On the New Stochastic Approach to Control the Investment Portfolio

Sergey A. Vavilov, Konstantin Yu. Ermolenko

Abstract — **The robust feed-back control schemes to provide the sustainable growth of investor capital under the absence of certain risks are introduced. These schemes are based on the current dynamics of the asset prices. It is assumed that the price of asset follows rather general stochastic differential equation. In contrast to the generally used self-financing strategy the control is realized within the framework of an open system. The latter implies the possibility to invest cash into the portfolio in the process of trading.**

Index Terms — **Portfolio management, assets trading, stochastic control methods, integrated volatility.**

I. Introduction

The attempts to apply classical methods of optimization based on the theory of optimal and adaptive control to realize the management of an investment portfolio very often tumble over serious problems. For instance the application of control theory as the stochastic version of dynamic programming approach [5], [6] implies the detailed information about the structure of factors in stochastic differential equations describing the dynamics of constituting portfolio assets. The latter information in contemporary financial markets seems hardly to be available. The methods of adaptive control theory are also not very often applicable because of the strongly nonstationary behavior of parameters of these or those modeling equations describing the dynamics of portfolio value.

Because of the aforesaid it is not surprising that the problem to create special control methods adapted to the investment portfolio management has long drawn the attention of researchers. Usually such methods imply the creation of control providing in a particular sense the positive dynamics of profit along with the minimization of quantitative and qualitative information about the structure of modeling equations. Moreover one of the most common models for assets pricing is the model of geometrical Brownian motion. Nevertheless when following this way to create the control of investment portfolio there arise a number of difficulties which may be formulated as follows. The heart of the matter is that the designing of control up till now has been based as a rule on the principles of self-financing strategy (see for instance [2], [7], [8], [10], [11]). The latter means that the purchase or sale of any assets automatically implies sale or purchase of a volume in the equivalent money terms of other assets constituting portfolio.

It is essential to note that realization of any circuit of management based on self-financing strategy implies (at least in terms of the literature available to the authors of the present work) the required number of assets in the portfolio significantly depends not only on the prices of struck bargains but also on the volatilities of corresponding assets.

The latter fact causes some inquires that seem to be an impediment in implementation of corresponding control systems. The point is that for majority of liquid assets the values of their volatilities have strongly non-stationary and pronounced palpitating character. It makes the tracking of their values with arbitrary precision in real time hardly possible. It is also important to keep in mind the property of delay inherent in each control system based on continuous model of pricing and the necessity to realize discrete procedure for their implementation. In this connection it is clear that the occurrence of essential mistakes is possible while defining the amount of assets included in a portfolio. How significantly such errors can affect the ultimate goal of management to provide the profitableness of portfolio remains not clear.

The aforesaid makes reasonable to pose the problem of creating the management of portfolio with a feed-back control based only on the prices of struck bargains to provide in some sense portfolio profitableness on a certain time interval and within the framework of the pricing model corresponding to geometrical Brownian motion.

The main goal of the present study is to solve the problem under consideration within the framework of a management alternative to self-financing strategy. It implies the possibility to invest additional cash from outside during the whole period of portfolio management. Moreover the release of cash as a result of trading allows its reinvestment to acquire new required assets.

The present paper is the revised and extended version of the authors' previous publication [14].

II. Formulation of the Problem and the Main Result

Originally consider elementary structure of the investment portfolio including only one type of assets. Assume that the

Manuscript received October 26, 2007.

S. A. Vavilov is with the Department of Applied Cybernetics, Faculty of Mathematics and Mechanics of St. Petersburg State University, Universitetsky prospekt, 28, Peterhof, Saint-Petersburg, 198504, Russia (e-mail: svavilov@mail.wplus.net).

K. Yu. Ermolenko is with the Department of Economic Cybernetics, Faculty of Economics of St. Petersburg State University, ul. Chaikovskogo, 62, Saint-Petersburg, 191194, Russia (e-mail: k.ermolenko@econ.pu.ru).

(Advance online publication: 19 February 2008)

price of asset x_t is a stochastic process on a time interval $[0,T]$ and follows stochastic differential equation

$$dx_t = c_t x_t dt + \sigma_t x_t dW_t, \quad (1)$$

where a factor of volatility $\sigma_t = \sigma(t,\omega)$ and $c_t = c(t,\omega)$ generally speaking are measurable random functions, W_t is a standard Wiener process.

Portfolio value is set by the dependence

$$f_t = a_t x_t + m_t, \quad (2)$$

where $a_t = a(t,\omega)$ is a measurable random function defining the number of assets, $m_t = m(t,\omega)$ is a measurable random function responding to some money equivalent which economical sense is given below.

Further, to avoid misunderstanding the realization of any random process in contrast to the process itself will be denoted by the corresponding letter with wave, as for example $\tilde{x}_t$ and x_t.

Consider the control defined for each moment t by the relationship

$$df_t = a_t dx_t + l(t,x_t)dt, \quad (3)$$

where dx_t is defined by the right hand side of equation (1) while the existence of stochastic differential df_t is supposed.

The second term in dependence (3) is interpreted as cash flow on the time interval dt invested and processed by the control system, while $l(t,x_t) \geq 0$. Consequently $l(t,x_t)$ is regarded as a regulator of the amount of cash processed by the control system and acts as control function.

Appling to the left and right hand sides of relationship (2) the procedure of calculating the stochastic differential, which implies the existence of stochastic differentials da_t and dm_t, one arrives to the relationship

$$df_t = a_t dx_t + x_t da_t + dx_t da_t + dm_t.$$

The latter one by making use of dependence (3) may be rewritten as follows

$$dm_t = -x_{t+dt} da_t + l(t,x_t)dt,$$

where x_{t+dt} is defined as $x_{t+dt} := x_t + dx_t$, or in the integral form

$$m_t = -\int_0^t x_{\tau+d\tau} da_\tau + \int_0^t l(\tau,x_\tau)d\tau. \quad (4)$$

Sufficient conditions to provide the existence of stochastic integral in relationship (4) as the limit of corresponding sums will be clarified below.

