Ishikawa Iteration Process for Nonself Nonexpansive Maps in Banach Spaces

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Abstract—**In this paper, we construct Ishikawa iterative scheme with errors for nonself nonexpansive maps and approximate fixed points of these maps through weak and strong convergenc of the scheme.**

Keywords: Fixed point, Nonexpansive nonself map, Opial property, Weak and strong convergence

1 Introduction

Let C be a nonempty subset of a real Banach space E. The map $T : C \to C$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C.$$

Nonexpansive selfmaps ever since their introduction, remained a popular area of research in various fields. Iterative construction of fixed points of these maps is a fascinating field of research. In 1967, Browder [1] studied the iterative construction of fixed points of nonexpansive selfmaps on closed and convex subsets of a Hilbert space. The Ishikawa iteration process in the context of self nonexpansive maps and asymptotically nonexpansive maps on closed convex subsets of a Banach space has been considered by a number of authors (see, for example, [2-5, 8-10] and the references therein). However, if the domain C of T is a proper subset of E and $T : C \to E$, then this iteration process may fail to be well defined (see, e.g, [7]).

A subset C of E is said to be a retract of E if there exists a continuous map $P : E \to C$ such that $Px = x$ for all $x \in C$. A map $P : E \to E$ is a retraction if $P^2 = P$. It easily follows that if a map P is a retraction, then $Py = y$ for all y in the range of P. $T : C \to C$ is demi-compact if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to p \in C$.

We construct the Ishikawa iteration scheme as follows:

$$\begin{aligned} x_{n+1} &= P(\alpha_n x_n + \beta_n T y_n + \gamma_n u_n), \\ y_n &= P(\acute{\alpha}_n x_n + \acute{\beta}_n T x_n + \acute{\gamma}_n v_n) \quad \text{for all } n \geq 1 \quad (1) \end{aligned}$$

where $x_1 \in C, T : C \to E$ is a nonself map, $P : E \to C$ is a nonexpansive retraction, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\acute{\alpha}_n\}, \{\acute{\beta}_n\}$ and $\{\acute{\gamma}_n\}$ are real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \acute{\alpha}_n + \acute{\beta}_n + \acute{\gamma}_n; \{u_n\}$ and $\{v_n\}$ are bounded sequences in C.

If $\gamma_n = \acute{\gamma}_n = 0$,then (1) becomes the iteration process studied by Shahzad[7] and if T is a selfmap, then (1) reduces to the following iteration scheme:

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \\ y_n &= \acute{\alpha}_n x_n + \acute{\beta}_n T x_n + \acute{\gamma}_n v_n \quad \text{for all } n \geq 1. \end{aligned}$$

In this paper, we approximate fixed points of nonself nonexpansive maps through weak and strong convergence of the scheme(1). Our theorems improve and generalize some previous results of selfmaps and nonself maps.

2 Preliminaries and Notations

Recall that a Banach space E is said to be uniformly convex if for each r with $0 < r \leq 2$, the modulus of convexity of E given by

$$\delta(r) = \inf\left\{1 - \frac{1}{2}\|x + y\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq r\right\},$$

satisfies the inequality $\delta(r) > 0$. For sequences, the symbol $\to$ (resp.$\rightharpoonup$) denotes norm (resp. weak) convergence. The space E is said to satisfy the *Opial's property* [6] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$.

A map $T : C \to E$ is called demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

We need the following known lemmas.

Lemma 1 *[11] Let $\{s_n\}$ and $\{t_n\}$ be two nonnegative real sequences satisfying*

$$s_{n+1} \leq s_n + t_n \text{ for all} n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} s_n$ exists.

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Lemma 2 *[1] Let C be a nonempty closed convex subset of a uniformly convex Banach space and let $T : C \to E$ be a nonexpansive map. Then $I - T$ is demiclosed at 0.*

Lemma 3 *[12] Let $p > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there is a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ satisfying $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda \|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda)g(\|x-y\|)$$

for all $x, y \in B_r[0]$, where $B_r[0] = \{x \in E : \|x\| \leq r\}$ and $w_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$ for all $\lambda \in [0,1]$.

3 Convergence Analysis

We establish a pair of lemmas to prove our convergence results.

