# Numerical Valuation of Fixed Rate Mortgages

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*Abstract*—**This paper considers the value of a fixed rate mortgage where the borrower has the choice to make early payment. The contract value is formulated in terms of integral equations with the optimal early exercise boundary embedded. A fast and effective algorithm has been established for solving the problem numerically. A novelty quadrature method has been derived to handle the singular integrals. Simulations have been made with a broad range of parameters to validate the performance of the algorithm. Analytical features of the problem are illustrated with numerical examples.** ***Keywords: fixed rate mortgage, early exercise, integral quadrature, numerical valuation***

## 1 The model and the problem formulation

Many option pricing problems are formulated as free boundary problems [21, 22, 23, 25]. The classical example is the valuation of American put option [5, 3, 18]. These free boundary problems usually don't have closed form solutions. Rather, efforts have been focused on finding fast and effective numerical schemes as well as the asymptotic expansions of the free boundary [25, 3, 19].

Here we consider an amortized mortgage contract with a given duration $T$ (years) and a fixed mortgage interest rate $c$ (year$^{-1}$), where the borrower is allowed to close the contract prematurely by settling the loan balance $M(\tau)$ at any time $\tau \leq T$. Here $m$ is the rate of payment, the amount of dollars that the borrower pays back to the bank per unit time. Because the loan is amortized, the borrower pays equal amount of $mdt$ (dollars) for each time period $dt$, which consists of both principal and interest. This kind of mortgage is popular in practice because it is easier for the borrower to make financial planning, relative to other kinds of loans with nonuniform installments. At any time $\tau$ during the term of the mortgage considered hereof, the outstanding loan balance owed by the borrower to the bank (lender), $M(\tau)$ is given by

$$M(\tau) = \frac{m}{c}\Big\{1 - e^{c(\tau - T)}\Big\},$$

which is the uniquely determined by the differential equation $dM(\tau) = cM(\tau)dt - md\tau$ with $M(T) = 0$.

From the borrower's point of view, he needs to decide, when he has sufficient amount of money, whether to settle the loan by paying off the balance or to invest on the financial market, expecting the return from the investment be more than enough to cover the cost of subsequent mortgage payments for certain amount of time. On the other hand, the lender may, for many good reasons, want to know the fair value of such a contract. For a bank (or a mortgage loan company) holding a large portion of such contracts in its portfolio, the value of these mortgages may have significant impact on its credit rating and financing cost.

Assume the borrower always has sufficient capital to pay back the outstanding balance at any time, then at any moment while the contact is in effect, the decision of the borrower on whether or not to close the contract depends on the rate of (short term) return that an investment can yield on the financial market. If an overall expected future return rate is low (relative to $c$) for certain period , one should choose to close the contract early. On the other hand, if an overall expected return rate is high relative to $c$ for certain period, one should choose to invest in the market with the amount of $M(t)$ less the current obligatory payment of $m$ per unit time.

In this paper, we assume the short term market return rate that the borrow can earn follows Vasicek model [24], $r_\tau$, which is described by the stochastic differential equation

$$dr_t = k(\theta - r_\tau)d\tau + \sigma\, dW_\tau$$

where $k, \theta$, and $\sigma$ are assumed to be positive known constants and $W_\tau$ is the standard Wiener process. Here the units for $k, \theta, \sigma$, and $W_t$ are year$^{-1}$, year$^{-1}$, year$^{-3/2}$ and year$^{1/2}$ respectively.

We follow the usual convention to use the time to maturity date of the contract, $t := T - \tau$, instead of real time $\tau$ for the convenience of mathematical analysis. To find such an optimal strategy, we introduce a function $V(r,t)$ being the (expected) value of the contract at time $t$ and current market return rate $r_t = x$. This value can be interpreted as an asset that the contract holder possesses, or a fair price that a third party buyer would have to offer to the contract holder in taking over the contract. Since the borrower has the choice to close the contract at any time $t$, in response to the market reality, by paying off $M(t)$, we have

$$0 \leq V(x,t) \leq M(t) \qquad \forall x \in \mathbf{R},\ t \geq 0.$$

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This automatically implies that $V(x,0)=0$ for all $x$. The value $V$ is calculated according to the borrower's optimal decision, and the optimal decision for the borrower is to close the mortgage contract at the first time that the short term market return rate $r_t$ is below $h(t)$, the unknown optimal early exercise boundary, or equivalently, the first time that $V(x,t)$ is equal to $M(t)$. According to standard mathematical finance theory [25, 21, 23], what we need to find is a $(h,V)$, which solves the following free boundary problem:

$$\begin{cases} \mathbf{L}(V)=m & \text{for } \ x>h(t), t>0, \\ V(x,t)=M(t) & \text{for } \ x\le h(t), t>0, \\ V_x(x,t)=0 & \text{for } \ x\le h(t), t>0, \\ V(x,0)=0 & \forall\ x\ge h(0)=c. \end{cases} \qquad (1)$$

where the operator $\mathbf{L}$ is defined as

$$\mathbf{L}(V)=\frac{\partial V}{\partial t}-\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2}-k(\theta-x)\frac{\partial V}{\partial x}+xV \qquad (2)$$

Similar problems have been discussed from option pricing viewpoint in [11, 13, 20, 17, 12, 7]. The mathematical well-posedness of the problem is shown in [16]; and the authors have proved that problem (1) admits a unique solution which is smooth up to to the free boundary $x=h(t)$. Efforts have also been made to solve similar problems numerically. For instance, a finite Crank-Nicolson finite difference approach is used in [1] to solve the partial differential equations. A bivariate binomial iteration scheme is proposed in [6]. A numerical integral equation method is recently introduced in [8], and trapezoid quadratures are applied to iteratively solve for the free boundary. One notices (see [15, 25, 26, 27], for instance) that usual numerical techniques such as finite difference or binomial method typically provide poor accuracy and stability, which are mainly attributed to the difficulty in handling free boundary conditions, in addition to low convergence rate. In this work we apply similar boundary integral approach (see, for instance, [10, 9, 14]) as used in [3, 8, 4] to solve for $V(x,t)$ numerically. We first provide the integral representations of the solution with the unknown free boundary embedded. Based on these integral identities, we are able to design a fast and effective algorithm to solve for the free boundary iteratively. Then the mortgage contract value $V$ is computed by numerical integrations once the early exercise boundary is known. A novelty numerical integration quadrature is derived to handle the highly singular integral equation at hand. The implementation and performance of our algorithm as well as insightful numerical examples are duly discussed.

