# Haar Wavelet Matrices Designation in Numerical Solution of Ordinary Differential Equations

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*Abstract* — **Wavelet transforms or wavelet analysis is a recently developed mathematical tool for many problems. Wavelets also can be applied in numerical analysis. In this paper, we apply Haar wavelet methods to solve ordinary differential equations with initial or boundary condition known. To avoid the tedious calculations and to promote the study of wavelets to beginners, we proposed a simple way to perform the calculations for the matrix representation. The procedure applied in this paper is taking the Haar Series for the highest order of differential and integrate the series. Four numerical examples are shown which including first, second, higher order differential equations with constant and variable coefficients. The results show that the proposed way are quite reasonable when compare to exact solution.**

*Index Terms* — **Haar wavelet methods, matrix representation, ordinary differential equations, computer algebra system**

## I. Introduction

Wavelet transform or wavelet analysis is a recently developed mathematical tool for signal analysis. To date, wavelets have been applied in numerous disciplines such as image compression, data compression, denoising data and many more [1]. In numerical analysis, wavelets also serve as a Galerkin basis to solve partial differential equations. Wavelet analysis involves tedious calculations. Practically, the calculation is done by using software with certain commands or special toolboxes. It may make the beginner feel intimidate [2]. Meanwhile, Haar function always has been choose for educational purpose, especially in many papers or books written on topic of introduction to wavelets [3-4]. Due to the powerful of wavelets in plenty fields, there are many works, such as in [5-6] in promoting the study of wavelet even in undergraduate level.

Meanwhile in numerical analysis, wavelet based algorithms have become an important tools because of the properties of localization. One of the popular families of wavelet is Haar wavelets. Due to its simplicity, Haar wavelets had become an effective tool for solving many problems, among that are Ordinary Differential Equations, ODEs and Partial Differential Equations, PDEs. In these works, the operator or matrix representation is expanded in a wavelet basis. Sometime, the works for writing up the operational matrices are quite tedious especially when ones intend to perform the calculation in high resolution. This will discourage the beginner to study how wavelet basis can be applied to solve differential equations, especially when in the works of encouraging the study of wavelets in undergraduate level which were done in [5-7].

In solving ordinary differential equations by using Haar wavelet related method, Chen and Hsiao [8-9] had derived an operational matrix of integration based on Haar wavelet. Lepik [10] had solved higher order as well as nonlinear ODEs by using Haar wavelet method. There are discussions by other researchers [11-13]. We are not going to compare with these distinguish scholars, but intend to come out with a simple procedure to solve ordinary differential equations which make use of the power of wavelets. With this, we hope that even for an undergraduate student can also perform the numerical calculation by using wavelets as a tool in order to solve ordinary differential equation problems without using any complicated algorithm but sufficient with the help of computer algebra system or Excel. With this, it is hoped that it will encourage the study of wavelet in undergraduate level.

## II. Haar Wavelet

Haar wavelet is the simplest wavelet. Haar transform or Haar wavelet transform has been used as an earliest example for orthonormal wavelet transform with compact support. The Haar wavelet transform is the first known wavelet and was proposed in 1909 by Alfred Haar. The Haar function can be described as a step function $\psi(x)$ and in Fig. 1 as follows :

$$\psi(x) = \begin{cases} 1 & 0 \le x \le 0.5 \\ -1 & 0.5 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

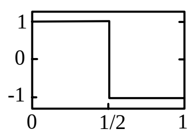


Fig. 1 Haar function

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This is also called mother wavelet. In order to perform wavelet transform, Haar wavelet uses translations and dilations of the function, i.e. the transform make use of following function

$$\psi(x) = \psi(2^j x - k) \quad (2)$$

Translation / shifting $\psi(x) = \psi(x - k)$

Dilation / scaling $\psi(x) = \psi(2^j x)$

where this is the basic works for wavelet expansion.

With the dilation and translation process as in Eq.(2), ones can easily obtain father wavelet, daughter wavelet, granddaughter wavelet and so on as in fig. 2.

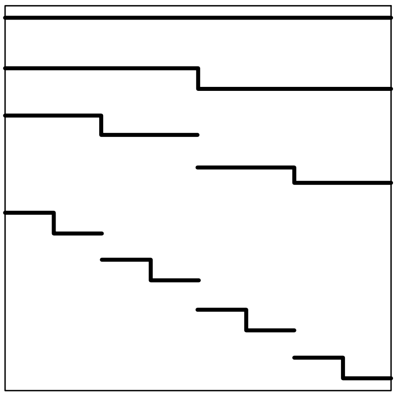


Fig. 2 Haar Wavelet (up to 2 resloution levels)

In the matrix form, the Haar matrix for resolution up to 2 levels is given below :

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

The pattern of the Haar wavelet undergo translation and dilation process and its matrix pattern are observed when the Haar wavelet been used in solving ordinary differential equations.

