**IAENG International Journal of Applied Mathematics, 38:3, IJAM_38_3_04**

# Numerical Method For The Incompressible Euler Equations

Peiangpob Mounnumprang

*Abstract*—**A numerical method to study the boundary value problem in which the governing equations are the steady Euler equations and the vorticity is given on the inflow parts of the domain boundary is developed. The Euler equations are implemented in terms of the stream function and vorticity. The convergence of the finite-difference equations to the exact solution is shown experimentally for the test problem by comparing the computational results with the exact solution on the sequence of grids. The numerical algorithm is illustrated with several examples of steady flow through a $2-D$ channel with two inflow and outflow parts of boundary. The analysis of calculations shows strong dependence of the flow structure on the vorticity given at the inflow parts of the boundary.**

***Index Terms*— numerical method, Euler equations, flowing-through problem, incompressible fluid**

## I. Introduction

A vortical flow of an ideal incompressible fluid in a given domain which boundary do not only consist of impermeable parts but also include the inflow and outflow parts rather interesting for its applications and for a long time remained not studied completely. We will call such kind of a problem as the "flowing-through" problem (sketch of the domain see in Figure 1 [1]). Kazhikhov and Ragulin [2] studied the existence and uniqueness of the boundary value problem where on the inflow parts of the boundary either three components of velocity or normal component of velocity and two tangent component of vorticity were prescribed, and on the outflow parts of the boundary either the normal component of velocity or pressure were imposed. A sufficiently full survey of works on the connection in a flowing-through problem has been provided by Antontsev et al. [3]

In this article we represent the well-posed formulation of flowing-through boundary-value problem for the Euler equation of an ideal incompressible fluid flow through a bounded $2-D$ domain. We assume that on inflow parts of the domain boundary the normal component of the velocity vector and the vorticity are known and on outflow parts of the domain boundary the values of normal component of velocity vector are known as well. For such type of the flowing through problem we developed numerical algorithm which is based on property of conservation of vorticity along stream line in steady $2-D$ flow.

Graduate School, Valaya Alongkorn Rajabhat University, Pathumthani, 13180, Thailand. (corresponding author to provide phone: 66-2-5293099).

## II. Methodology

Let $\Omega$ be a bounded domain in $R^2$ whose boundary consists of the three kinds of parts. The inflow parts of domain boundary are denoted by $\Gamma_i^1,\ i=1,...,L,$ and the outflow parts of the boundary are denoted by $\Gamma_j^1,\ j=1,...,K$. The impermeable parts of boundary are denoted by $\Gamma_m^0,\ m=1,...,M$. The steady motion of a homogeneous ideal incompressible fluid is described by the Euler equations

$$uu_x + vu_y = -p_x,$$
$$uv_x + vv_y = -p_y,$$
$$u_x + v_y = 0,$$

where $(x,y)$ denoted the Cartesian coordinates of points on $\Omega$, $\vec{u}=(u,v)$ denoted the components of velocity vector on the $x$ and $y$ directions respectively, $p$ is the pressure.

We reformulate the flowing-through boundary value problem in terms of the stream function and vorticity

$$\Delta\psi = -\omega,$$
$$\frac{\partial(\psi,\omega)}{\partial(x,y)} = 0 \text{ in } \Omega$$

where $\psi$ is the stream function and $\omega$ is the vorticity (see in Figure 2).

The approximate solution of problem will be found by an iterative method. Algorithm to find solution of boundary value problem is as follows the Figure 3.

To solve Poisson equation for the stream function with Direchlet boundary conditions we used height order compact finite difference scheme, see for example [4]. The corresponding finite difference equation was solved by SOR method. The convergence of the numerical algorithm is confirmed by a test problem with known analytical solution.