## 2 Integral equation formulation of $V$

One can simplify the nonlinear operator defined in (2) by transformations. First let $\psi(x,t):=\dfrac{c}{m}\Big[M(t)-V(x,t)\Big]$, then the PDE in (1) is equivalent to

$$\psi_t-\frac{\sigma^2}{2}\psi_{xx}-k(\theta-x)\psi_x+x\psi = (x-c)(1-e^{-ct}),$$

where the financial interpretation of $\psi$ is a dimensionless quantity measuring the advantage of deferring closing the mortgage by investing in market. $M(t)-V(x,t)$ represents the amount of premium loss if the contract is closed at the current time $t$ and market return rate $x$ and if it is actually not optimal to do so. We remark that $\psi(x,t)\le 1-e^{-ct}$. This upper bound of $\psi(x,t)$ corresponds to the fact that $V\ge 0$. Indeed, one can show that $1-e^{-ct}$ is a super-solution so that by comparison

$$\psi(x,t)<1-e^{-ct} \qquad \forall\, x\in\mathbf{R}, t>0.$$

Also, differentiating in $x$ one sees that

$$\Big\{\frac{\partial}{\partial t}-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}-k(\theta-x)\frac{\partial}{\partial x}+(x+k)\Big\}\psi_x = 1-e^{-ct}-\psi\ge 0 \qquad \text{if} \quad \psi>0.$$

The maximum principle then implies that $\psi_x(x,t)\ge 0$ for all $x\in\mathbf{R}, t\ge 0$. Therefore, there exists a function $h:(0,-\infty)\to[-\infty,\infty)$ such that for each $t>0$,

$$\psi(x,t)>0 \qquad \Longleftrightarrow \qquad x>h(t).$$

Let $\phi(x,t):=e^{-g(x,t)}\psi(x,t)$, with $g(x,t)$ to be to be determined shortly. When $\phi>0$ we have $\psi>0$ and equation for $\psi$ is transformed to the following equation for $\phi$:

$$\phi_t-\tfrac{\sigma^2}{2}\phi_{xx}-[\sigma^2 g_x+k(\theta-x)]\phi_x+q\phi = (x-c)(1-e^{-ct})e^{-g}$$

where

$$q:=g_t-\tfrac{\sigma^2}{2}g_{xx}-g_x\{\tfrac{\sigma^2}{2}g_x+k(\theta-x)\}+x.$$

We want to find a special $g$ such that $q\equiv 0$. To this end we choose

$$g(x,t) \ = \ \frac{k}{\sigma^2}\Big(x+\frac{\sigma^2}{2k^2}-\theta\Big)^2+\Big(k+\frac{\sigma^2}{2k^2}-\theta\Big)t.$$

The equation for $\phi$ becomes

$$\phi_t-\tfrac{\sigma^2}{2}\phi_{xx}-\{kx+\frac{\sigma^2}{k}-k\theta\}\phi_x=(x-c)(1-e^{-ct})e^{-h}$$

Finally, we make the change of variables

$$y=\tfrac{k^{1/2}e^{kt}}{\sigma}\Big[x+\tfrac{\sigma^2}{k^2}-\theta\Big],$$
$$s=e^{2kt},$$
$$W(y,s)=\tfrac{2\sqrt{\pi}k^{3/2}}{\sigma}\phi(x,t).$$

After these steps of change of variables and after changing the boundary conditions accordingly, the original system (1.1) for $(V,h)$ now becomes the following system for $(W,\eta)$

$$\begin{cases} W_s - W_{yy} = f(y,s) & \text{if } \; y > \eta(s), s > 1, \\ W(y,s) = 0 & \text{if } \; y \le \eta(s), s > 1, \\ W_y(y,s) = 0 & \text{if } \; y \le \eta(s), s > 1, \\ W(y,1) = 0 & \forall y \ge \eta(1) = \omega. \end{cases} \tag{3}$$

where

$$f(y,s) = \sqrt{\pi}(s^{\frac{c}{2k}} - 1)s^{-2-\frac{\sigma^2}{4k^3}-\frac{c-\theta}{2k}}$$
$$\times(y - \frac{\sqrt{k}}{\sigma}(c-\theta+\frac{\sigma^2}{k^2})\sqrt{s})e^{-(\frac{y}{\sqrt{s}}-\frac{\sigma}{2k^{3/2}})^2},$$
$$\omega = \frac{\sigma}{k^{\frac{3}{2}}} + \frac{(c-\theta)k^{\frac{1}{2}}}{\sigma}.$$

Since the fundamental solution associated with the operator $\partial_s - \frac{1}{4}\partial^2_{yy}$ is known as $\frac{e^{-y^2/s}}{\sqrt{\pi s}} := \Gamma(y,s)$. Using Green's identity, the solution $W$ to the differential equation in (3) can be expressed as

$$W(y,s) = \int_1^s \int_{\eta(\xi)}^{\infty} \Gamma(y-\rho, s-\xi) f(\rho,\xi)\, d\rho d\xi \tag{4}$$

Without loss of generality, we assume $m = 1$. Translate integral representation for $W$ in (3) into the integral representation for $V$, and simplify terms, we have

$$V(x,t) = M(t) - \int_0^t \int_{h(\tau)}^{\infty} \frac{1}{\sqrt{\pi\alpha}} e^{\beta_1 + \frac{1-s}{k}y - \frac{[\beta_2+x-ys]^2}{\alpha}}$$
$$\times(1-\frac{y}{c})(e^{-c\tau}-1)dyd\tau, \tag{5}$$

where $s, \alpha, \beta_1, \beta_2$ are functions of $(x,y;t,\tau)$ defined by

$$s = e^{k(t-\tau)} \tag{6}$$

$$\alpha = \frac{\sigma^2}{\theta}(s^2-1) \tag{7}$$

$$\beta_1 = \frac{3\sigma^2}{4k^3}s^2 + \frac{1}{k}(\theta - \frac{\sigma^2}{k^2})s + (-\theta + \frac{\sigma^2}{2k^2} + k)$$
$$\times(t-\tau) + \frac{1}{k}(-\theta + \frac{\sigma^2}{4k^2})$$

$$\beta_2 = \frac{\sigma^2}{2k^2}s^2 + (\theta - \frac{\sigma^2}{k^2})s + (-\theta + \frac{\sigma^2}{2k^2})$$