The set of fixed points of T is denoted by

$$F(T) = \{x \in C : Tx = x\}.$$

Lemma 4 *Let C be a nonempty convex subset of a normed space E and let $T : C \to E$ be a nonexpansive map with $F(T) \neq \phi$. If $\{x_n\}$ is defined as in (1) where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\acute{\alpha}_n\}, \{\acute{\beta}_n\}$ and $\{\acute{\gamma}_n\}$ are real sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \acute{\alpha}_n + \acute{\beta}_n + \acute{\gamma}_n, \sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \acute{\gamma}_n < \infty; \{u_n\}$ and $\{v_n\}$ are bounded sequences in C, then $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F(T)$.*

Proof. Let $q \in F(T)$. Then $M = \max\left\{\sup_{n\geq 1} \|u_n - q\|, \sup_{n\geq 1} \|v_n - q\|\right\} < \infty$. Consider

$$\begin{aligned}
\|x_{n+1} - q\| &= \|P(\alpha_n x_n + \beta_n T y_n + \gamma_n u_n) - q\| \\
&\leq \|\alpha_n x_n + \beta_n T y_n + \gamma_n u_n - q\| \\
&\leq \|\alpha_n (x_n - q) + \beta_n (T y_n - q) + \gamma_n (u_n - q)\| \\
&\leq \alpha_n \|x_n - q\| + \beta_n \|T y_n - q\| + \gamma_n \|u_n - q\| \\
&\leq \alpha_n \|x_n - q\| + \beta_n \|y_n - q\| + \gamma_n M \\
&= \beta_n \left\|P(\acute{\alpha}_n x_n + \acute{\beta}_n T x_n + \acute{\gamma}_n) - q\right\| \\
&\quad + \alpha_n \|x_n - q\| + \gamma_n M \\
&\leq (\acute{\alpha}_n \beta_n + \beta_n \acute{\beta}_n + \alpha_n) \|x_n - q\| \\
&\quad + (\gamma_n + \beta_n \acute{\gamma}_n) M \\
&\leq \left[(1 - \acute{\beta}_n)\beta_n + \beta_n \acute{\beta}_n + (1 - \beta_n)\right] \|x_n - q\| \\
&\quad + (\gamma_n + \beta_n \acute{\gamma}_n) M \\
&= \|x_n - q\| + (\gamma_n + \beta_n \acute{\gamma}_n) M.
\end{aligned}$$

Hence by Lemma 1, $\lim_{n\to\infty} \|x_n - q\|$ exists.

Lemma 5 *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T : C \to E$ be a nonexpansive map with $F(T) \neq \phi$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\acute{\alpha}_n\}, \{\acute{\beta}_n\}$ and $\{\acute{\gamma}_n\}$ be real sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \acute{\alpha}_n + \acute{\beta}_n + \acute{\gamma}_n, 0 \leq \beta_n \leq \beta < 1, \sum_{n=1}^{\infty} \beta_n, = \infty, \limsup_{n\to\infty} \acute{\beta}_n < 1, \sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \acute{\gamma}_n < \infty; \{u_n\}$ and $\{v_n\}$ are bounded sequences in C. Then for the sequence $\{x_n\}$ given by (1), we have $\liminf_{n\to\infty} \|x_n - T x_n\| = 0$.*

Proof. For any $q \in F(T)$, we have

$$\begin{aligned}
\|y_n - q\| &= \left\|P(\acute{\alpha}_n x_n + \acute{\beta}_n T x_n + \acute{\gamma}_n v_n) - q\right\| \\
&\leq \left\|\acute{\alpha}_n x_n + \acute{\beta}_n T x_n + \acute{\gamma}_n v_n - q\right\| \\
&\leq \acute{\alpha}_n \|x_n - q\| + \acute{\beta}_n \|T x_n - q\| + \acute{\gamma}_n \|v_n - q\| \\
&\leq (\acute{\alpha}_n + \acute{\beta}_n) \|x_n - q\| + \acute{\gamma}_n \|v_n - q\|,
\end{aligned}$$

which shows that $\{y_n\}$ is bounded. Consequently, $\{x_n - q, T y_n - q\} \in B_r[0] \cap C$ for some $r > 0$. Denote by H, the max of
(i) $\sup_{n\geq 1} \|x_n - q\|^2$
(ii) $\sup_{n\geq 1} \left(\|v_n - x_n\|^2 + 2\left\|\acute{\beta}_n (T y_n - q) + (1 - \acute{\beta}_n)(x_n - q)\right\| \|v_n - x_n\|^2\right)$
and
(iii) $\sup_{n\geq 1} \left(\|u_n - x_n\|^2 + 2\|\beta_n (T y_n - q) + (1 - \beta_n)(x_n - q)\| \|u_n - x_n\|^2\right)$