**(Advance online publication: 21 August 2008)**

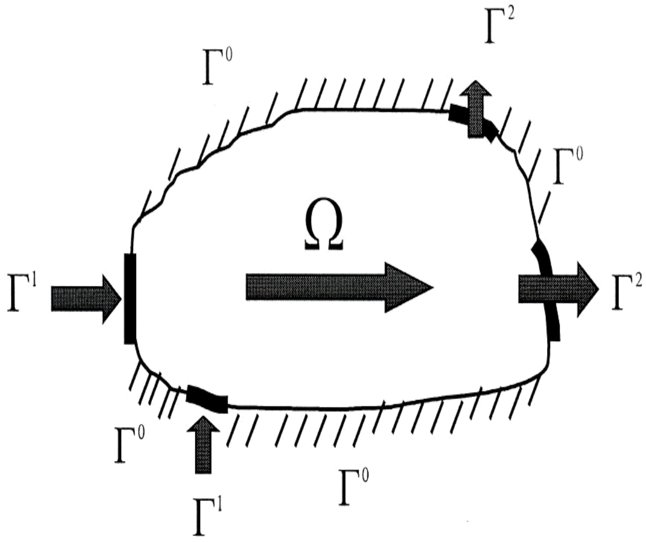



Figure 1

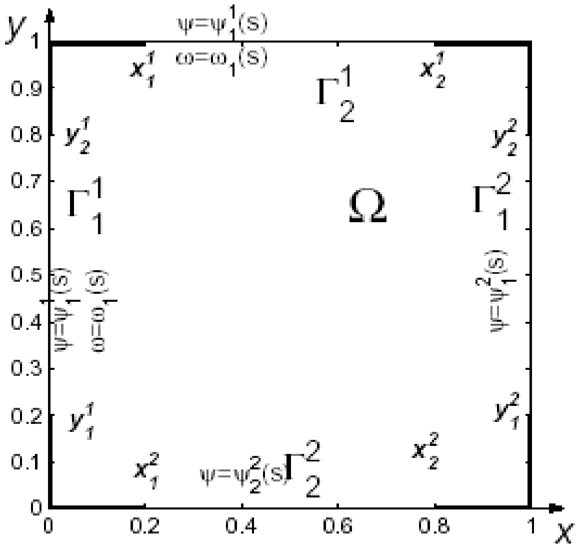



Figure 2

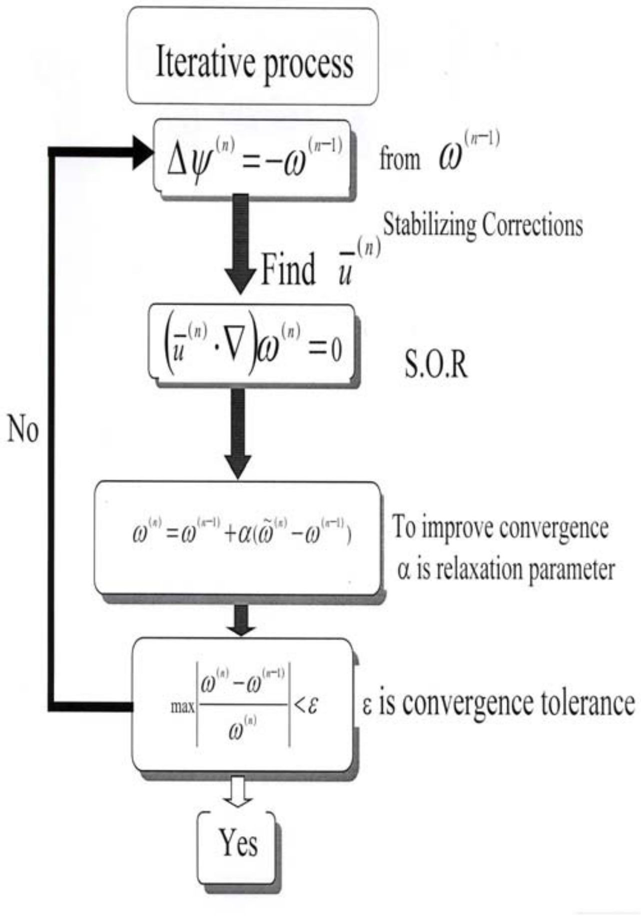



Figure 3

**Table 1**: Norm of errors for stream function and vorticity. Rate of convergence for test problem

| Grid | $err = \left\|\psi^h - \psi_{exact}\right\|_\infty$ | Rate | $err = \left\|\psi^h - \psi_{exact}\right\|_\infty$ | Rate |
|---|---|---|---|---|
| 6 x 6 | 0.173E-05 | - | 0.345E05 | - |
| 11 x 11 | 0.979E-07 | 4.14 | 0.196E-06 | 4.13 |
| 21 x 21 | 0.579E-08 | 4.08 | 0.116E-07 | 4.08 |
| 41 x 41 | 0.413E-09 | 3.81 | 0.862E-09 | 3.81 |

III. NUMERICAL RESULTS

Table1 shows the infinity norm of the absolute errors which are obtained from the grid systems having $N \times N$ nodes. The rate of convergence is defined by equation

$$m = \frac{\ln(err_{N_1}) - \ln(err_{N_2})}{\ln(N_1) - \ln(N_2)},$$

where $m$ is rate of convergence. This confirms that the finite difference scheme is of fourth-order accuracy