Let $V(x,t) = M(t) - U(x,t)$. The inside integral on the right hand side of (5) is a convolution of a linear function of $y$ and an exponential of a quadratic form of $y$, which can be explicitly calculated. And the result of such a calculation gives

$$U(x,t) = \frac{\sigma}{2c\sqrt{\pi k}} \int_0^t (e^{-c\tau}-1)e^{\beta_3 - \hat{h}^2} s^{-1}\sqrt{1-s^2}d\tau$$
$$+\frac{1}{2c}\int_0^t (e^{-c\tau}-1)e^{\beta_3-\hat{h}^2}s^{-1}[(-c+\theta-+\frac{\sigma^2}{2k^2})$$
$$+(x-\theta+\frac{\sigma^2}{k^2})s^{-1} - \frac{\sigma^2}{2k^2}s^{-2}]d\tau, \tag{8}$$

where

$$\beta_3 = (-\theta + \frac{\sigma^2}{2k^2} + k)(t-\tau) + (\frac{\theta}{k} - \frac{3\sigma^2}{4k^3} - \frac{x}{k})$$
$$+(-\frac{\theta}{k} + \frac{\sigma^2}{k^3} - \frac{x}{k})s^{-1} - \frac{\sigma^2}{4k^3}s^{-2}$$

$$\hat{h} = \begin{cases} \frac{s}{\sqrt{\alpha}}\{(h(\tau) - \theta + \frac{\sigma^2}{2k^2}) - (-\frac{\theta}{k} + \frac{\sigma^2}{k^2} \\ \quad +x)s^2 + \frac{\sigma^2}{2k^2}s^{-2}\}, \quad \text{if} \quad \tau \ne t; \\ 0, \quad \text{if} \quad \tau = t. \end{cases} \tag{9}$$

A direct differentiation with respect to $x$ yields a representation of $U_x(x,t)$ in terms of a sum of two integral terms, hereafter called $I$ and $II$ for notational simplicity:

$$U_x(x,t) = I + II \tag{10}$$

where

$$I = \frac{1}{2c\sqrt{\pi}}\int_0^t \frac{s}{\sqrt{\alpha}}(1-e^{-c\tau})e^{\beta_4 - \hat{h}^2}G_1(x,t;y,\tau)d\tau \tag{11}$$

$$II = \frac{1}{2ck}\int_0^t (1-e^{-c\tau})e^{\beta_4}G_1(x,t;y,\tau)\text{Erfc}(\hat{h})d\tau, \tag{12}$$

$$\beta_4 = (-\theta + \frac{\sigma^2}{2k^2})(t-\tau) + (-\frac{\theta}{k} - \frac{3\sigma^2}{4k^3} - \frac{x}{k})$$
$$+(\frac{\theta}{k} + \frac{\sigma^2}{k^3} - \frac{x}{k})s^{-1} - \frac{\sigma^2}{4k^3}s^{-2} \tag{13}$$

$$G_1(x,t;y,\tau) = -\frac{\sigma^2}{k^2} + [2(h(\tau)-c) + \frac{\sigma^2}{k^2}]s^{-1}$$
$$+\frac{\sigma^2}{k^2}s^{-2} - \frac{\sigma^2}{k^2}s^{-3} \tag{14}$$

$$G_2(x,t;y,\tau) = (-c+\theta-\frac{\sigma^2}{2k^3})+(k-c-x+\theta-\frac{3\sigma^2}{2k^3})s^{-1}$$

$$+(x-2\theta-\frac{3\sigma^2}{2k^3})s^{-2}-\frac{\sigma^2}{2k^3}s^{-3} \quad (15)$$

Now all the free boundary conditions for $V$ as specified in (1) can be written in terms of $U$ as

$$U(h(t),t) = 0, \quad (16)$$

$$U_x(h(t),t)) = 0, \quad (17)$$

where both $U$ and $U_x$ are functionals of $h$, whose integral representations are given by (8) and (10). Based on these integral identities, one can construct numerical iteration algorithms to find $(V,h)$. Having this said, we would like to point out that $G_1(x,t;y,\tau)$ appearing in first integral term on the right hand side of (10) is highly singular as $\tau \to t$. This singularity indeed leads to poor performance of our algorithm in terms of accuracy and stability, as verified by our numerical simulations. This numerical difficulty is solved by an improved integral quadrature outlined in the subsequent section (3.3).

## 3 Numerical Methods and Algorithm

We seek to numerically solve for $h(t)$ from the integral identity (17). Since $U_x$ is an operator from $h \in C^1((0,\infty))$ to $U_x(x,t)$ defined by (10), the problem is to find $h \in C([0,\infty)) \cap C^\infty((0,\infty))$ such that $U_x[h](t) \equiv 0, \forall t \geq 0$. We remark that both $U$ and $U_x$ are functionals of $h$, i.e., for fixed $t$, $U(h(t),t)$ and $U_x(h(t),t)$ are not only affected by $h(t)$, but also $h(\tau), \forall \tau < t$. First of all, from (1.1), we know that $h(0) = c$. Financially this means that as time approaches to maturity of the contract, the optimal early exercise boundary must approach to the mortgage rate $c$, otherwise an arbitrage opportunity will be possible. Then at each moment $t > 0$, the value of the free boundary $h(t)$ must be chosen such that the integral identity (17) holds. This provides the theoretical foundation for our numerical schemes.

### 3.1 Newton's iteration Scheme

We now apply Newton's method to solve for the unknown function $h$ iteratively. Suppose we have already found $h$ for $t = t_0, t_1, t_2, ..., t_{n-1}$, and want to find $h$ at $t = t_n$ . Start with an initial guess $h^1(t_n)$, say $h^1(t_n) \equiv h(t_{n-1})$. Plug $h(t_i)$, $i = 1, ..., n$, back into the integral equation (10), we will have the value of $U_x(h^1(t_n), t_n)$. If $U_x(h^1(t_n), t_n) = 0$, then the iteration ends. If $U_x(h^1(t_n), t_n) \neq 0$, we update the initial guess of $h(t_n)$ to $h(t_n) = h^2(t_n)$ with reasonable amount of correction. Then we plug this updated $h(t_i)$, $i = 1, ..., n$, back into the integral equation (10) again, and check if $U_x(h^2(t_n), t_n) = 0$, and the procedure repeats until a positive integer $j$ is reached such that$U(h^j(t_n), t_n) = 0$ for specified error tolerance level. This Newton's scheme can be summarized by

$$h^j(t_n) = h^{j-1}(t_n) - \frac{U_x(h^{j-1}(t_n), t_n)}{2f(h^{j-1}(t_n), t_n)},$$

where

$$f(h^{j-1}(t_n), t_n) = \frac{1}{\sigma^2}(1-e^{-ct_n})(1-\frac{h^{j-1}(t_n)}{c}) \quad (18)$$

represents the rate of corrections needed to be made for updating $h(t_{j-1})$ at interation step $j-1$. The definition of $f$ is provided in the next subsection (3.2). We remark that in the first interval $(0, \Delta t]$, one can simply pick the very first initial guess $h^1(t_1) = h(0) = c$.