The first term in dependence (4) taken with minus is the value of assets constituting portfolio as the result of effected trading and it will be further referred to as portfolio cost.

Define profit $\tilde{p}_t$ for the observable value of price $\tilde{x}_t$ as the difference between the current price of assets and the portfolio cost

$$\tilde{p}_t = \tilde{a}_t \tilde{x}_t - \int_0^t \tilde{x}_{\tau+d\tau} d\tilde{a}_t. \quad (5)$$

Keeping in mind formulas (2), (3), (4) the latter dependence is equivalent to the relationship

$$\tilde{p}_t = \tilde{f}_t - \int_0^t l(\tau,\tilde{x}_\tau)d\tau, \quad (6)$$

where $\tilde{f}_t$ is a portfolio value for the observable price.

For the initial instant the portfolio is considered to be empty containing neither assets nor cash.

Consider the notions of the lower and upper bounds of sensitivity which are considered as the respective borders of the price band symmetric with respect to the price of the first bargain struck by the control system. Further, it is supposed the price of the asset is inside the pointed out band during the whole period of control $[0,T]$. For the utility and brevity of calculations the price of asset will be made dimensionless and scaled with respect to the lower bound of sensitivity, thus, defining the aforementioned price band as an interval $(1,\beta)$ where $\beta > 1$ is fixed.

We say that the control provides profitableness of an investment portfolio on the time interval $[0,T]$ if $\tilde{p}_T > 0$.

Pose the problem of the existence and realization of portfolio control to provide its profitableness on a given time interval $[0,T]$.

Theorem. Let the following conditions hold on the time interval $[0,T]$, where $T > 0$:

1. The price of asset x_t follows stochastic differential equation (1), moreover volatility σ_t is considered as a nonrandom function of time and consequently one can put down $\sigma_t = \tilde{\sigma}_t$.

2. Integrated volatility is subjected to the following condition of growth: $\int_\tau^T \sigma_s^2 ds \geq \gamma(T-\tau)$ for arbitrary $\tau \in [0,T]$, where γ is strictly positive number.

3. The observable realization of price $\tilde{x}_t$ does not pierce the borders of the price band $(1,\beta)$, where $\beta > 1$ is an arbitrary finite number.

Then if fixed T and β correspond to sufficiently large γ there exists control providing the profitableness of an investment portfolio on the time interval $[0,T]$. Moreover, within the framework of such control the amount of assets in the portfolio for each instant depends on the prices of struck bargains but does not depend explicitly on the volatility values.

Proof. Seek unknown f_t as the function of two variables $f_t = f(t,x_t)$, where x_t follows equation (1). Applying to $f(t,x_t)$ Ito's formula and comparing it with ratio (3) one arrives to the dependences

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_t^2 x_t^2 \frac{\partial^2 f}{\partial x_t^2} = l(t,x_t), \quad (7)$$

$$a_t = \frac{\partial f}{\partial x_t}. \quad (8)$$

The control $l(t, x_t)$ is set according to the relationship

$$l(t, x_t) = r(t)\varphi(x_t), \quad (9)$$

where $\varphi(x)$ is the eigenfunction corresponding to the first eigenvalue λ_1 of the following Sturm-Liouville problem

$$\frac{d^2\varphi}{dx^2} + \frac{\lambda_1^2}{x^2}\varphi = 0, \quad (10)$$

$$\varphi(1) = \varphi'(\beta) = 0. \quad (11)$$

The structure of function $r(t)$ will be clarified bellow.

As for the initial instant of control $t = 0$ the portfolio is empty then

$$f(0, x_t) = 0. \quad (12)$$

Besides the following boundary conditions are introduced

$$\frac{\partial f}{\partial x_t} \to 0 \text{ as } x_t \to \beta, \quad (13)$$

$$f(t, x_t) \to 0 \text{ as } x_t \to 1. \quad (14)$$

Owing to ratio (8) the fulfillment of boundary condition (13) implies the system of control takes long position, i.e. $a_t \geq 0$, and tends to get rid of assets when the price converges to the upper bound of sensitivity.

To clarify boundary condition (14) make use of relationships (2) and (4). The management efficiency in some cases implies the portfolio cost to exceed the cash flow spent for the acquisition of assets, i.e. the inequality $m_t < 0$ is to be valid. In particular, as it is shown below, when asset price converges to the lower bound of sensitivity it is reasonable that the whole amount of cash released in the process of the effected sales to be reinvested in purchasing of the assets, namely $a_t \to -\frac{m_t}{x_t}$ as $x_t \to 1$, what precisely matches, owing to relationship (2), the fulfillment of boundary condition (14).

Taking into account relationship (9) seek solution to the initial-boundary value problem (7), (12), (13), (14) in the form

$$f(t, x_t) = K(t)\varphi(x_t),$$

where $K(t)$ is the unknown function.

As the result of trivial transformations ultimately one arrives to the relationship

$$f(t, x_t) = \int_0^t e^{\frac{1}{2}\lambda_1^2 \int_\tau^t \sigma_s^2 ds} r(\tau) d\tau \cdot \varphi(x_t), \quad (15)$$

while the value of λ_1 and structure of function $\varphi(x_t)$ are determined by the formulas [9]

$$\lambda_1^2 = b^2 + \frac{1}{4}, \; \varphi(x_t) = \sqrt{x_t}\sin(b \ln x_t), \quad (16)$$

where b is the minimal strictly positive root of the equation

$$tg(b \ln \beta) = -2b. \quad (17)$$

By introducing the new variable $z = b \ln \beta$ equation (17) may be rewritten as follows

$$tg(z) = -\frac{2z}{\ln \beta}. \quad (18)$$

The graphical solution to the derived transcendental equation is presented at fig. 1.