There are few important characteristics of Haar wavelet. First, it is a piecewise constant functions. Second, it is the simplest orthonormal wavelets, (Not all wavelets are orthonormal !) Third, it has the compact support [0, 1]. The first and third characteristics made Haar wavelet cannot be applied directly to solve ODE . The piecewise constant function means it is actually not continuous, and thus it cannot be differentiated in the points of discontinuity.

There is two possibilities to overcome these problems. One way is to regularize the piecewise constant Haar function using interpolation splines. However, this is not easy to do, thus the simplicity of Haar wavelet get loss. We did not apply this way in our work.

The second way is proposed by Chen and Hsiao [8], which is expand the highest derivative in the differential equation into Haar series. Other derivatives are obtained through integrations. The whole system is discretized by collocation method. The collocation method here is actually refer to segmentation process.

## III. Haar wavelet Method for ordinary differential equations

For solving linear ordinary differential equation with $n^{\text{th}}$ order, say

$$A_1 y^{(n)}(x) + A_2 y^{(n-1)}(x) + \cdots + A_n y(x) = f(x),$$

where $x \in [A, B]$ and initial conditions

$y^{(n-1)}(A)$, $y^{(n-2)}(A)$ $\cdots$, $y(A)$ are known.

We follow the work done by Lepik [10]. Say we intend to do until $j$ level of resolution, hence we let $m = 2(2^j)$. The interval $[A, B]$ will be divided into $m$ subintervals, hence $\Delta x = \dfrac{B-A}{m}$ and the matrices are in the dimension of $m \times m$.

Here we suggest the step by step procedures for easy understanding. Mainly, there are 5 steps as shown in the procedure as follow.

Procedure :

Step 1: Let $y^{(n)}(x) = \sum_{i=1}^{m} a_i h_i(x)$ where $h$ is haar matrix and $a_i$ is the wavelet coefficients.

Step 2: Obtain appropriate $v$ order of $y(x)$ by using

$$y^{(v)}(x) = \sum_{i=1}^{m} a_i P_{n-v,i}(x) + \sum_{\sigma=0}^{n-v-1} \frac{1}{\sigma!}(x-A)^{\sigma} y_0^{(v+\sigma)}$$

Step 3: Replace $y^{(n)}(x)$ and all the value of $y^{(v)}(x)$ into the problem.

Step 4: Calculate the wavelet coefficients, $a_i$ .

Step 5: Obtain the numerical solution for $y(x)$ .

Step 2 is the key procedure where matrix $P_{n-v,i}(x)$ will be counted. If ones intend to do the calculation until level $j$ of resolution, ones will obtain the matrix $P_{n-v,i}(x)$ (let $n - v = \alpha$ ) as in the pattern shown in fig. 3 where $C = B - A$

In the designation of matrix $P$ in step 2 of the procedure above, we found that the trend is very interesting, Firstly, the elements can be simplified by common factor. Secondly, there is some elements need to be counted and thirdly, there is some elements which can be obtained by ‘copy and paste’. This trend is similar to the graph of Haar wavelet family in Figure 2 that we have shown in Section II.

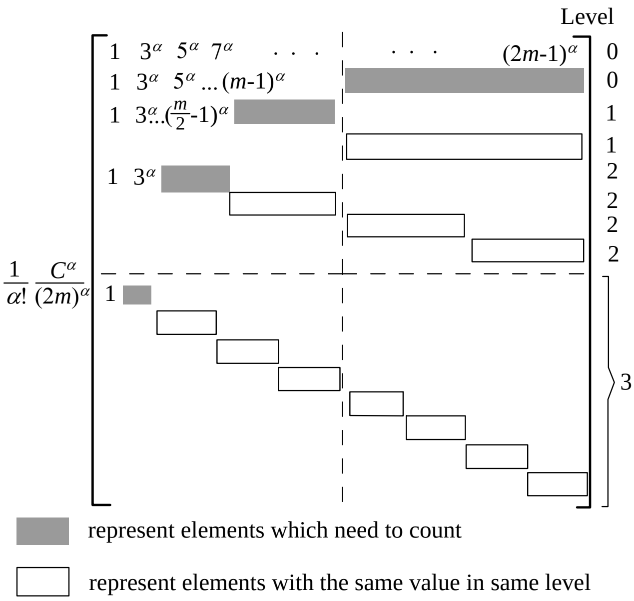


Fig. 3: Designation of matrix *P*.