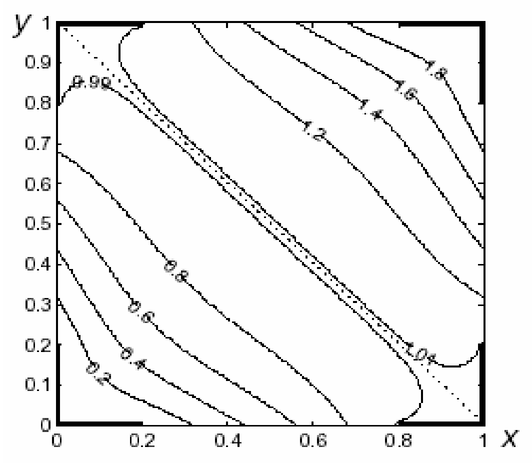
$$\Gamma_1^1:\ \omega = \omega_1(s) = A_\omega^1 \sin\left(K_1 \frac{s - s_1^1}{\tilde{s}_1^1 - s_1^1}\right);$$

$$\Gamma_2^1:\ \omega = \omega_2(s) = A_\omega^2 \sin\left(K_2 \frac{s - s_2^1}{\tilde{s}_2^1 - s_2^1}\right);$$

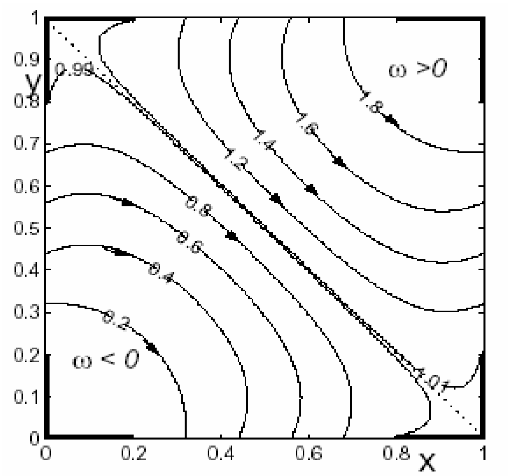
Figures 4-7 illustrate the effect of varying the boundary values of vorticity on both inflow parts of boundary. Figures 4 and 5 show streamlines and pressure contours for case where the values of vorticity prescribed on inflow parts have different sign. There are negative vorticity $(A_\omega^1 = -10, K_1 = 1)$ on $\Gamma_1^1$ and positive vorticity

$(A_\omega^2 = 10, K_2 = 1)$ on $\Gamma_2^1$ for Figure 4. On the contrary, there are positive vorticity $(A_\omega^1 = 10, K_1 = 1)$ on $\Gamma_1^1$ and negative vorticity $(A_\omega^2 = -10, K_2 = 1)$ on $\Gamma_2^1$ for Figure 5.

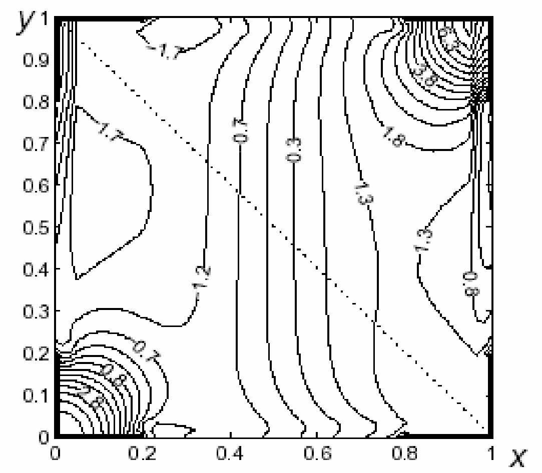
Figure 6 plots streamlines and pressure contours for $A_\omega^1 = A_\omega^2 = 10, K_1 = K_2 = 1$. In these cases the pattern of flow are not more symmetric. However, it is possible to get flow pattern in Figure 6 by rotation with respect diagonal between vortexes $(x = 0, y = 1)$ and $(x = 1, y = 0)$. In these cases the patterns of flow remain symmetric about diagonals of square similar to the potential flow (see in Figure 7).