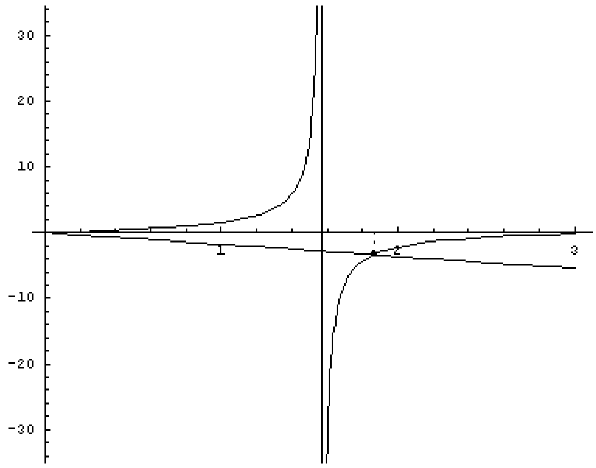

Fig. 1. Graphical solution to the transcendental equation.

Note that relationships (16), (17) describe the whole set of eigenvalues and eigenfunctions of Sturm-Liouville problem (10), (11). However the choice of the first eigenvalue provides, as one can easily note, the corresponding eigenfunction to be separated from zero inside the specified price band $(1, \beta)$.

Relationships (8), (15) define the amount of assets in the portfolio according to the formula

$$\tilde{a}_t = \left(\frac{\partial f}{\partial x_t}\right)_{x_t = \tilde{x}_t} = \int_0^t e^{\frac{1}{2}\lambda_1^2 \int_\tau^t \sigma_s^2 ds} r(\tau) d\tau \cdot \varphi'(x_t)\Big|_{x_t = \tilde{x}_t}. \quad (19)$$

Partition the time interval $[0, T]$ in n parts as follows $0 = T_0 < T_1 < \ldots < T_n = T$.

Define function $r(\tau)$ as the limit of pointwisely converging sequence of functions determined by the relationship

$$r_n(\tau) = \frac{u_n(\tau)}{\varphi(\tilde{x}_\tau)} e^{-\frac{1}{2}\lambda_1^2 \int_\tau^{T_i} \sigma_s^2 ds}, \quad (20)$$

where $\tau \in (T_{i-1}, T_i]$, $u_n(\tau)$ are given functions, while the sequence $u_n(\tau)$ as $n \to +\infty$ is supposed to converge pointwisely to the function $u(\tau)$ for a uniform partition.

Substituting in (19) instead of $r(\tau)$ sequence (20) one arrives to the relationship

$$\tilde{a}_{T_j}^n = \sum_{i=1}^{j} \int_{T_{i-1}}^{T_i} \frac{u_n(\tau)}{\varphi(\tilde{x}_\tau)} d\tau \cdot \varphi'(x_t)\Big|_{x_t = \tilde{x}_{T_j}},$$

or

$$\tilde{a}_{T_j}^n = \int_0^{T_j} \frac{u_n(\tau)}{\varphi(\tilde{x}_\tau)} d\tau \cdot \varphi'(x_t)\Big|_{x_t = \tilde{x}_{T_j}}.$$

Ultimately, realizing limit transition as $n \to +\infty$ and within the framework of uniform partition one arrives to the formula describing the continuous distribution of the amount of assets in time under the observable realization of asset price $\tilde{x}_t$:

$$\tilde{a}_t = \int_0^t \frac{u(\tau)}{\sqrt{\tilde{x}_\tau}\,\sin(b\ln\tilde{x}_\tau)} d\tau \cdot \frac{\partial}{\partial x_t}\left(\sqrt{x_t}\,\sin(b\ln x_t)\right)\Big|_{x_t=\tilde{x}_t} . \quad (21)$$

By making use of the same arguments and taking into account relationships (6), (9), (15), (20) write down the value of profit at the moment of time T for the observable realization of asset price $\tilde{x}_t$:

$$\tilde{p}_T = \int_0^T \frac{u(t)}{\sqrt{\tilde{x}_t}\,\sin(b\ln\tilde{x}_t)} dt \cdot \sqrt{\tilde{x}_T}\,\sin(b\ln\tilde{x}_T) - \\ -\int_0^T e^{-\frac{1}{2}\lambda_1^2\int_t^T \sigma_s^2 ds} u(t)dt \quad . \quad (22)$$

Note that the transition to the control function $u(t)$ makes it possible to get rid of the explicit dependence on $\tilde{x}_t$ in the second term of formula (22)corresponding to the cash flow processed by the control system by the moment T .