The pattern of design of the matrices *P* is similar to Haar wavelet. For calculation of *P* matrices, we focus on the elements need to be counted as shown in fig. 3. Here, we suggest the following algorithm for counting the elements which are required.

Algorithm :

$$\frac{1}{\alpha!}\frac{1}{(2m)^\alpha}\left[(C(\frac{m}{2^L}+2l-1))^\alpha - 2(C(2l-1))^\alpha\right]$$

$\alpha = n - v$

$L = 0,1,2,\cdots, j$ (Level of Haar wavelet)

$l = 1,2,3,\cdots,\dfrac{m}{2(2^L)}$

and where $C = B - A$

In the matrix *P* shown in the fig 3, we factored out the common factor $\dfrac{1}{\alpha!}\dfrac{C^\alpha}{(2m)^\alpha}$ for all the elements in the matrix which we calculated using the algorithm above, for the reason of simplicity.

The good thing here is we need to obtain the matrix $P_1$, or $P_2$ and so on for certain level of wavelet once only, and the same matrix can be stored and to be used for other different ODE if there is $P_1$, or $P_2$ and so on needed to be obtained in this Step 2 if the interval of the problem and the level of resolution is same.

## IV. Numerical Examples

Three examples are shown. All the calculation is done by using 3 levels of Haar wavelet.

Example 1:

Solve the equation $y''(x) + y(x) = \sin x + x\cos x$,

$x \in [0,1]$ with $y(0) = 1$, $y'(0) = 1$.

Exact solution :

$$y(x) = \cos x + \frac{5}{4}\sin x + \frac{1}{4}(x^2 \sin x - x\cos x)$$

By using

Step 1 : $y''(x) = \sum_{i=1}^{m} a_i h_i(x)$

Step 2: $y(x) = \sum_{i=1}^{m} a_i P_{2,i}(x) + \sum_{\sigma=0}^{2-0-1} \frac{1}{\sigma!}(x-0)^\sigma y_0^{(\sigma)}$

$$= \sum_{i=1}^{m} a_i P_{2,i}(x) + 1 + x$$

Step 3 : Hence, we get

$$y''(x) + y(x) = \sin x + x\cos x$$

$$\sum_{i=1}^{m} a_i h_i(x) + \sum_{i=1}^{m} a_i P_{2,i}(x) + 1 + x = \sin x + x\cos x$$

$$\sum_{i=1}^{m} a_i\left[h_i(x) + P_{2,i}(x)\right] = \sin x + x\cos x - 1 - x$$

The matrix *P* is shown below.

$$\frac{1}{2!}\frac{1}{32^2}\begin{pmatrix}
1 & 3^2 & 5^2 & 7^2 & 9^2 & 11^2 & 13^2 & 15^2 & 17^2 & 19^2 & 21^2 & 23^2 & 25^2 & 27^2 & 29^2 & 31^2 \\
1 & 3^2 & 5^2 & 7^2 & 9^2 & 11^2 & 13^2 & 15^2 & 287 & 343 & 391 & 431 & 463 & 487 & 503 & 511 \\
1 & 3^2 & 5^2 & 7^2 & 79 & 103 & 119 & 127 & & & & & & & & \\
 & & & & & & & & 1 & 3^2 & 5^2 & 7^2 & 79 & 103 & 119 & 127 \\
1 & 3^2 & 23 & 31 & & & & & & & & & & & & \\
 & & & & 1 & 3^2 & 23 & 31 & & & & & & & & \\
 & & & & & & & & 1 & 3^2 & 23 & 31 & & & & \\
 & & & & & & & & & & & & 1 & 3^2 & 23 & 31 \\
1 & 7 & & & & & & & & & & & & & & \\
 & & 1 & 7 & & & & & & & & & & & & \\
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 & & & & & & & & & & 1 & 7 & & & & \\
 & & & & & & & & & & & & 1 & 7 & & \\
 & & & & & & & & & & & & & & 1 & 7
\end{pmatrix}$$

Step 4: Solving the system of linear equation. We obtain wavelet coefficients, $a_i$ .

Step 5: Obtain the numerical solution for $y(x)$ as in Table 1.