From Lemma 3 and the scheme (1), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|P(\alpha_n x_n + \beta_n T y_n + \gamma_n u_n) - q\|^2 \\
&\leq \|\alpha_n x_n + \beta_n T y_n + \gamma_n u_n - q\|^2 \\
&= \|\beta_n (T y_n - q) + (1 - \beta_n)(x_n - q) + \gamma_n (u_n - x_n)\|^2 \\
&\leq \|\beta_n (T y_n - q) + (1 - \beta_n)(x_n - q)\|^2 + \gamma_n H \\
&\leq \beta_n \|T y_n - q\|^2 + (1 - \beta_n) \|x_n - q\|^2 \\
&\quad - w_2(\beta_n) g(\|x_n - T y_n\|) + \gamma_n H \\
&\leq \beta_n \|T y_n - q\|^2 + (1 - \beta_n) \|x_n - q\|^2 \\
&\quad - \beta_n (1 - \beta_n) g(\|x_n - T y_n\|) + \gamma_n H \\
&\leq \beta_n \|y_n - q\|^2 + (1 - \beta_n) \|x_n - q\|^2 \\
&\quad - \beta_n (1 - \beta) g(\|x_n - T y_n\|) + \gamma_n H \beta_n \acute{\beta}_n \|T x_n - q\|^2 \\
&\quad + \beta_n (1 - \acute{\beta}_n) \|x_n - q\|^2 + (1 - \beta_n) \|x_n - q\|^2 \\
&\quad - \beta_n (1 - \beta_n) g(\|x_n - T y_n\|) + (\gamma_n + \acute{\gamma}_n) H \\
&\leq \|x_n - q\|^2 - \beta_n (1 - \beta_n) g(\|x_n - T y_n\|) + (\gamma_n + \acute{\gamma}_n) H \\
&\quad \|x_n - q\|^2 - \beta_n (1 - \beta) g(\|x_n - T y_n\|) + (\gamma_n + \acute{\gamma}_n) H
\end{aligned}$$

That is,

$$\begin{aligned}
\beta_n (1 - \beta) g(\|x_n - T y_n\|) &\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
&\quad + (\gamma_n + \acute{\gamma}_n) H \qquad (2)
\end{aligned}$$

Let m be a positive integer such that $m \geq 1$. Summing up the terms from 1 to m on both sides of (2), we have

$$\begin{aligned}
(1 - \beta) \sum_{n=1}^{m} \beta_n g(\|x_n - T y_n\|) &\leq \|x_1 - q\|^2 - \|x_{m+1} - q\|^2 \\
&\quad + H \sum_{n=1}^{m} (\gamma_n + \acute{\gamma}_n) \qquad (3)
\end{aligned}$$

Let $m \to \infty$ in (3). Then we have

$$(1-\beta) \sum_{n=1}^{\infty} \beta_n g(\|x_n - T y_n\|) \leq \|x_1 - q\|^2 + H \sum_{n=1}^{\infty} (\gamma_n + \acute{\gamma}_n)$$