When the control function $u(t)$ is represented by any a priori given piecewise constant nonnegative function which is not identically equal to zero, the first term in formula (22) is strictly positive while the second term may be taken arbitrary small because of the second condition of the Theorem. Thus, the constructed portfolio management really provides the profitableness of an investment portfolio on the time interval $[0,T]$ that makes the proof of the Theorem completed. ♦

Remark 1. It is worth noting that relationship (21) explicitly does not depend both on c_t and on volatility σ_t from equation (1). Thus, to construct the required management defining the amount of assets in portfolio there is no necessity to identify the pointed out factors to provide the profitableness of portfolio. On the other hand from formula (22) one can see that the increasing of integrated volatility leads to the essential growth of profit in time. ♦

Remark 2. One can easily check that nonnegative values of the function $u(t)$ provide the system of control to take the long position, i.e. $\tilde{a}_t \geq 0$ for arbitrary t . ♦

Remark 3. Note that the constructed management provides under certain conditions the profitableness of portfolio but the optimality of such management is not guaranteed. In other words the existence of some other management providing higher profitableness is possible. ♦

Remark 4. Note that the stochastic integral on the right hand side of formula (5) is regarded as the limit of the following sums sequence

$$\tilde{S}_n(t) = \sum_{i=1}^{j} \tilde{x}_{T_i}\left(\tilde{a}_{T_i} - \tilde{a}_{T_{i-1}}\right) \quad (23)$$

for $t \in \left(T_{j-1}, T_j\right]$, obtained in the process of the time interval $[0,T]$ partition $0 = T_0 < T_1 < \ldots < T_n = T$ and converging in $L_2[0,T]$ norm as $n \to +\infty$. The values $\tilde{a}_{T_i} = a(T_i, \tilde{x}_{T_i})$ are defined by formula (21). Thus, the supposition of sums (23) convergence in $L_2[0,T]$ norm imposes certain restrictions on the process x_t . By making use of Ito's formula the stochastic integral on the right hand side of relationship (4) may be presented as the sum of Riemann integral determined on the trajectories of random process x_t and Ito's integral:

$$\int_0^t x_{\tau+d\tau} da_\tau = \int_0^t \psi_1(\tau, x_\tau)d\tau + \int_0^t \psi_2(\tau, x_\tau)dW_\tau , \quad (24)$$

where ψ_1, ψ_2 are smooth functions defined by the relationship $a_t = a(t, x_t)$ according to the formula

$$a_t = \int_0^t \frac{u(\tau)}{\sqrt{\tilde{x}_\tau}\,\sin(b\ln\tilde{x}_\tau)} d\tau \cdot \frac{\partial}{\partial x_t}\left(\sqrt{x_t}\,\sin(b\ln x_t)\right).$$

Thus, to provide the existence of corresponding integrals in relationship (24) one may use standard sufficient conditions either in the form of restrictions on the process x_t itself or in the form of restrictions on the factors c_t and $\sigma_t = \sigma(t)$ of stochastic differential equation (1). One can easily show that the existence of integrals on the right hand side of relationship (24) when $t \in [0,T]$ implies the convergence of sums (23) in $L_2[0,T]$ norm almost sure. ♦

Remark 5. Note that the value of γ to provide the Theorem statement may be individual for different realizations $\tilde{x}_t$ of the random process x_t . ♦

Remark 6. It is worth noting that the Theorem statement remains valid in the case when volatility $\sigma_t = \sigma(t,\omega)$ is a random function of time, but it should not depend on the process x_t .

III. Some usefull estimates

It is reasonable that the construction of control function $u(t)$ as well as the width of the price band want further detailing as they are to be matched to the duration of investment, the distribution of invested cash flow in time and the global dynamics of integrated volatility within the framework of condition 2 of the Theorem.

Take into account the situation when the whole amount of cash, which is denoted by V , is deposited on the broker's account of an investor. In this situation the control function $u(\tau)$ is considered to be a constant and one may put $u(\tau) = u_0$. The time horizon of investments $[0,T]$ either is given in advance or to be chosen while the profitableness of an investment portfolio is to be provided on the pointed out time interval.

Take into consideration the integrated volatility

$$J(t) = \int_0^t \tilde{\sigma}_s^2 ds$$

on the time interval $[0,T]$. There is strong evidence, confirmed by the numerous experimental data for the broad class of high liquid assets, that the value of integrated volatility oscillates in the vicinity of a linear function and, consequently, admits approximation

$$J(t) \approx \alpha \cdot t, \tag{25}$$

where α is an a priori known at the initial moment $t=0$ quantity. Moreover, the value of α may be different for different type of assets.

Now suppose that relationship (25) is fulfilled not approximately but precisely and correspondingly $\gamma=\alpha$. Then the following estimates are valid

$$\tilde{p}_T = \int_0^T \frac{u_0}{\sqrt{\tilde{x}_t}\sin(b\ln\tilde{x}_t)}dt \cdot \sqrt{\tilde{x}_T}\sin(b\ln\tilde{x}_T) - \int_0^T e^{-\frac{1}{2}\lambda_1^2\int_t^T \sigma_s^2 ds} u_0 dt =$$

$$= \int_0^T \frac{u_0}{\sqrt{\tilde{x}_t}\sin(b\ln\tilde{x}_t)}dt \cdot \sqrt{\tilde{x}_T}\sin(b\ln\tilde{x}_T) - \int_0^T e^{-\frac{1}{2}\lambda_1^2\alpha(T-t)} u_0 dt \geq$$

$$\geq u_0 \int_0^T \frac{dt}{\sqrt{\tilde{x}_t}\sin(b\ln\tilde{x}_t)} \cdot \sqrt{\tilde{x}_T}\sin(b\ln\tilde{x}_T) - \frac{2u_0}{\lambda_1^2\alpha}.$$

If one admits that the specific scale of price variance is sufficiently less than the price of the first bargain struck by the control system: $\frac{|\tilde{x}_t - \tilde{x}_0|}{\tilde{x}_0} << 1$, then it is not hard to estimate the time interval $[0,T]$ to provide the profitableness of an investment portfolio

$$T \approx \frac{2}{\lambda_1^2 \cdot \alpha}. \tag{26}$$

Moreover, in this case it is possible to estimate the value of u_0 keeping in mind the general amount of investments V:

$$V \approx \frac{2u_0}{\lambda_1^2 \cdot \alpha}.$$

Actually dependence (26) determines the ratio between T, β and γ under conditions of the Theorem. Thus, the value of T from relationship (26) defines the distinctive time interval of investments to provide under the pointed out additional conditions the portfolio profitableness. Simultaneously it determines the value of λ_1^2 and consequently the value of β, i.e. the width of the price band in accordance with transcendental equation (18). Ultimately, the whole amount of investments V defines the value of control function u_0. Note that the diminishing of the price band width leads to the decreasing of T value but on the other hand increases the risk of the price to pierce the band borders.