### 3.2 Rate of correction for Newton's iteration

Strictly speaking $U_{xx}$ has a jump at $x = h(t)$ for each fixed $t$. Because of this, one cannot simply differentiate (10 ) to find $U_{xx}(h(t),t)$. However, one can find the limits of $U_{xx}(x,t)$ as $x$ approaches $h(t)$ from both above and below the free boundary $h(t)$. And we are going to use the average of these two limits to approximate the rate of correction for each step of Newton's iteration. First, we know that, for fixed $t > 0$,

$$\lim_{x \to h(t)_-} U_{xx}(x,t) \equiv 0, \quad (19)$$

because in the early exercise region (where $x \leq h(t)$) $U = M(t) - V \equiv 0$. To calculate $\lim_{x \to h(t)} U_{xx}$ in the continuation region (i.e. the region in $t-x$ plane where $x > h(t)$), we first notice, by regularity argument, that $U_x$ exists and is continuous for all $x \in R$. In particular, $\lim_{x \to h(t)_+} U_x = U_x(h(t),t) \equiv 0$. Now let $V \to M(t)$ in the PDE in (1), we have

$$\lim_{x \to h(t)_+} U_{xx} = \lim_{x \to h(t)_+} V_{xx} = \frac{2}{\sigma^2}(1-e^{-ct})(1-\frac{x}{c}).$$

We use the average of $\lim_{x \to h(t)_+} U_{xx}$ and $\lim_{x \to h(t)_-} U_{xx}$ to approximate the proper amount of rate of correction needed for updating the initial guess of $h(t_n)$ at each step of iterations for each $t = t_n$, and the expression is given in (18). Numerical experiments show that the Newton's iteration designed upon this correction converges very fast. On average, it takes less than 2 iterations to converge to true solution with a error tolerance level of $10^{-9}$.

### 3.3 An Improved Integration Quadrature

Since the integrand of the first integral term in (10) is highly singular as $\tau \to t$, usual numerical integration methods will unlikely produce stable algorithm with high convergence. To proceed, notice that the integral needing special care is essentially of the form

$$\int_0^t F(t,\tau)\frac{1}{\sqrt{1-s^{-2}}}d\tau := E(t), \quad (20)$$

where $s := s(t,\tau)$ is defined by (6), and $F(t,\tau)$ is smooth and uniformly bounded on $[0,t]$.

One possible way to numerically evaluate (20) is to appeal to the Taylor expansion

$$\frac{1}{\sqrt{1-s^{-2}}} = 1 + \frac{1}{2}s^{-2} + \frac{3}{8}s^{-4} + \frac{5}{16}s^{-6} + \frac{35}{128}s^{-8}....$$

While it sounds plausible, the main problem with this Taylor expansion approach is that the series diverges as $\tau \to t$. Due to the complexity of the $F(t,\tau)$ in terms of the integral in (10), it is not clear whether it is possible to achieve a prescribed accuracy requirement with only finite terms of the Taylor series, or how many terms to keep, if it is indeed possible.

Alternatively, one can rewrite (20) as

$$E(t) = -\frac{1}{k}\int_{\tau=0}^{\tau=t} F(t,\tau)s^2 d\sqrt{1-s^{-2}}, \qquad (21)$$

after which usual numerical integration rules, say, trapezoid quadrature, can be applied. Numerical simulations show that our subsequent algorithm, if substituted with the integration quadrature based on (21), is still stable, but tend to converge only half as fast as the algorithm implemented with the improved quadrature outlined as follows.

To find such an integral quadrature, let

$$z = \arccos(e^{-\lambda(t-\tau)}),$$

with $\lambda$ to be determined later. A direct computation gives $d\tau = -\frac{\sqrt{1-e^{-2\lambda(t-\tau)}}}{\lambda e^{-\lambda(t-\tau)}}dz$, and (20) becomes

$$\begin{aligned} E(t) &= -\int_{\tau=0}^{\tau=t} \frac{F(t,\tau)}{\sqrt{1-s^{-2}}}\frac{\sqrt{1-e^{-2\lambda(t-\tau)}}}{\lambda e^{-\lambda(t-\tau)}}dz \\ &= -\int_{\tau=0}^{\tau=t} \sqrt{\frac{1-e^{-2\lambda(t-\tau)}}{1-s^{-2}}}\frac{F(t,\tau)}{\lambda e^{-\lambda(t-\tau)}}dz. \end{aligned}$$

Let

$$P(t,\tau) = \begin{cases} \sqrt{\dfrac{1-e^{-2\lambda(t-\tau)}}{1-e^{-2k(t-\tau)}}}, & \text{if } \tau < t; \\ \sqrt{\frac{\lambda}{k}}, & \text{if } \tau = t, \end{cases} \qquad (22)$$

then (20) becomes

$$E(t) = -\int_{\tau=0}^{\tau=t} F(t,\tau)\frac{P(t,\tau)}{\lambda e^{-\lambda(t-\tau)}}dz. \qquad (23)$$

Clearly, the integrand of the integral in (23), as a function in $\tau$ for $t$ fixed, is smooth and uniformly bounded on $[0,t]$ on condition that $\lambda > 0$ and $F(t,\tau)$ is smooth and uniformly bounded.

In theory, one can choose any $\lambda > 0$ to apply the integral quadrature (23) to evaluate (10), the integral equation we are interested in. However, a particular choice of

$$\lambda = \theta - \frac{\sigma^2}{2k^2} \qquad (24)$$

is convenient for this purpose since such a choice of $\lambda$ will lead to vanishing the $(-\theta + \frac{\sigma^2}{2k^2})(t-\tau)$ components in $\beta_4$. With $\lambda$ defined in (24), we apply the integration quadrature (23) to the first integral term in (10), get

$$I = \frac{1}{\sqrt{\pi}c}\int_0^t (e^{-c\tau}-1)e^{\beta_5}\frac{\sqrt{\theta}}{\sigma\lambda}G_1 P e^{\hat{h}^2}dz,$$

where

$$\beta_5 = (-\frac{\theta}{k} - \frac{3\sigma^2}{4k^3} - \frac{x}{k}) + (\frac{\theta}{k} + \frac{\sigma^2}{k^3} - \frac{x}{k})s^{-1} - \frac{\sigma^2}{4k^3}s^{-2} \quad (25)$$

To evaluate $II$, we factor out $s^{-1}$, i.e., $e^{-\theta(t-\tau)}$, then we get

$$e^{-\theta(t-\tau)}d\tau = -\frac{\sqrt{1-e^{2(\lambda-\theta)(t-\tau)}}}{\lambda}e^{(\lambda-\theta)(t-\tau)}dz.$$

