Minimum Degree Conditions for Graphs to be (g, f, n)-Critical Graphs

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Abstract—**Let G be a graph of order p, and let a and b and n be nonnegative integers with $1 \le a \le b$, and let g and f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for all $x \in V(G)$. A (g, f)-factor of graph G is defined as a spanning subgraph F of G such that $g(x) \le d_F(x) \le f(x)$ for each $x \in V(G)$. Then a graph G is called a (g, f, n)-critical graph if after deleting any n vertices of G the remaining graph of G has a (g, f)-factor. In this paper, we prove that every graph G is a (g, f, n)-critical graph if its minimum degree is greater than $p+a+b-2\sqrt{(a+1)p-bn+1}$. Furthermore, it is showed that the result in this paper is best possible in some sense.**

Index Term*–*graph, minimum degree,* (g, f)*-factor,* (g, f, n)*-critical graph

1 Introduction

In this paper, we consider a finite graph G with vertex set $V(G)$ and edge set $E(G)$, which has neither loops nor multiple edges. For any vertex x of G, we denote by $d_G(x)$ the degree of x in G. We denote by $\delta(G)$ the minimum vertex degree of G. For any $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$ and $G-S = G[V(G)\setminus S]$.

Let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for each $x \in V(G)$. A (g, f)-factor of graph G is defined as a spanning subgraph F of G such that $g(x) \le d_F(x) \le f(x)$ for each $x \in V(G)$ (Where of course d_F denotes the degree in F). If $g(x) = f(x)$ for each $x \in V(G)$, then a (g, f)-factor is called an f-factor. If $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then a (g, f)-factor is called an $[a, b]$-factor. If $g(x) = f(x) = k$ for all $x \in V(G)$, then a (g, f)-factor is called a k-factor. A graph G is called a (g, f, n)-critical graph if after deleting any n vertices of G the remaining graph of G has a (g, f)-factor. If G is a (g, f, n)-critical graph, then we also say that G is (g, f, n)-critical. If $g(x) = f(x)$ for each $x \in V(G)$, then a (g, f, n)-critical graph is an (f, n)-critical graph. If $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then a (g, f, n)-critical graph is an (a, b, n)-critical graph. If $a = b = k$, then an (a, b, n)-critical graph is simply called a (k, n)-critical graph. In particular, a $(1, n)$-critical graph is simply called an n-critical graph. The other notations and definitions not given in this paper can be found in [1].

Q. Yu [2] gave the characterization of n-critical graphs. O. Favaron [3] studied the properties of n-critical graphs. G. Liu and Q. Yu [4] studied the characterization of (k, n)-critical graphs. The characterization of (a, b, n)-critical graphs with $a < b$ was given by G. Liu and J. Wang [5]. S. Zhou [6,7,8,13] gave some sufficient conditions for graphs to be (a, b, n)-critical graphs. J. Li [9] showed two sufficient conditions for graphs to be (a, b, n)-critical graphs. S. Zhou [10] obtained a sufficient condition for graphs to be (g, f, n)-critical graphs. The characterization of (g, f, n)-critical graphs was given by J. Li and H. Matsuda [11]. In this paper, we obtain two new sufficient conditions for graphs to be (g, f, n)-critical graphs. The main results will be given in the following section. The following results on k-factors and (a, b, n)-critical graphs and (g, f, n)-critical graphs are known.

In [12], Y. Egawa and H. Enomoto proved the following result for the existence of k-factors.

Theorem 1 [12] *Let $k \ge 2$ be an integer, and let G be a graph of order n, kn is even. If*

$$\delta(G) > n + 2k - 2\sqrt{kn+1},$$

then G has a k-factor.

In [5], G. Liu and J. Wang gave a necessary and sufficient condition for graphs to be (a, b, n)-critical graphs.