For different high liquid assets the value of α corresponding to the one year time unit interval varies in the range between 0.15 and 0.3. In table 1 the results of numerical evaluation for T as the dependence of α and β are presented.

TABLE 1. THE RESULTS OF NUMERICAL EVALUATION FOR T AS THE DEPENDENCE OF α AND β.

α	β	T, months
0,2	1,1	0,44
0,2	1,2	1,61
0,2	1,3	3,32
0,2	1,4	5,44
0,2	1,5	7,86
0,2	1,6	10,51
0,2	1,7	13,31
0,2	1,8	16,23
0,2	1,9	19,23
0,2	2,0	22,28

Note, that the described control system can provide portfolio profitableness even on the time intervals when the trend of the asset price slumps but under the condition of the integrated volatility sufficiently sharp growth. To demonstrate this fact consider the chart at fig. 2 presenting price dynamics of depositary receipts (ADR) on Unified Energy System of Russia (UESR) shares in the period from 9.04.2007 till 3.07.2007. Simultaneously the chart at fig. 3 demonstrates the effected trading profit dynamics.

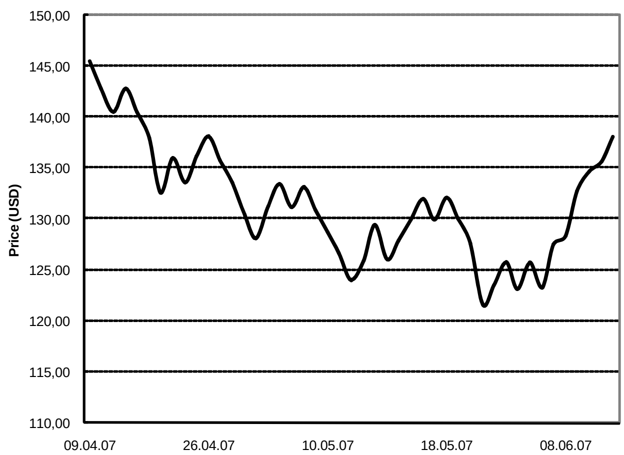

Fig. 2 Price dynamics of the UESR Depositary Receipts in the period from 9.04.2007 till 3.07.2007.

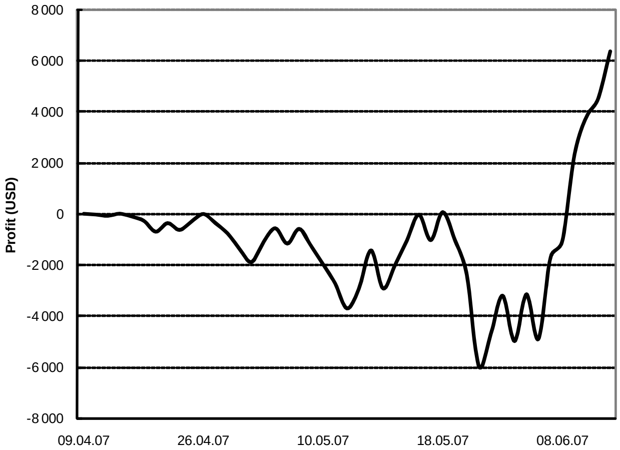

Fig. 3 The effected trading profit dynamics of the portfolio constituted by UESR Depositary Receipts in the period from 9.04.2007 till 3.07.2007.

It is remarkable that the price of the first bargain struck by the control system was 145.46 dollars per one ADR while the price of the last one was 138.07 dollars.

Pose the question if the proposed control system is arbitrage-free. Pay attention that the possibility of a price to pierce the low border of the price band may be regarded as the basic risk factor. To remove it expand the lower border of the price band to its utmost limit, i.e. to zero, thus realizing the limit transition $\beta \to +\infty$. Analyzing transcendental equation (18) one can easily see that in this case its minimal strictly positive root $z \to \pi$ and consequently $\lambda_1^2 \to \frac{1}{4}$. Note that second term in formula (22) taken with minus may be interpreted as the value of cash flow with the discount factor $\Delta = \frac{\lambda_1^2}{2}\alpha$ when relationship (25) is supposed to be fulfilled precisely. Thus, in the absence of the basic risk factor one arrives to the rate of return providing by the control system defined as follows $\Delta = \frac{\alpha}{8}$. For a one year time unit one may put $\alpha \approx 0.3$ and consequently $\Delta \approx 3.75\%$. Simultaneously the time interval defined by relationship (26) on which the portfolio profitableness is provided constitutes approximately 27 years. It is worth noting that the estimates obtained are very close to the basic characteristics such as the rate of return and the time to maturity for American Treasury Bills. Estimate the rate of return by making use of the different arguments. Define the rate of return as the ratio of the profit value on the time interval $[0, t]$ to the non-discounted cash flow $V = \frac{2u_0}{\lambda_1^2\alpha}$ multiplied by t:

$$\Delta = \left(u_0 t - \frac{2u_0}{\lambda_1^2\alpha}\right) \Big/ \left(\frac{2u_0}{\lambda_1^2\alpha}\cdot t\right) = \frac{1}{T} - \frac{1}{t},$$

where $T = \frac{2}{\lambda_1^2\alpha}$. As $t \to +\infty$ the rate of return $\Delta \to \frac{1}{T}$, i.e. to the quantity obtained earlier.

