Altered Jacobian Newton Iterative Method for Nonlinear Elliptic Problems

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Abstract—**We present an Altered Jacobian Newton Iterative Method for solving nonlinear elliptic problems. Effectiveness of the proposed method is demonstrated through numerical experiments. Comparison of our method with Newton Iterative Method is also presented. Convergence of the Newton Iterative Method is highly sensitive to the initialization or initial guess. Reported numerical work shows the robustness of the Altered Jacobian Newton Iterative Method with respect to initialization.**

Keywords: Newton, Jacobian, non-linear, elliptic, Krylov solver

1 Introduction

The past fifty to sixty years have seen a considerable advancement in methods for solving linear systems. Krylov subspace method is the result of the huge effort by the researchers during the last century. It is one among the top ten algorithms of the 20th century. There exists optimal linear solvers [5]. But, still there is no optimal nonlinear solver or the one that we know of. Our research is in the field of optimal solution of nonlinear equations generated by the discretization of the nonlinear partial differential equation [1; 2; 3; 4]. Let us consider the following nonlinear elliptic partial differential equation [4]

$$\operatorname{div}(-\mathrm{K}\operatorname{grad} p) + f(p) = s(x,y) \qquad \textbf{in} \quad \Omega \qquad (1)$$

$$p(x,y) = p^{\mathbf{D}} \qquad \textbf{on} \quad \partial\Omega_{\mathbf{D}} \qquad (2)$$

$$g(x,y) = (-\mathrm{K}\,\nabla p)\cdot\hat{\mathbf{n}} \qquad \textbf{on} \quad \partial\Omega_{\mathbf{N}} \qquad (3)$$

Here, Ω is a polyhedral domain in $\mathbb{R}^{\mathrm{d}}$ ($\mathrm{d} = 2,\ 3$), the source function $s(x,y)$ is assumed to be in $L^2(\Omega)$, and the medium property K is uniformly positive. In the equations (2) and (3), $\partial\Omega_{\mathbf{D}}$ and $\partial\Omega_{\mathbf{N}}$ represent Dirichlet and Neumann part of the boundary, respectively. $f(p)$ represents nonlinear part of the equation. p is the unknown function. The equations (1), (2) and (3) models a wide variety of processes with practical applications. For example, pattern formation in biology, viscous fluid flow phenomena, chemical reactions, biomolecule electrostatics and crystal growth [6; 7; 8; 9; 10; 11].

There are various methods for discretizing the equations (1), (2) and (3). To mention a few: Finite Volume, Finite Element and Finite Difference methods [1]. These methods converts nonlinear partial differential equations into a system of algebraic equations.

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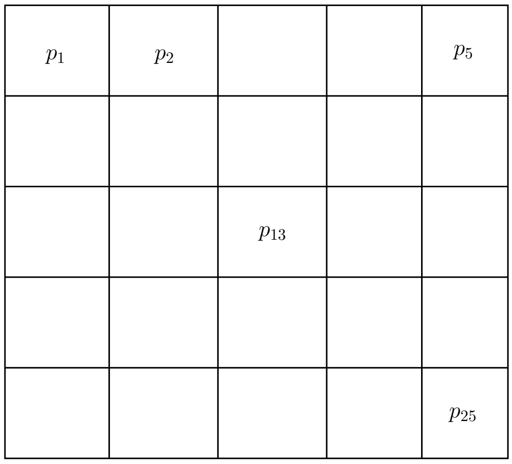

Figure 1: A 5×5 mesh.

Let discretization of the nonlinear partial differential equations result in a system of nonlinear algebraic equations $\mathbf{A}(\mathbf{p}) = 0$. Each cell in the mesh produces a nonlinear algebraic equation [1; 4]. Thus, discretization of the equations (1), (2) and (3) on a mesh with n cells result in n nonlinear equations, and let these equations are given as

$$\mathbf{A}(\mathbf{p}) = \begin{pmatrix} A_1(\mathbf{p}) \\ A_2(\mathbf{p}) \\ \vdots \\ A_n(\mathbf{p}) \end{pmatrix}. \qquad (4)$$

Figure 1 depicts a 5×5 mesh with the unknown $\mathbf{p}$. Since, the mesh contains 25 cells. Thus, the vector $\mathbf{A}(\mathbf{p})$ for this mesh will consist of 25 nonlinear algebraic equations [see 4]. In the next section, Altered Jacobian Newton Iterative Method is presented.