Theorem 2 [5] *Let a, b and n be nonnegative integers with $1 \le a < b$, and G a graph of order $p \ge a+n+1$. Then G is (a, b, n)-critical if and only if*

$$b|S| + d_{G-S}(T) - a|T| \ge bn, \text{ or}$$
$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \le b|S| - bn$$

for all subset S of $V(G)$ with $|S| \ge n$, where $T = \{x :$

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$x \in V(G) \setminus S, d_{G-S}(x) \leq a-1\}$ and $p_j(G-S) = |\{x \in V(G) \setminus S : d_{G-S}(x) = j\}|$.

In [10], S. Zhou showed a sufficient condition for graphs to be (g, f, n)-critical graphs.

Theorem 3 [10] *Let G be a graph, and let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$. If $g(x) \leq d_G(x)$ and $f(x)(d_G(y) - n) \geq d_G(x)g(y)$ for each $x, y \in V(G)$, then G is a (g, f, n)-critical graph. Here n is a nonnegative integer.*

2 The Proof of Main Theorems

In this paper, we obtain a new sufficient condition for graphs to be (g, f, n)-critical graphs. Our result is the extension of Theorem 1.

Theorem 4 *Let G be a graph of order p, and let a, b and n be nonnegative integers such that $1 \leq a < b$, and let g and f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) < f(x) \leq b$ for each $x \in V(G)$. If*

$$\delta(G) > p + a + b - 2\sqrt{(a+1)p - bn + 1}, \qquad (1)$$

then G is a (g, f, n)-critical graph.

In Theorem 4, if $n = 0$, then we obtain the following corollary.

Corollary 1 *Let G be a graph of order p, and let a and b be integers such that $1 \leq a < b$, and let g and f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) < f(x) \leq b$ for each $x \in V(G)$. If*

$$\delta(G) > p + a + b - 2\sqrt{(a+1)p + 1},$$

then G has a (g, f)-factor.

According to Corollary 1 and the definition of (g, f, n)-critical graph, we obtain easily the following result.

Theorem 5 *Let G be a graph of order p, and let a, b and n be nonnegative integers such that $1 \leq a < b$, and let g and f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) < f(x) \leq b$ for each $x \in V(G)$. If*

$$\delta(G) > p + a + b - 2\sqrt{(a+1)p + 1} + n,$$

then G is a (g, f, n)-critical graph.

Let S and T be disjoint subsets of $V(G)$. We write $e_G(S,T) = |\{xy \in E(G) : x \in S, y \in T\}|$, $f(S) = \sum_{x \in S} f(x)$, $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$, and $g(T) = \sum_{x \in T} g(x)$. Our proof of Theorem 4 relies heavily on the following theorem.

Theorem 6 [11] *Let G be a graph, $n \geq 0$ an integer, and let g and f be two integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$. Then G is a (g, f, n)-critical graph if and only if*

$$\begin{aligned} \delta_G(S,T) &= f(S) + d_{G-S}(T) - g(T) \\ &\geq \max\{f(N) : N \subseteq S, |N| = n\} \end{aligned}$$

for all disjoint subsets S and T of $V(G)$ with $|S| \geq n$.

Proof of Theorem 4. Suppose a graph G satisfies the condition of the theorem, but it is not a (g, f, n)-critical graph. Then, by Theorem 6, there exist disjoint subsets S and T of $V(G)$ with $|S| \geq n$ such that

$$\begin{aligned} \delta_G(S,T) &= f(S) + d_{G-S}(T) - g(T) \\ &\leq \max\{f(N) : N \subseteq S, |N| = n\} - 1. \end{aligned} \qquad (2)$$

We choose subsets S and T such that $|T|$ is minimum and S and T satisfy (2).

Claim 1. $d_{G-S}(x) \leq g(x) - 1 \leq b - 2$ for each $x \in T$.

Proof. Suppose that there exists a vertex $x \in T$ such that $d_{G-S}(x) \geq g(x)$. Then the subsets S and $T - \{x\}$ satisfy (2), which contradicts the choice of T. Completing the proof of Claim 1.