IV. Experimental and Theoretical Evidence of the Pricing Model Adequacy

Introduce the notion of theoretical profit for each instant $t \in [0, T]$ in accordance with formula (22)

$$\tilde{p}_t = \int_0^t \frac{u(\tau)}{\sqrt{\tilde{x}_\tau}\,\sin(b\ln\tilde{x}_\tau)}d\tau\cdot\sqrt{\tilde{x}_t}\,\sin(b\ln\tilde{x}_t) - \\ -\int_0^t e^{-\frac{1}{2}\lambda_1^2\int_\tau^t \sigma_s^2 ds} u(\tau)d\tau. \qquad (27)$$

Note that theoretical value of profit according to formula (27) may be estimated from above and below by making use of the limit values of volatility ($\sigma_t = 0$ and $\sigma_t = +\infty$) as follows

$$\tilde{p}_t^L \le \tilde{p}_t \le \tilde{p}_t^H,$$

where

$$\tilde{p}_t^L = \int_0^t \frac{u(\tau)}{\sqrt{\tilde{x}_\tau}\,\sin(b\ln\tilde{x}_\tau)}d\tau\cdot\sqrt{\tilde{x}_t}\,\sin(b\ln\tilde{x}_t) - \int_0^t u(\tau)d\tau,$$

$$\tilde{p}_t^H = \int_0^t \frac{u(\tau)}{\sqrt{\tilde{x}_\tau}\,\sin(b\ln\tilde{x}_\tau)}d\tau\cdot\sqrt{\tilde{x}_t}\,\sin(b\ln\tilde{x}_t).$$

Define the notion of real profit on the basis of formula (5) as the limit of the following sums sequence

$$\tilde{g}_n(t) = \tilde{a}_{T_i}\tilde{x}_{T_i} - \sum_{i=1}^{j}\tilde{x}_{T_i}\left(\tilde{a}_{T_i} - \tilde{a}_{T_{i-1}}\right) \qquad (28)$$

for $t \in \left(T_{j-1}, T_j\right]$, obtained in the process of the time interval $[0, T]$ partition: $0 = T_0 < T_1 < ... < T_n = T$ and converging in $L_2[0, T]$ norm as $n \to +\infty$ to the function denoted as $\tilde{g}(t)$. The values $\tilde{a}_{T_i} = a\left(T_i, \tilde{x}_{T_i}\right)$ are defined by formula (21). Note that the value of real profit does not depend explicitly on the volatility.

For high liquid assets traded both at American and Russian financial markets the pointed out estimates of theoretical profit have a high degree of direct correlation with real profit. Moreover, in all cases considered up till now when $u(\tau) = u_0 > 0$ the real profit curve was strictly in the range between $\tilde{p}_t^L$ and $\tilde{p}_t^H$. This fact, at least indirectly, confirms the adequacy of the introduced model.

The examples of the respective charts are presented at fig. 4 and 5 when the management of two portfolios containing respectively stocks of UESR and Dell Computers Corporation takes place.

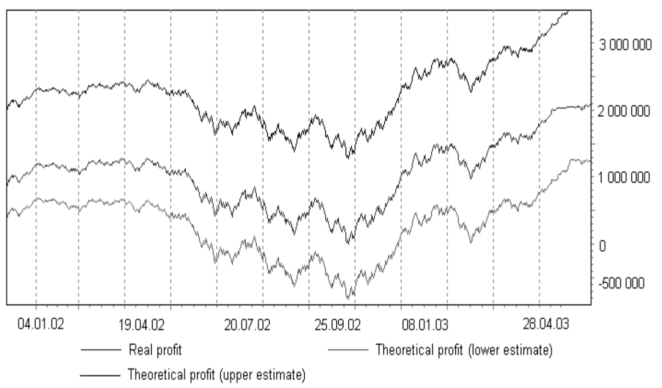

Fig. 4 Charts of theoretical profit estimates (in US dollars) and of real profit dynamics corresponding to the management of portfolio constituted by UESR stocks in the period from 3.01.2001 till 30.05.2003.

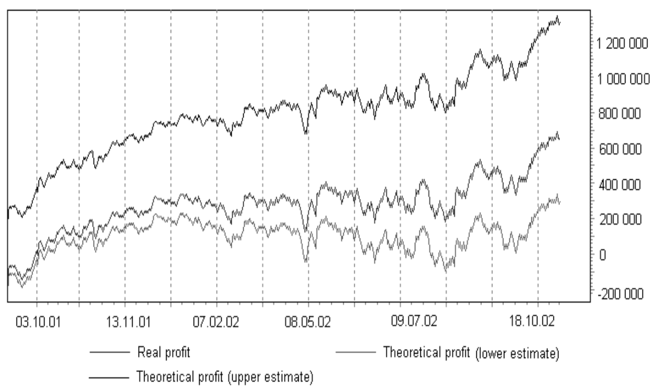

Fig. 5 Charts of theoretical profit estimates (in US dollars) and of real profit dynamics corresponding to the management of portfolio constituted by Dell Computers Corporation stocks in the period from 7.05.2001 till 19.11.2003.

It is worth noting that admissibility of condition 1 of the Theorem may be indirectly confirmed not only by experimental data but also in the framework of certain theoretical arguments presented below.

Take into account the pricing model $x_t = \exp(h_t)$, where x_t is a price of an asset while h_t is the auto-regression process of arbitrary finite order with nonrandom factors

$$h_t = c_1 h_{t-\Delta} + c_2 h_{t-2\Delta} + \ldots + c_n h_{t-n\Delta} + \varepsilon_t , \qquad (29)$$

where ε_t is a white noise, Δ is a given time interval.