2 Altered Jacobian Newton Iterative Method

We are interested in finding the vector $\mathbf{p}$ for which the operator $\mathbf{A}(\mathbf{p})$ vanish. Let us first formulate the Newton Iterative Method. The Taylors expansion of nonlinear

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operator $\mathbf{A}(\mathbf{p})$ around some initial guess $\mathbf{p}_0$ is

$$\mathbf{A}(\mathbf{p}) = \mathbf{A}(\mathbf{p}_0) + \boldsymbol{J}(\mathbf{p}_0)\,\Delta\mathbf{p} + hot, \tag{5}$$

where *hot* stands for higher order terms. That is, terms involving higher than the first power of $\Delta\mathbf{p}$. Here, difference vector $\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0$. The Jacobian $\boldsymbol{J}$ is a $n \times n$ linear system evaluated at the $\mathbf{p}_0$. The Jacobian $\boldsymbol{J}$ in the equation (5) is given as follows

$$\boldsymbol{J} = \left[\frac{\partial A_i}{\partial p_j}\right] = \begin{pmatrix} \frac{\partial A_1}{\partial p_1} & \frac{\partial A_1}{\partial p_2} & \cdots & \frac{\partial A_1}{\partial p_n} \\ \frac{\partial A_2}{\partial p_1} & \frac{\partial A_2}{\partial p_2} & \cdots & \frac{\partial A_2}{\partial p_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial A_n}{\partial p_1} & \frac{\partial A_n}{\partial p_2} & \cdots & \frac{\partial A_n}{\partial p_n} \end{pmatrix}. \tag{6}$$

Since, we are interested in the zeroth of the non-linear vector function $\mathbf{A}(\mathbf{p})$. Thus, setting the equation (5) equals to zero and neglecting higher order terms will result in the following well known Newton Iteration Method (NIM)

$$\begin{aligned} \boldsymbol{J}(\mathbf{p}_k)\,\Delta\mathbf{p}_{k+1} &= -\mathbf{A}(\mathbf{p}_k), \\ \mathbf{p}_{k+1} &= \mathbf{p}_k + \Delta\mathbf{p}_{k+1}, \quad k = 0, \ldots, n. \end{aligned} \tag{7}$$

Newton Iterative Method may not always converge, and it's convergence depends on the initialization $\mathbf{p}_0$. If the initial guess if far from the exact solution, the Newton's method may not converge.

Let us modify the Jacobian (6) as follows

$$\boldsymbol{J}^f \stackrel{\text{def}}{=} \begin{pmatrix} \frac{\partial A_1}{\partial p_1} + A_1(p_1) & \frac{\partial A_1}{\partial p_2} & \cdots & \frac{\partial A_1}{\partial p_n} \\ \frac{\partial A_2}{\partial p_1} & \frac{\partial A_2}{\partial p_2} + A_2(p_2) & \cdots & \frac{\partial A_2}{\partial p_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial A_n}{\partial p_1} & \frac{\partial A_n}{\partial p_2} & \cdots & \frac{\partial A_n}{\partial p_n} + A_n(p_n) \end{pmatrix} \tag{8}$$

Here, $A_i(p_j)$ is the i^{th} element of the vector $\mathbf{A}$ evaluated at p_j. Based on the above definition of the Altered Jacobian, we propose the following Alterned Jacobian Newton Iterative Method (AJNIM)

$$\begin{aligned} \boldsymbol{J}^f(\mathbf{p}_k)\,\Delta\mathbf{p}_{k+1} &= -\mathbf{A}(\mathbf{p}_k), \\ \mathbf{p}_{k+1} &= \mathbf{p}_k + \Delta\mathbf{p}_{k+1}, \quad k = 0, \ldots, n. \end{aligned} \tag{9}$$

In the next section, we compare methods (7) and (9). Dependence of the convergence of the Newton Iterative Method (7) on initialization is notoriously well documented. Numerical work shows the robustness of the Altered Jacobian Newton Iterative Method for different initial guesses.

3 Numerical Experimentation

Without loss of generality let us assume that K is unity, and the boundary is of Dirichlet type. Let $f(p)$ be $10^4\,p\,\exp(p)$. Thus, the equations (1), (2) and (3) are written as

$$-\nabla^2 p + 10^4\,p\,\exp(p) = f \qquad \textbf{in} \quad \Omega, \tag{10}$$

$$p(x,y) = p^{\mathrm{D}} \qquad \textbf{on} \quad \partial\Omega_{\mathrm{D}}. \tag{11}$$

For computing the true error and convergence behavior of the methods, let us further assume that the exact solution of the equations (10) and (11) is the following bubble function