Again $(\lambda-\theta)(t-\tau)$ will lead to vanishing some terms in $\beta_4$. Furthermore, from $\frac{d\cos(z)}{d\tau} = \lambda e^{-\lambda(t-\tau)}$, we have $e^{-\theta(t-\tau)}d\tau = \frac{e^{(\lambda-\theta)(t-\tau)}}{\lambda}d\cos(z)$, so the second integral in (10) becomes

$$II = \frac{1}{2c\lambda}\int_{\tau=0}^{\tau=t}(1-e^{-c\tau})e^{\beta_5}G_2(x,y;t,\tau)\text{Erfc}(\hat{h})d\cos(z).$$

To summarize, we have

$$\begin{aligned} U_x(x,t) &= \frac{1}{\sqrt{\pi}c}\int_{\tau=0}^{\tau=t}(e^{-c\tau}-1)e^{\beta_5}\frac{\sqrt{\theta}}{\sigma\lambda}G_1 P e^{\hat{h}^2}dz \\ &+ \frac{1}{2c\lambda}\int_{\tau=0}^{\tau=t}(1-e^{-c\tau})e^{\beta_5}\text{Erfc}(\hat{h})d\cos(z). \end{aligned} \qquad (26)$$

And this is the integration quadrature that is implemented in our numerical algorithm. While it is designed to handle the singular integral at hand, it works for other similar situations as specified in (20). We remark that we choose $\lambda = \theta - \frac{\sigma^2}{2k^2}$ simply because it leads to a simpler expression of $U_x$. When $\theta - \frac{\sigma^2}{2k^2} = 0$ for extreme parameter values, the integration quadrature (23) still works since we are free to choose a different $\lambda > 0$. We also remark that our statistical simulation shows, for risk free return rate in real economy, $\sigma << k$ and $\frac{\sigma^2}{2k^2} << \theta$. For instance, using maximum likelihood estimation to calibrate the parameters of the Vasicek model assuming it is the model governing the movement of the 10-year treasury notes yield in the U.S. market, we find that $\theta = 0.049, \sigma = 0.009$, and $k = 0.767$, which gives $\frac{\sigma^2}{2k^2} = 0.000028 << \theta$.

Table 1: $T = 1, c = 0.06, \theta = 0.04, k = 1, \sigma = 0.01$, $h(T) = (h(T)^\sharp + 579) \times 10^{-4}$

| $N$ | 32 | 64 | 128 | 256 |
|---|---|---|---|---|
| $h(T)^\sharp$ | 0.1019 | 0.3470 | 0.4352 | 0.4668 |
| Error ($10^{-4}$) | N/A | 0.2451 | 0.0882 | 0.0316 |
| Rate | NA | NA | 2.7776 | 2.7954 |
| $N$ | 512 | 1024 | 2048 | |
| $h(T)^\sharp$ | 0.4781 | 0.4820 | 0.4835 | |
| Error ($10^{-4}$) | 0.0112 | 0.0040 | 0.0014 | |
| Rate | 2.8064 | 2.8138 | 2.8208 | |

Table 2: $T = 1,\ c = 0.06,\ \theta = 0.05,\ k = 1,\ \sigma = 0.01$, $h(T) = (h(T)^\sharp + 570) \times 10^{-4}$

| $N$ | 32 | 64 | 128 | 256 |
|---|---|---|---|---|
| $h(T)^\sharp$ | 0.0433 | 0.1780 | 0.2260 | 0.2430 |
| Error ($10^{-4}$) | N/A | 0.1347 | 0.0480 | 0.0170 |
| Rate | NA | NA | 2.8090 | 2.8178 |
| $N$ | 512 | 1024 | 2048 | |
| $h(T)^\sharp$ | 0.2491 | 0.2512 | 0.2519 | |
| Error ($10^{-4}$) | 0.0060 | 0.0021 | 0.0008 | |
| Rate | 2.8223 | 2.8241 | 2.8285 | |

### 3.4 Convergence and Validation

We validate our numerical method by running the program with different parameter values for the Vasicek model and different interest rates and maturities for the mortgage contract. Of course it is not possible to test all continuous values of all parameters, but we try with reasonable discrete values. We do numerical experiments by changing only one parameter, say mortgage rate $c$, at one time, and keep other parameters fixed. The error tolerances we set for Newton's iteration is $10^{-9}$, where $\triangle t$ is the mesh size for time discretization. In the following tables, $t = T$ is time to maturity of the contract, say, $t = T = 15$ (years) means the example is of a 15-year mortgage contract as of now; $N$ is the number of meshes used in discretization of time interval $[0, T]$ ( Unless otherwise stated, the meshes are uniformly spaced); $h(T)$ is the numerical solution for previously specified error tolerance at $t = T$; "Error" means the relative error between the numerical solutions of $h(T)$ for previous $N$ and current $N$, i.e., by how much $h(T)$ changes as $N$ changes; "Rate" means numerical convergence rate, i.e., the ratio of "Error" for previous $N$ and "Error" for current $N$.

The Table 1, 2, and 3 are the results for relatively small $T = 1$ (years). We run the program for different values of $\theta$ as other parameters fixed and get a very stable convergence rate above 2.8.

The Table 4, 5, and 6 are results for relatively large $T = 15$ (years). We run the program for different values of $\theta$ as other parameters fixed and again get a very stable convergence rate about 2.8. Although the conver-

Table 3: $T = 1, c = 0.06, \theta = 0.06, k = 1, \sigma = 0.01$, $h(T) = (h(T)^\sharp + 555) \times 10^{-4}$

| $N$ | 32 | 64 | 128 | 256 |
|---|---|---|---|---|
| $h(T)^\sharp$ | 0.1312 | 0.2337 | 0.2710 | 0.2845 |
| Error ($10^{-4}$) | N/A | 0.1025 | 0.0374 | 0.0135 |
| Rate | NA | NA | 2.7418 | 2.7687 |
| $N$ | 512 | 1024 | 2048 | |
| $h(T)^\sharp$ | 0.2894 | 0.2911 | 0.2917 | |
| Error ($10^{-4}$) | 0.0048 | 0.0017 | 0.0006 | |
| Rate | 2.7885 | 2.7978 | 2.8118 | |