which implies on the basis of $\sum_{n=1}^{\infty} \beta_n = 0$ that

$$\liminf_{n\to\infty} g(\|x_n - Ty_n\|) = 0.$$

By vitue of the properties of the function g, we conclude that

$$\liminf_{n\to\infty} \|x_n - Ty_n\| = 0.$$

Since P is a nonexpansive retraction, so we have

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - Ty_n\| + \|Tx_n - Ty_n\| \\ &\leq \|x_n - Ty_n\| + \|x_n - y_n\| \\ &= \|x_n - Ty_n\| \\ &\quad + \left\|Px_n - P(\acute{\alpha_n} x_n + \acute{\beta_n} Tx_n + \acute{\gamma_n} v_n)\right\| \\ &\leq \|x_n - Ty_n\| \\ &\quad + \left\|x_n - (\acute{\alpha_n} x_n + \acute{\beta_n} Tx_n + \acute{\gamma_n} v_n)\right\| \\ &= \|x_n - Ty_n\| + \acute{\beta_n} \|x_n - Tx_n\| + \acute{\gamma_n} \|x_n - v_n\| \\ &\leq \|x_n - Ty_n\| + \acute{\beta_n} \|x_n - Tx_n\| + \acute{\gamma_n} H \end{aligned}$$

That is,

$$(1 - \acute{\beta_n}) \|x_n - Tx_n\| \leq \|x_n - Ty_n\| + \acute{\gamma_n} H$$

By taking $\liminf$ on both sides of the above inequality, we have

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{4}$$

Theorem 1 *Let E be a uniformly convex Banach space satisfying the Opial property and let C, T and $\{x_n\}$ be as in Lemma 5. If $F(T) \neq \phi$, then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. By Lemma 5, $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$. Therefore, there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $\lim_{m\to\infty} \|x_m - Tx_m\| = 0$. For $q \in F(T)$, $\lim_{n\to\infty} \|x_n - q\|$ exists as proved in Lemma 3. Since E is refexive, there exists a subsequence $\{x_{m_i}\}$ of $\{x_m\}$ converging weakly to some $z_1 \in C$. Now $\lim_{m\to\infty} \|x_m - Tx_m\| = 0, I - T$ is demiclosed at 0 by Lemma 2 and hence we obtain $Tz_1 = z_1.$ That is, $z_1 \in F(T)$. In order to show that $\{x_m\}$ converges weakly to z_1, take a subsequence $\{x_{m_j}\}$ of $\{x_m\}$ converging weakly to some $z_2 \in C$. Again, as before, we can prove that $z_2 \in F(T)$. Next, we prove that $z_1 = z_2$. Suppose that $z_1 \neq z_2$. Then by the Opial property

$$\begin{aligned} \lim_{m\to\infty} \|x_m - z_1\| &= \lim_{m_i\to\infty} \|x_{m_i} - z_1\| \\ &< \lim_{m_i\to\infty} \|x_{m_i} - z_2\| \\ &= \lim_{m\to\infty} \|x_m - z_2\| \\ &= \lim_{m_j\to\infty} \|x_{m_j} - z_2\| \\ &< \lim_{m_j\to\infty} \|x_{m_j} - z_1\| \\ &= \lim_{m\to\infty} \|x_m - z_1\|, \end{aligned}$$

a contradiction. This proves that $\{x_m\}$ converges weakly to a fixed point of T. As $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F(T)$, therefore $\{x_n\}$ converges weakly to a fixed point of T.

Theorem 2 *Let E be a uniformly convex Banach space and let C, T and $\{x_n\}$ be as in Lemma 5. If T is demicompact and $F(T) \neq \phi$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. As in the proof of Lemma 5, $\lim_{m\to\infty} \|x_m - Tx_m\| = 0$ for a subsequence $\{x_m\}$ of $\{x_n\}$. Suppose that T is demi-compact. Since $\{x_m\}$ is bounded and $\lim_{m\to\infty} \|x_m - Tx_m\| = 0$, therefore there exists a subsequence $\{x_i\}$ of $\{x_m\}$ such that $\{x_i\}$ converges strongly to $z \in C$. In view of (4), z is a fixed point of T.As $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F(T)$, so $x_n \to z$.

Remark 1 *(i) Lemma 5 improves Lemma 5 due to Khan and Fukhar-ud-din [5] in the case of one map and Theorem 3.3 of Shahzad [7].*
(ii) Theorem 1 improves Theorem 3.2 of Takahashi and Tamura [9] and also Theorem 1 of Khan and Fukhar-ud-din [5] in the case of one map.
(iii) Theorem 2 improves Theorem 3.7 [7] and Theorem 4.2 of Takahashi and Tamura [9] in the setting of a uniformly convex Banach space.

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