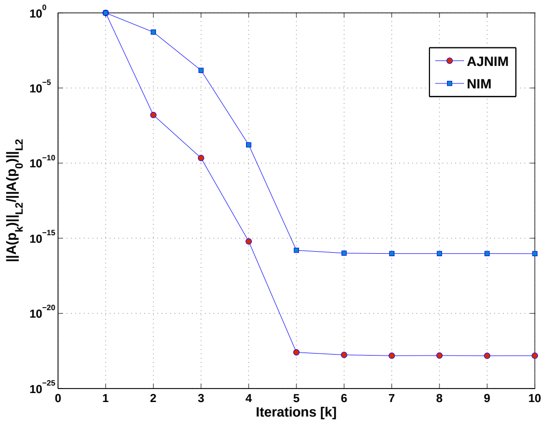
$$p = x\,(x-1)\,y\,(y-1).$$

Let our domain be a unit square. Thus, $\Omega = [0,1] \times [0,1]$. We are discretizing equations (10) and (11) on a 20×20 mesh by the method of Finite Volumes [1; 2; 4; 12]. Discretization results in a nonlinear algebraic vector (4) with 400 nonlinear equations.

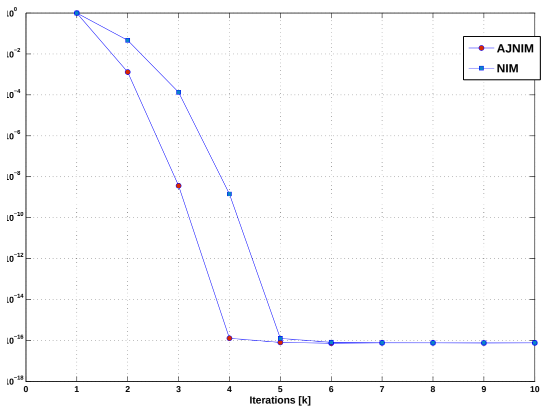
If the method is convergent, L_2 norm of the difference vector, $\Delta\mathbf{p}$, and the residual vector, $\mathbf{A}(\mathbf{p})$, converge to zero [see 12]. We are reporting convergence of both of these vectors. For better understanding the error reducing property of these methods, we report variation of $\|\mathbf{A}(\mathbf{p_k})\|_{L_2}/\|\mathbf{A}(\mathbf{p_0})\|_{L_2}$ and $\|\Delta(\mathbf{p_k})\|_{L_2}/\|\Delta(\mathbf{p_0})\|_{L_2}$ with iterations (k).

We performed three experiments with different initialization in the algorithms (7) and (9). In the first test, let the initial vector be zero for both the algorithms. Figure 2 reports the result. Figure 2(a) presents convergence of the residual vector while the Figure 2(b) presents convergence of the difference vector. In these figures, NIM stands for Newton Iterative Method while AJNIM stands for Alterted Newton Iterative Method. These figures show that both the methods converges at the same rate (quadratically), but still the Altered Jacobian Newton Iterative Method is better in reducing the error.

In the second case, let us select initial vector whose elements 10. Figure 3 presents comparison of the two methods for an initial vector whose elements are 10. It can be seen in the Figures 3(a) and 3(b) that the Altered Jacobian Newton Iterative Method converges faster than the Newton Iterative Method. Let us finally take an initial vector with elements equal to 100. Figure 4 presents comparison of the two methods for an initial vector whose elements are 100. The Figures 4(a) and 4(b) show that the Newton Iterative Method is not converging while the Altered Jacobian Newton Iterative Method still converges. The table 1 presents error after 10 iterations of the two methods. These experiments does show the independence of the convergence of the Altered Jacobian Newton Iterative Method with respect to initialization. We saw that the Newton Iterative Method converges quadratically for

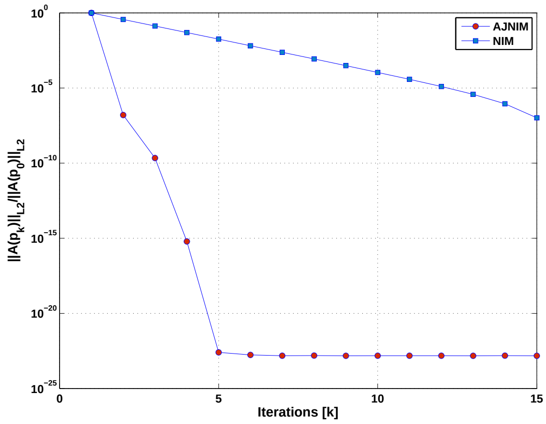


(a) Iteration vs $\|\mathbf{A}(\mathbf{p_k})\|_{L_2}/\|\mathbf{A}(\mathbf{p_0})\|_{L_2}$.