Table 4: $T = 15, c = 0.08, \theta = 0.07, k = 0.5, \sigma = 0.01$, $h(T) = (h(T)^\sharp + 73) \times 10^{-3}$

| $N$ | 32 | 64 | 128 | 256 |
|---|---|---|---|---|
| $h(T)^\sharp$ | 0.3782 | 0.5196 | 0.5695 | 0.5871 |
| Error ($10^{-3}$) | N/A | 0.1414 | 0.0499 | 0.0176 |
| Rate | NA | NA | 2.8310 | 2.8388 |
| $N$ | 512 | 1024 | 2048 | |
| $h(T)^\sharp$ | 0.5933 | 0.5955 | 0.5962 | |
| Error ($10^{-3}$) | 0.0062 | 0.0022 | 0.0008 | |
| Rate | 2.8398 | 2.8383 | 2.8365 | |

Table 5: $T = 15, c = 0.08, \theta = 0.08, k = 0.5, \sigma = 0.01$, $h(T) = (h(T)^\sharp + 67) \times 10^{-3}$

| $N$ | 32 | 64 | 128 | 256 |
|---|---|---|---|---|
| $h(T)^\sharp$ | 0.2307 | 0.3779 | 0.4418 | 0.4675 |
| Error ($10^{-3}$) | N/A | 0.1472 | 0.0639 | 0.0256 |
| Rate | NA | NA | 2.3017 | 2.4932 |
| $N$ | 512 | 1024 | 2048 | |
| $h(T)^\sharp$ | 0.4773 | 0.4810 | 0.4824 | |
| Error ($10^{-3}$) | 0.0099 | 0.0037 | 0.0014 | |
| Rate | 2.6035 | 2.6755 | 2.7214 | |

Table 6: $T = 15, c = 0.08, \theta = 0.09, k = 0.5, \sigma = 0.01$, $h(T) = (h(T)^\sharp + 49) \times 10^{-3}$

| $N$ | 32 | 64 | 128 | 256 |
|---|---|---|---|---|
| $h(T)^\sharp$ | -1.8244 | -1.0251 | 0.0485 | 0.6076 |
| Error ($10^{-3}$) | N/A | 0.7993 | 1.0736 | 0.5592 |
| Rate | NA | NA | 0.7445 | 1.9200 |
| $N$ | 512 | 1024 | 2048 | |
| $h(T)^\sharp$ | 0.8482 | 0.9442 | 0.9809 | |
| Error ($10^{-3}$) | 0.2406 | 0.0960 | 0.0367 | |
| Rate | 2.3240 | 2.5072 | 2.6122 | |

Table 7: $\theta = 0.05, k = 0.15, \sigma = 0.015$

| $c$ | $N = 128$ | $N = 256$ | $N = 512$ | $N = 1024$ |
|---|---|---|---|---|
| 0.01 | 2.7705 | 2.8835 | 2.9062 | 2.8998 |
| 0.02 | 2.6989 | 2.7929 | 2.8253 | 2.8355 |
| 0.03 | 2.6111 | 2.7127 | 2.7620 | 2.7882 |
| 0.04 | 2.5097 | 2.6399 | 2.7105 | 2.7521 |
| 0.05 | 2.3260 | 2.5309 | 2.6419 | 2.7063 |
| 0.06 | 1.0619 | 1.9848 | 2.3489 | 2.5317 |
| 0.07 | 3.6402 | 3.5464 | 3.4403 | 3.3336 |
| 0.08 | 3.1452 | 3.0838 | 3.0249 | 2.9758 |
| 0.09 | 3.0215 | 2.9882 | 2.9522 | 2.9211 |
| 0.10 | 2.9568 | 2.9420 | 2.9187 | 2.8969 |

| $c$ | $N = 2048$ | $N = 4096$ | $N = 8192$ |
|---|---|---|---|
| 0.01 | 2.8869 | 2.8729 | 2.8521 |
| 0.02 | 2.8361 | 2.8377 | 2.8342 |
| 0.03 | 2.8017 | 2.8128 | 2.8164 |
| 0.04 | 2.7766 | 2.7948 | 2.8041 |
| 0.05 | 2.7475 | 2.7730 | 2.7899 |
| 0.06 | 2.6373 | 2.7002 | 2.7413 |
| 0.07 | 3.2331 | 3.1451 | 3.0704 |
| 0.08 | 2.9368 | 2.9073 | 2.8854 |
| 0.09 | 2.8964 | 2.8778 | 2.8639 |
| 0.10 | 2.8789 | 2.8652 | 2.8549 |

gence rates in Table 5 and 6 are a little bit below 2.8 at $N = 2048$, they eventually do surpass 2.8 if we keep increasing $N$, the number of meshes for time dicretization. As one can imagine, as $T$ increases, it takes a larger $N$ to reach a stationary level of convergence. The next Table 7 gives the convergence rates we have observed for $T = 30$ (years), which is, to our best knowledge, the longest mortgage duration market can offer. This time we run the program for different values of $c$ while other parameters are fixed.

Similar experiments have been be tried and similar results have been obtained. From all such experiments, one can see that although the convergence rates for different sets of parameters may start at different values, their stationary values are about $2.8 \sim 3.0$ after the mesh being refined with sufficiently large $N$.

## 4 Numerical solution and discussions

In this section we present some numerical solutions obtained from our algorithm, from which some insightful analytical features about the mortgage problem can be naturally drawn.

Shown in Figure 4 are the plots of optimal early exercise boundary for different parameter values. These plots suggest that the boundary is smoothly decreasing in $c$ for $t$ and other parameters fixed. And the subtraction of $(c - \theta)$ has influence on the rate of such a decreasing. Hinted by the convexity of similar problems, say American put option with zero-dividend yield [4], one may tend to postulate that $h(t)$ is also convex. From a viewpoint of numerical analysis, convexity is a desired feature for free boundary problems because it may be useful for deriving certain iteration schemes to approximate the free boundary numerically [3]. To date, a formal proof on the convexity of the boundary is yet to be achieved. But our numerical simulations tend to suggest that the boundary is indeed convex. In Figure 4, we numerically demonstrate the behavior of $\hat{h}(\tau, t)$ in $\tau$ for fixed $t$. We think $\hat{h}$ is important because its appears, as a whole, in several places in the integral equation (10). First we see that that $\hat{h}(t,t) \equiv 0$, $\forall t > 0$ fixed; and $\hat{h}(\tau, t) \equiv \frac{\sqrt{k}}{\sigma}(h(\tau) - \theta + \frac{\sigma^2}{2k^2})$ for $t \to \infty$, $\forall \tau \geq 0$ fixed, in particular, $\lim_{t\to\infty} \hat{h}(0,t) \equiv \frac{\sqrt{k}}{\sigma}(c - \theta + \frac{\sigma^2}{2k^2})$. From Figure 4, we also see that $\hat{h}$ is not necessarily convex.

