more efficient than the corresponding GME when $\alpha < 0.8$. Also, the ME estimator of the characteristic exponent, although less efficient than the corresponding GME, it is a close competitor at least when the true parameter value of α is not close to zero. As a conclusion, and apart from providing good initial guess for the GME, the simple moment estimator may be preferred over the more complex GME if the sample size is not large and the true characteristic exponent is around the value $\alpha = 1/2$.

Our real–data application, employs the SPS laws in the modelling of the intrinsic time process in the Athens Stock Exchange. It is well known that for a typical stock, market activity is highly volatile within the trading day, having a long right tail. In particular, the opening of each trading day is followed by a period of intense market activity. Then follows a ‘regular’ period, and the market closes with a peak of transactions at the end of the day. We have employed daily data on the stock of ‘Alpha Bank’, a major private bank, for the period Jan. 2–May 30, 2003. In particular the volume of transactions was broken into 10–minute time intervals within each trading

$\alpha\downarrow$	$n=20$	$n=40$	$n=100$	$n=20$	$n=40$	$n=100$
0.2	0.19	0.54	0.68	0.06	0.04	0.03
	15.8	29.3	67.9	394	95.7	45.2
0.4	0.15	0.15	0.16	0.10	0.08	0.07
	6.41	8.05	12.3	12.3	8.68	7.25
0.5	0.10	0.10	0.10	0.09	0.08	0.06
	5.05	4.80	5.43	5.95	4.90	4.29
0.6	0.07	0.07	0.07	0.08	0.07	0.07
	3.46	3.26	3.16	3.76	3.27	3.05
0.8	0.04	0.04	0.03	0.03	0.03	0.02
	2.44	2.27	2.19	2.20	1.97	1.86
0.9	0.02	0.01	0.01	0.01	0.006	0.003
	3.45	3.31	3.22	1.97	1.72	1.58

Table 1: Normalized mean squared error of the moment estimator (left part) and the generalized moment estimator (right part) of α (top entry) and c (bottom entry).

Date	1/15	1/20	2/03	2/21	3/05
$\hat{\alpha}_n$	0.354	0.643	0.620	0.439	0.605
Date	3/17	4/04	4/22	5/07	5/23
$\hat{\alpha}_n$	0.596	0.567	0.512	0.586	0.319

Table 2: Date (Month/Day) and α–estimates for the stock price data.

day, resulting in 30 observations per day. Then, for these data the estimates of α and c were obtained by solving the system of equations in (5). We have tried several particular dates, corresponding to different days of the week, and different months. Selected results are shown in Table 2. Interestingly, in each case the second equation in (5) yielded an estimate $\hat{\alpha}_n$ well within the acceptable domain $(0,1)$, thus providing some confidence that indeed some SPS law is the underlying random mechanism.

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