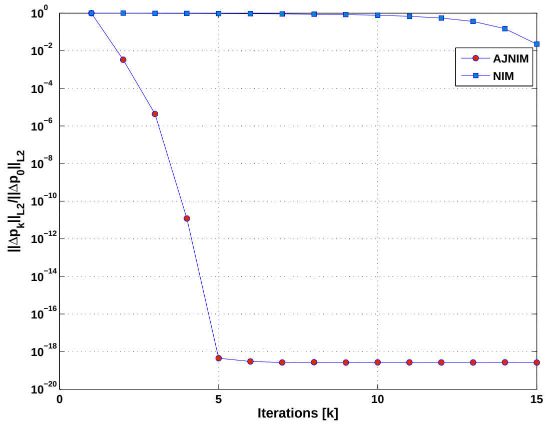


(b) Iteration vs $\|\Delta(\mathbf{p_k})\|_{L_2}/\|\Delta(\mathbf{p_0})\|_{L_2}$.

Figure 2: Initial guess is the zero vector. Here, NIM stands for Newton Iterative Method while AJNIM stands for Altered Jacobian Newton Iterative Method.

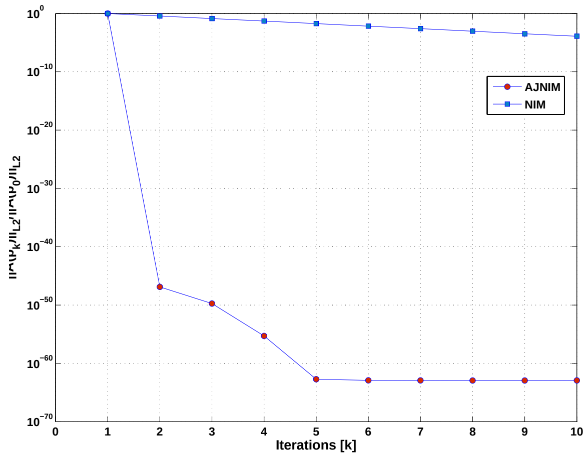


(a) Iteration vs $\|\mathbf{A}(\mathbf{p_k})\|_{L_2}/\|\mathbf{A}(\mathbf{p_0})\|_{L_2}$.

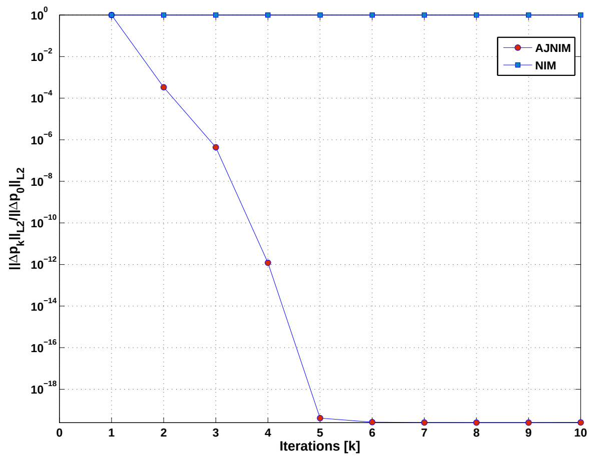


(b) Iteration vs $\|\Delta(\mathbf{p_k})\|_{L_2}/\|\Delta(\mathbf{p_0})\|_{L_2}$.

Figure 3: Initial vector is of size 400 with each elements equal to 10.



(a) Iteration vs $\|\mathbf{A}(\mathbf{p_k})\|_{L_2}/\|\mathbf{A}(\mathbf{p_0})\|_{L_2}$.



(b) Iteration vs $\|\Delta(\mathbf{p_k})\|_{L_2}/\|\Delta(\mathbf{p_0})\|_{L_2}$.

Figure 4: Here, each elements of the initial vector consists of 100.

the first case (initial guess is zero vector) but its convergence rate decreases as we selected other initial guesses. On the other hand, for all the initial guesses the Altered Jacobian Newton Iterative Method converges quadratically.

Table 1: Error by Altered Jacobian Newton Iterative Method (ALT NIM) and Newton Iterative Method (NIM) after 10 iterations. Here, initial guess vector is 100.

Method	$\|\Delta\mathbf{p}\|_{L_2}$	$\|\mathbf{A}(\mathbf{p})\|_{L_2}$
NIM	19.7828	1.658×10^{44}
AJNIM	5.058×10^{-17}	1.573×10^{-15}

4 Conclusions

We have developed a nonlinear algorithm named Altered Jacobian Newton Iterative Method for solving system nonlinear equations formed from discretization of nonlinear elliptic problems. Presented numerical work shows that the Altered Jacobian Newton Iterative Method is robust with respect to the initialization.

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