If $T = \emptyset$, then by (2), $f(S) - 1 \geq \max\{f(N) : N \subseteq S, |N| = n\} - 1 \geq \delta_G(S,T) = f(S)$, a contradiction. Hence, $T \neq \emptyset$. Let

$$h = \min\{d_{G-S}(x) : x \in T\}.$$

According to Claim 1, we have

$$0 \leq h \leq b - 2,$$

and

$$\delta(G) \leq h + |S|. \qquad (3)$$

According to (2) and $|S| + |T| \leq p$, we get that

$$\begin{aligned} bn - 1 &\geq \max\{f(N) : N \subseteq S, |N| = n\} - 1 \\ &\geq \delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \\ &\geq (a+1)|S| + d_{G-S}(T) - (b-1)|T| \\ &\geq (a+1)|S| + h|T| - (b-1)|T| \\ &= (a+1)|S| - (b-h-1)|T| \\ &\geq (a+1)|S| - (b-h-1)(p-|S|) \\ &= (a+b-h)|S| - (b-h-1)p. \end{aligned}$$

Thus, we obtain

$$|S| \leq \frac{(b-h-1)p + bn - 1}{a+b-h}. \qquad (4)$$

In view of (3) and (4), we have

$$\delta(G) \leq h + |S| \leq h + \frac{(b-h-1)p + bn - 1}{a+b-h}. \qquad (5)$$

Let $f(h) = h + \frac{(b-h-1)p+bn-1}{a+b-h}$. In the range of $h \le a+b$, the function $f(h)$ attains its maximum value at $h = a + b - \sqrt{(a+1)p - bn + 1}$. Since $0 \le h \le b-2$, then we have

$$f(h) \le f(a+b-\sqrt{(a+1)p-bn+1}) = p+a+b-2\sqrt{(a+1)p-bn+1},$$

that is,

$$\delta(G) \le p+a+b-2\sqrt{(a+1)p-bn+1},$$

this contradicts (1).

From the argument above, we deduce the contradiction. Hence, G is a (g, f, n)-critical graph.
Completing the proof of Theorem 4.

Remark. Let us show that the condition $\delta(G) > p+a+b-2\sqrt{(a+1)p-bn+1}$ in Theorem 4 can not be replaced by $\delta(G) \ge p+a+b-2\sqrt{(a+1)p-bn+1}$. Let $a = 2, b = 3$, and $n \ge 0$ an integer. Let $H = K_{n+1} \bigvee (K_a \cup K_a)$. Then $p = 2a+n+1$ and $\delta(H) = a+n$. Thus, we obtain easily $p+a+b-2\sqrt{(a+1)p-bn+1} = a+n$, that is, $\delta(H) = p+a+b-2\sqrt{(a+1)p-bn+1}$. Let $S = V(K_{n+1}) \subseteq V(H)$, $T = V(K_a \cup K_a) \subseteq V(H)$. Since $a \le g(x) < f(x) \le b$ and $b = a+1$, then we have $g(x) = a$ and $f(x) = b = a+1$. Thus, we get

$$\begin{aligned} \delta_H(S,T) &= f(S) + d_{H-S}(T) - g(T) \\ &= b|S| + (a-1)|T| - a|T| \\ &= b|S| - |T| \\ &= b(n+1) - 2a \\ &= bn + b - 2a \\ &= bn - 1 \qquad (Since\ a = 2\ and\ b = 3) \\ &< bn = \max\{f(N) : N \subseteq S, |N| = n\}. \end{aligned}$$

By Theorem 6, H is not a (g, f, n)-critical graph. In the above sense, the result of Theorem 4 is best possible.

We may adopt the similar way to argue the condition $\delta(G) > p+a+b-2\sqrt{(a+1)p+1}+n$ in Theorem 5, and the condition $\delta(G) > p+a+b-2\sqrt{(a+1)p+1}+n$ in Theorem 5 is the best possible in some sense.

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