Next we present examples of solutions of mortgage contract value $V(x,t)$ and demonstrate the differential behavior of $V(x,t)$ in changing $x$ for fixed $t = T$. These features, though some are intuitive, are typically not easy to verify, if not facilitated with numerical simulations. Figure 3 is an output of $V$ against $x$ for different values of mortgage rate $c$ as other parameters are fixed. The left plot is for 30-year mortgages and the right one is for 15-year mortgages. It is apparent from these plots that the longer the mortgage duration $T$ is, the higher the contract value, given other parameters fixed. And also, the higher the mortgage rate $c$ is, the lower its value $V(x,t)$, given $m$ and other conditions are the same. From Figure 4, we see that $V(x,t)$ is increasing in $k$, where $k$ measures the speed by which the market interest $r_t$ is reverted to the long term mean $\theta$ if it moves away from $\theta$. From Figure 5, we see that $V(x,t)$ is decreasing in $\sigma$, where $\sigma$ measures the volatility of of market interest $r_t$.

Figure 5 tends to suggest that $V(x,t)$ goes to zero as $x$ increases for fixed $t$, yet does not describe true asymptotic behavior of $V(x,t)$ as $x \to \infty$ because values of $x$ are not large enough. In the following Figure 6, we provide several plots of extremely large $x$, which seem to suggest that V(x,t) decays exponentially to zero $x \to \infty$.

## 5 Conclusion

A type of amortized fixed rate mortgage is numerically solved with a fast and effective algorithm. A novelty quadrature method has been derived and used to handle the singular integral equations. The performance of the numerical method is validated with vast simulations. Analytical features of the solution are illustrated with numerical examples.

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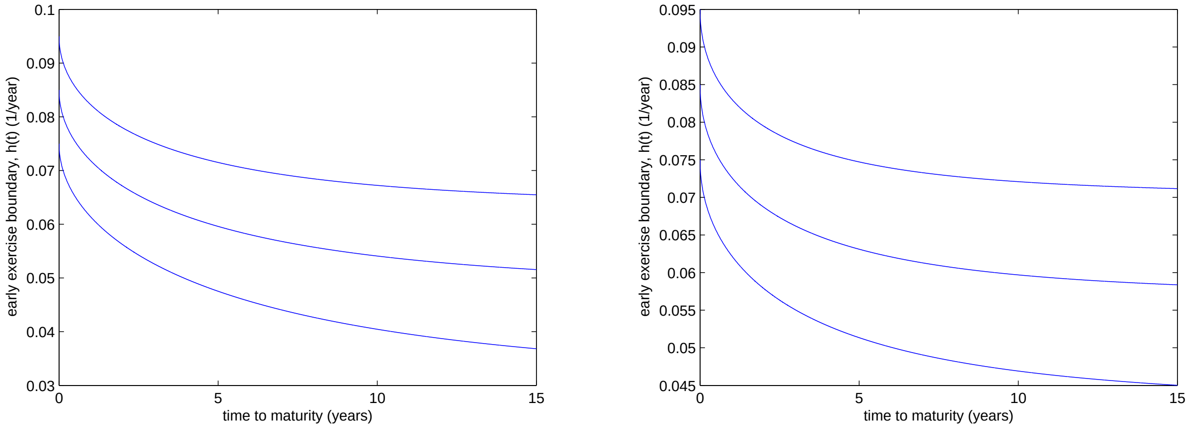


Figure 1: $c = 0.095, 0.085, 0.075$ (top to bottom), $\theta = 0.07$ (left), $0.05$ (right), $k = 0.25, \sigma = 0.030$.

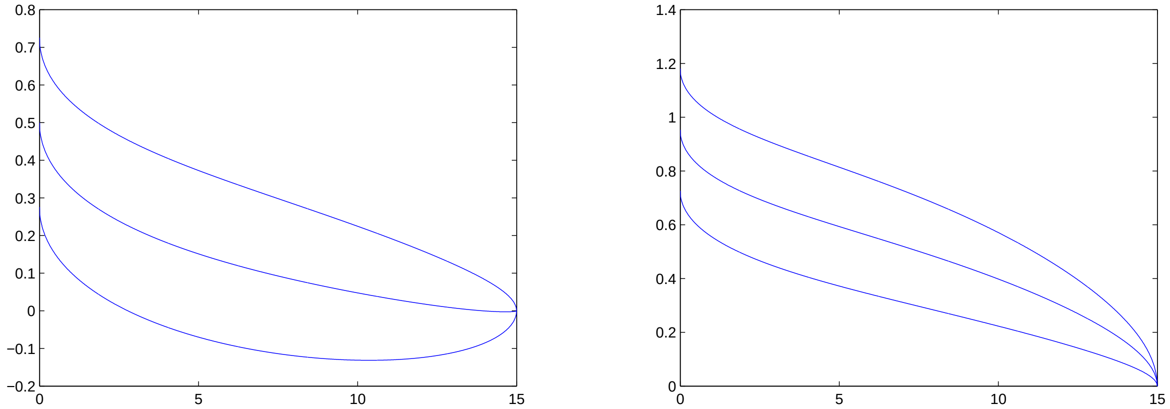

Figure 2: $c = 0.095, 0.085, 0.075$ (top to bottom), $\theta = 0.07$ (left), $0.05$ (right), $k = 0.15, \sigma = 0.015$.

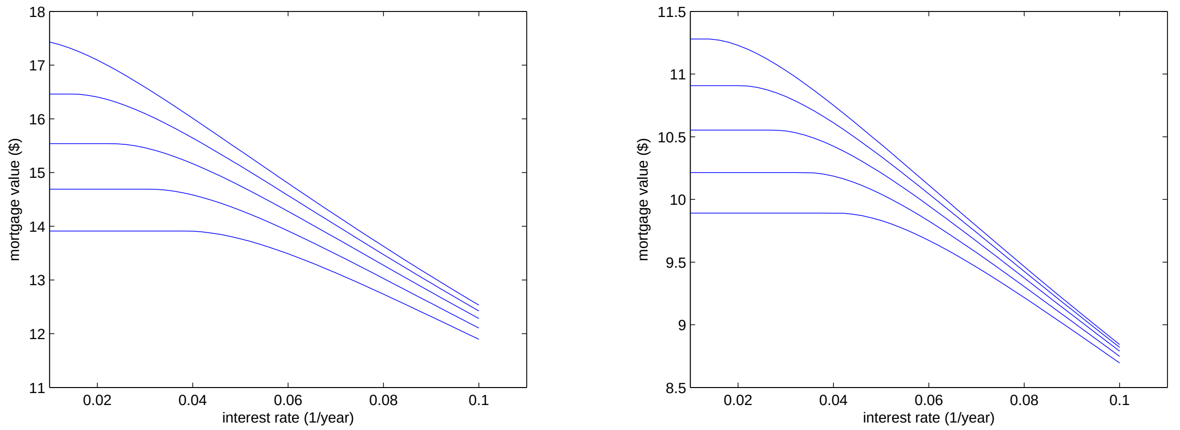


Figure 3: A comparison of 30-year and 15-year mortgage contract values. $c = 0.04, 0.045, 0.05, 0.055, 0.06$ (top to bottom), $\theta = 0.05, k = 0.15, \sigma = 0.015, m = 1$, $T = 30$ (left), $T = 15$ (right).