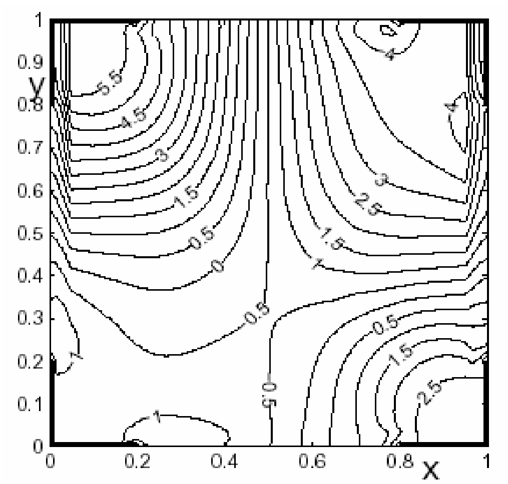
Streamlines
Figure 5.1



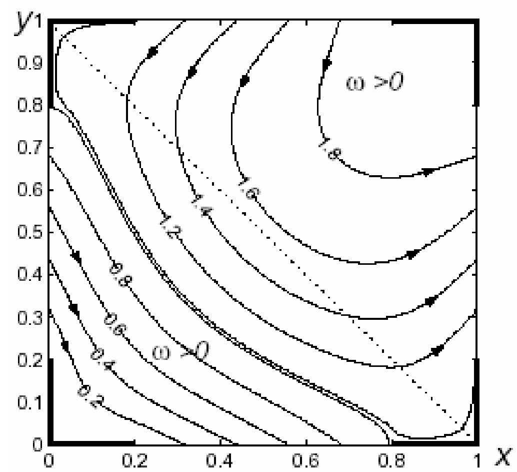
Streamlines
Figure 4.1



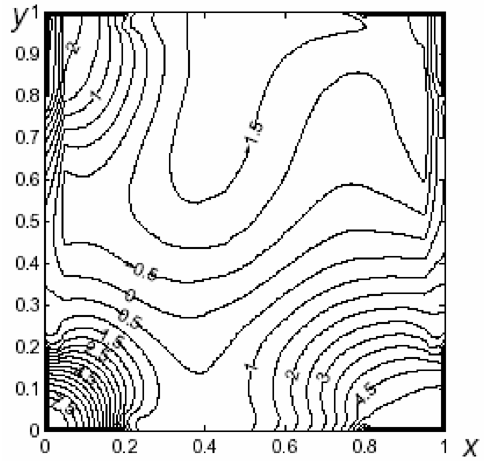
Pressure Contours
Figure 5.2



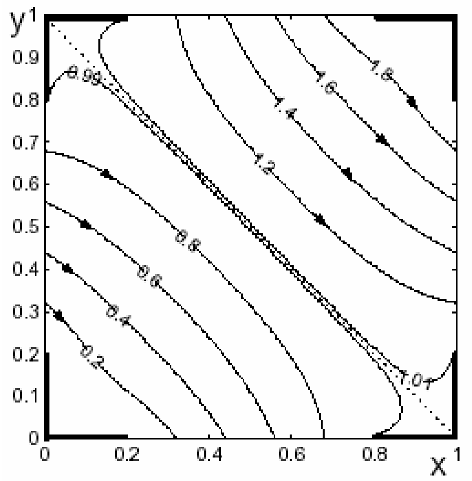
Pressure Contours
Figure 4.2



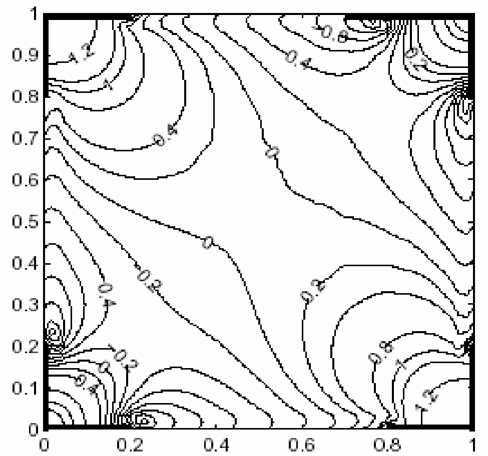
Streamlines
Figure 6.1




Pressure Contours
Figure 6.2




Streamlines
Figure 7.1




Pressure Contours
Figure 7.2

## IV. CONCLUSION

In this paper we introduce a new method for the approximate solution to the Euler equation of steady $2-D$ inviscid flow.

- Method is based on law of vorticity conservation along streamline.
- We have studied flowing through boundary value problem where on inflow parts of boundary domain the values of vorticity are given.
- The convergence of the finite difference method is confirmed by a test problem with known analytical solution.
- Numerical calculations are performed for 2-D inviscid flow through unite square domain with two inflow and two outflow parts of boundary.
- Dependence of flow pattern on the boundary values of vorticity at inflow parts is demonstrated.
- The basic idea of developed here method may be easily utilizes with finite element, finite volume method and etc.

## ACKNOWLEDGMENT

The authors would like to thank Thai Research Fund for supporting this work. Also, I would like to thank Assoc. Prof. Dr. Nikolay Moshkin for giving the idea of this work.

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