As it is well known [4], relationship (29) can be rewritten in finite differences

$$a_1 \nabla^n h_t + a_2 \nabla^{n-1} h_t + \ldots + a_n \nabla h_t + a_{n+1} h_t = \varepsilon_t , \qquad (30)$$

where a_i are the factors defined by the values c_i while $\nabla^{k+1} h_t = \nabla(\nabla^k h_t)$ and $\nabla h_t = h_t - h_{t-\Delta}$.

Write down the continuous analogue of the difference equation (30) as follows

$$a_1 \Delta^n \frac{d^n h_t}{dt^n} + a_2 \Delta^{n-1} \frac{d^{n-1} h_t}{dt^{n-1}} + \ldots + a_n \Delta \frac{dh_t}{dt} + a_{n+1} h_t = \varepsilon_t . \qquad (31)$$

The solution to Cauchy problem for equation (31) on the interval $[0,t]$ can be presented in the form

$$h_t = \varphi(t) + \int_0^t G(t-s) dW_s , \qquad (32)$$

where $\varphi(t)$ is a deterministic function while $G(t-s)$ is the transfer function as an integrand in Ito's integral which is also not random. Consider t to be fixed and introduce the new notation $F(s) = G(t-s)$ for the convenience of calculations. As $F(s)$ is a deterministic function it is possible to apply integration by parts to Ito's integral in expression (32). Thus, the following transformations are valid

$$\int_0^t G(t-s) dW_s = \int_0^t F(s) dW_s = F(t) W_t - \int_0^t W_s F'(s) ds =$$
$$= G(0) W_t + \int_0^t W_s G'(t-s) ds = G(0) \int_0^t dW_s + \int_0^t W_s G'(t-s) ds$$

that leads to the relationship

$$d\left[\int_0^t G(t-s) dW_s\right] = G(0) dW_t + W_t G'(0) dt + + \left\{\int_0^t W_s G''(t-s) ds\right\} dt .$$

Ultimately, keeping in mind (32) the following representation for h_t may be written down

$$dh_t = \left[\varphi'(t) + W_t G'(0) + \int_0^t W_s G''(t-s) ds\right] dt + G(0) dW_t .$$

Applying to the function $x_t = \exp(h_t)$ Ito's formula one directly arrives to the relationship

$$dx_t = \left\{\left[\varphi'(t) + W_t G'(0) + \int_0^t W_s G''(t-s) ds\right] + \frac{1}{2} G^2(0)\right\} x_t dt + G(0) x_t dW_t ,$$

which is precisely the special case of equation (1) where

$$c_t = \varphi'(t) + W_t G'(0) + \int_0^t W_s G''(t-s) ds + \frac{1}{2} G^2(0) ,$$
$$\sigma_t = G(0) .$$

Thus, volatility σ_t does not depend on W_t and consequently on x_t. Accordingly, when the number of terms and the values of factors in auto-regression process (29) vary the volatility value $\sigma_t = G(0)$ is subjected to the transition.

V. The Procedure to Calculate Integrated Volatility

Consider the direct method of integrated volatility $\int_0^t \sigma_s^2 ds$ evaluation when $t \in [0,T]$ on the basis of relationships (27) and (28) omitting the intermediate stage of calculating the volatility σ_s itself inside the time interval $[0,T]$.

Note that there has been published substantial amount of literature concerning the problem of calculating integrated volatility including both original papers (see, for instance, [3]) and detailed reviews (see, for instance, [1]). Under rather general conditions it was shown the convergence in probability of specially constructed sums (realized variance) defined by the statistical data to the value of integrated volatility

$$\sum_{i=1}^{n}\left(\ln\frac{\tilde{x}_{t_i}}{\tilde{x}_{t_{i-1}}}\right)^2 \to \int_0^t \tilde{\sigma}_s^2 ds \text{ as } n\to+\infty .$$

Here the value $\tilde{x}_{t_i}$ corresponds to the price of the bargain struck at the instance t_i on the time interval $[0,t]$.

Nevertheless in a number of recent publications [15], [16] it was shown that the proposed algorithm is not robust towards different kind of errors arising as the result of the market microstructure effects, in particular, because of the bid-ask spread existence. Thus, the observable values are not $\tilde{x}_{t_i}$ but $\tilde{y}_{t_i} = \tilde{x}_{t_i} + \varepsilon_{t_i}$, where ε_{t_i} are independent random variables with zero mathematical expectations and finite dispersions. It turned out that under the pointed out circumstances the realized variance is not more the consistent estimate for integrated volatility. It is remarkable that the experimental data provides strong evidence of this theoretical inference.

One of the possible ways to design robust algorithm of integrated volatility evaluation is to make use of the following arguments. Equate the expression for theoretical profit (27) to the limit value $\tilde{g}(t)$ of real profit defined by sequence (28). Thus, considering $u(\tau)\equiv 1$ one arrives to the integral equation with respect to the unknown function $\nu(t)$

$$\int_0^t \nu(\tau)d\tau - \chi(t)\nu(t) = h(t) , \qquad (33)$$

where $\nu(t) = e^{\frac{1}{2}\lambda_1^2\int_0^t \sigma_s^2 ds} - 1$,

$$\chi(t) = \int_0^t \frac{d\tau}{\sqrt{\tilde{x}_\tau}\sin(b\ln\tilde{x}_\tau)}\sqrt{\tilde{x}_t}\sin(b\ln\tilde{x}_t) - g(t) ,$$

$h(t) = \chi(t) - t$.

Note that the transformations performed provide the fulfillment of condition $\nu(0)=0$.