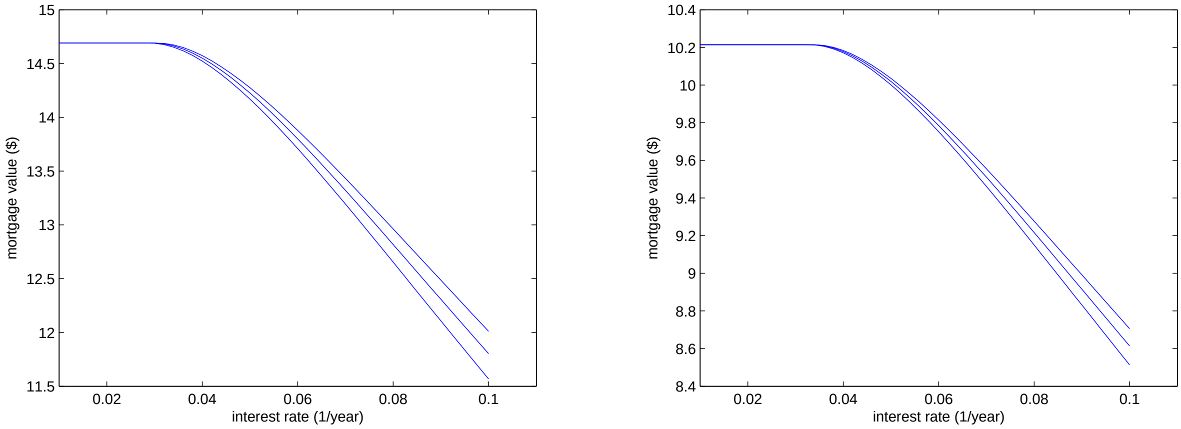


Figure 4: A comparison of 30-year and 15-year mortgage contract values. $c = 0.05$, $\theta = 0.055$, $k = 0.14, 0.12, 0.10$ (top to bottom), $\sigma = 0.015, m = 1$, $T = 30$ (left), $T = 15$ (right).

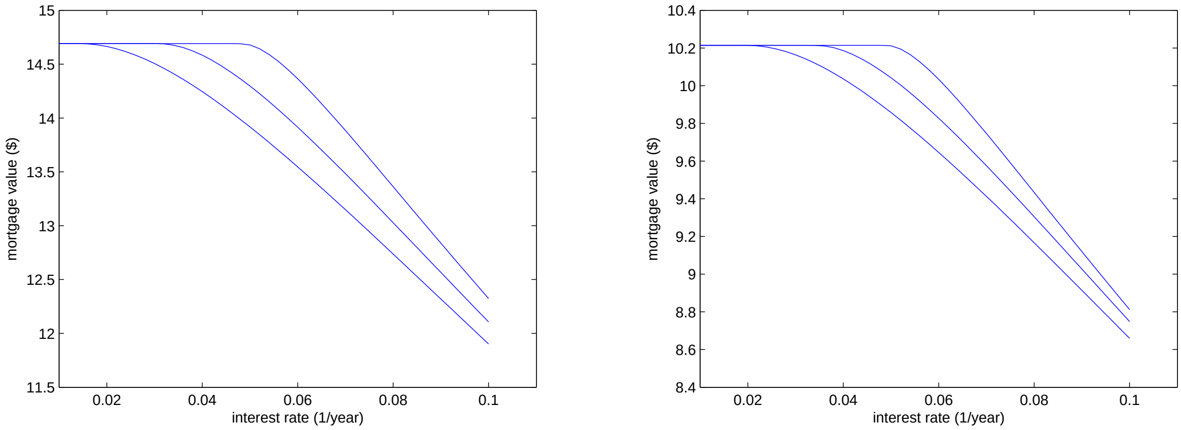


Figure 5: A comparison of 30-year and 15-year mortgage contract values. $c = 0.05$, $\theta = 0.055$, $k = 0.15$, $\sigma = 0.005, 0.015, 0.025$ (top to bottom), $m = 1$, $T = 30$ (left), $T = 15$ (right).

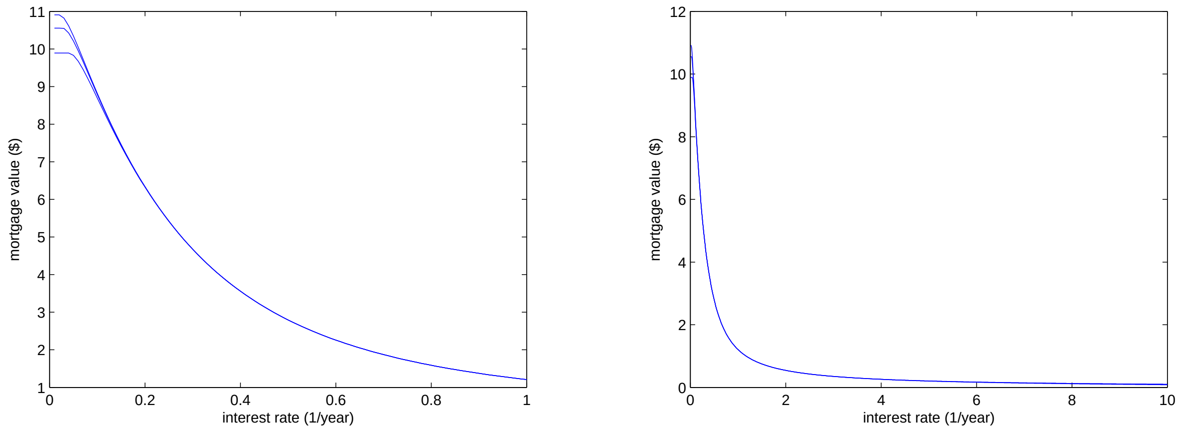


Figure 6: A 15-year mortgage value $V$ as a function of interest rate $x$. $c = 0.05, 0.055, 0.06$ (from top to bottom), $\theta = 0.05, k = 0.15, \sigma = 0.015, m = 1$. Note the three curves are not distinguishable for large very large $x$ in the left plot.

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