The derived integral equation is a standard ill-posed problem as the functions $\chi(t)$ and $h(t)$ are not differentiable because of their dependence on the trajectory of Wiener process.

The desired function $\nu(t)$ for equation (33) is searched in the space $L_2[0,T]$ on the set of functions of bounded variation M. This set is a compact one owing to the Helly's second theorem [12] thus providing the boundedness of the inverse operator A^{-1} corresponding to the operator on the left hand side of relationship (33)

$$A\nu = \int_0^t \nu(\tau)d\tau - \tilde{\chi}(t)\nu(t) .$$

The existence of the inverse operator A^{-1} is provided by the easily verified triviality of the operator A kernel.

Thus, owing to the well known results [13] the inverse problem (33) has the unique quasisolution.

Recall that element $\tilde{\nu}\in M$ is called a quasisolution to equation (33) if the following relationship is valid

$$\rho_{L_2}\left(A\tilde{\nu},h\right) = \inf_{\nu\in M}\rho_{L_2}\left(A\nu,h\right) .$$

Because any continuous functions $\chi(t)$ and $h(t)$ in L_2 metrics may be approximated with arbitrary precision by continuously differentiable functions the above mentioned infimum may be made arbitrary small.

Thus, the integrated volatility on the time interval $[0,t]$ may be expressed via function $\nu(t)$ as follows

$$\int_0^t \sigma_s^2 ds = \frac{2}{\lambda_1^2}\ln\left[\nu(t)+1\right], \quad 0\le t\le T .$$

It is worth noting that the numerical solution to the derived integral equation (33) obtained on the basis of variational approach is robust towards small perturbations of functions $\chi(t)$ and $h(t)$ in L_2 metrics.

CONCLUSIONS

Within the framework of the proposed portfolio management the control system is not supposed to forecast the direction of price dynamics but carries out the effective procedure to diminish the weighted average price of the assets constituting portfolio. The usage of the considered strategy implies, in essence, two kinds of risks. The first one is stipulated by the possibility of a price to pierce the lower border of the corresponding price band. This kind of risks may be supervised by purchasing put options with strikes in the vicinity of the price band lower border. The second kind of risks deals with the possibility of the temporal sharp slowdown of the integrated volatility growth. Such situation may take place because of the approximate character of relationship (25). This kind of risks may lead to the temporal suspension of the control system functioning.

Finally, it is worth noting that the control system when a short position takes place may be constructed by making use of the analogues arguments. It suffices to scale prices with respect not to the lower as in the case of a long position trading but to the upper border of the corresponding price band and reiterating the previous arguments to shift boundary conditions (13) and (14) each one for another. Thus, basic formulas (21) and (22) preserve their structure but the selected root of transcendental equation (18) turns out to be different as in this case the quantity $\ln\beta$ is evidently to be negative.

REFERENCES

[1] Andersen T.G., Bollerslev T., Diebold F.X. Parametric and nonpara-metric measurement of volatility. *Handbook of Financial Econometrics*. Amsterdam: North Holland. 2002.

[2] Back K., Pliska S. "On the fundamental theorem of asset pricing with an infinite state space." *Journal of Mathematical Economics* 20, 1991. pp. 1-33.

[3] Barndorf-Nielsen O.E., Shephard N. Econometric analysis of realized volatility and its use in estimating stochastic volatility models. *Journal of Royal Statistical Society*, 2002, 64, part 2, pp. 253-280.

[4] Box G.E.P., Jenkins G.M. *Time series analysis. Forecasting and Control*. San-Francisco, California: Holden-Day, 1970.

[5] Duffie D. *Dynamic Asset Pricing Theory*. Princeton, NJ: Princeton University Press. 1996.

[6] Fleming W.H., Rishel R.W. *Deterministic and Stochastic Optimal Control*. New York: Springer-Verlag. 1975.

[7] Frittelli M. and Lakner P. "Arbitrage and free lunch in a general financial market model: the fundamental theorem of asset pricing." *Mathematical Finance*, 1995. pp. 89-94.

[8] Harrison J.M. and Kreps. D.M. "Martingales and arbitrage in multiperiod securities markets." *Journal of Economic Theory* 20, 1979. pp. 381-408.

[9] Kamke. E. *Differentialgleichungen: Losungsmethoden und Losungen, I, Gewohnliche Differentialgleichungen*, B. G. Teubner, Leipzig. 1977.

[10] Karatzas I. *Lectures on the Mathematics of Finance*. American Mathematical Society. 1997.

[11] Karatzas I., Lehoczky J.P., Shreve S.E. "Optimal portfolio and consumption decisions for a "small investor" on a finite horizon." *Journal of Control and Optimization* 25, 1987. pp. 1557-1586.

[12] Kolmogorov A.N., Fomin S.V. *Elements of functions and functional analysis theory*. Moscow, Nauka, 1989 (in Russian).

[13] Tikhonov A.N., Arsenin V.Ya. *The methods to study ill-posed problems*. Moscow, Nauka, 1986 (in Russian).

[14] Vavilov S.A., Ermolenko K.Yu. "The management of an investment portfolio in financial markets within the framework of an approach alternative to self-financing strategy." *Proceedings of World Congress on Engineering 2007*, London, U.K., 2007. pp. 1006-1009.

[15] Zhang L., Mykland P.A., Ait-Sakhalia Y. A tale of two time scales: determining integrated volatility with noisy high-frequency data. *Journ. Amer. Stat. Ass.* 2005, 100, No.472, pp. 1394-1411.

[16] Zhang L. Efficient estimation of stochastic volatility using noisy observations: a multi-scale approach. *Bernoulli*, 2006, 12, No. 6, pp. 1019-1044.