Table 1 : Numerical solution for Example 1

| $x$ ( /32) | Solution | Exact solution | Absolute error |
|---|---|---|---|
| 1 | 1.030777 | 1.030767 | 0.000010 |
| 3 | 1.089527 | 1.089496 | 0.000031 |
| 5 | 1.144880 | 1.144700 | 0.000180 |
| 7 | 1.196844 | 1.196643 | 0.000201 |
| 9 | 1.247001 | 1.245594 | 0.001407 |
| 11 | 1.293240 | 1.291819 | 0.001421 |
| 13 | 1.337136 | 1.335577 | 0.001559 |
| 15 | 1.378678 | 1.377118 | 0.001560 |
| 17 | 1.427640 | 1.416676 | 0.010964 |
| 19 | 1.465381 | 1.454467 | 0.010914 |
| 21 | 1.501598 | 1.490681 | 0.010917 |
| 23 | 1.536265 | 1.525485 | 0.010780 |
| 25 | 1.570336 | 1.559012 | 0.011324 |
| 27 | 1.602462 | 1.591364 | 0.011098 |
| 29 | 1.633465 | 1.622605 | 0.010860 |
| 31 | 1.663308 | 1.652763 | 0.010545 |

Example 2 shows the solution of higher order differential equation by using Haar wavelet method. The result shows that the method is applicable for higher order DE, even for variable coefficient.

Example 2:
Solve the equation $y^{(4)}(x) + xy(x) = 16\sin 2x + x\sin 2x$,
$x \in [0,1]$ with $y(0)=0$, $y'(0)=2$, $y''(0)=0$, $y'''(0)=-8$
Exact solution : $y(x) = \sin 2x$
By carry out Step 1 to 3, we obtain

$$\sum_{i=1}^{m} a_i \left[h_i(x) + xP_{4,i}(x)\right] + 2x^2 - \frac{4}{3}x^4 = 16\sin 2x + x\sin 2x$$

By carry out Step 4 to 5, we obtained the results as in Table 2.

Table 2 : Numerical solution for Example 2

| $x$ ( /32) | Solution | Exact solution | Absolute error |
|---|---|---|---|
| 1 | 0.062459 | 0.062459 | 0.000000 |
| 3 | 0.186405 | 0.186403 | 0.000002 |
| 5 | 0.307463 | 0.307439 | 0.000024 |
| 7 | 0.423743 | 0.423676 | 0.000067 |
| 9 | 0.533859 | 0.533303 | 0.000556 |
| 11 | 0.635637 | 0.634607 | 0.001030 |
| 13 | 0.727687 | 0.726009 | 0.001678 |
| 15 | 0.808581 | 0.806081 | 0.002500 |
| 17 | 0.887019 | 0.873575 | 0.013444 |
| 19 | 0.946546 | 0.927437 | 0.019109 |
| 21 | 0.992728 | 0.966827 | 0.025901 |
| 23 | 1.024951 | 0.991129 | 0.033822 |
| 25 | 1.042978 | 0.999966 | 0.043012 |
| 27 | 1.046518 | 0.993198 | 0.053320 |
| 29 | 1.035738 | 0.970932 | 0.064806 |
| 31 | 1.010991 | 0.933514 | 0.077477 |

Example 3 involves ODE with exponential coefficients. The results is compare with the results obtained by series solution. Due to the behavior of exponential, we use a relatively small step size if compared to previous two examples, which is until 4 level of resolution.

Example 3:
Solve the equation $y'(x) + e^x y(x) = x^2$,
$x \in [0,1]$ with $y(0) = 4$.
Series solution : $y(x) = 4 - 4x + x^3 + \frac{1}{12}x^4$ (up to first four nonzero terms only)
By carry out Step 1 to 3, we obtain

$$y'(x) + e^x y(x) = \sum_{i=1}^{m} a_i \left[h_i(x) + e^x P_{1,i}(x)\right] = x^2 - 4e^x$$

By carry out Step 4 to 5, we obtained the results as in Table 3 .

Table 3 : Numerical solution for Example 3

| $x$ ( /64) | Solution | Series solution | Difference |
|---|---|---|---|
| 1 | 3.936517 | 3.937504 | 0.000987 |
| 3 | 3.807570 | 3.812603 | 0.005034 |
| 5 | 3.674638 | 3.687980 | 0.013342 |
| 7 | 3.537674 | 3.563820 | 0.026146 |

Example 4:
Solve the equation $y''(x) + y(x) = \sin x + x\cos x$,
$x \in [0,1]$ with $y(0) = 1$, $y(1) = 1.667433$.
Exact solution : $y(x) = \cos x + \frac{5}{4}\sin x + \frac{1}{4}(x^2 \sin x - x\cos x)$

$$y'(x) + e^x y(x) = \sum_{i=1}^{m} a_i \left[h_i(x) + e^x P_{1,i}(x)\right] = x^2 - 4e^x$$

Compare to Example 1, slightly different consideration is done in Step 2 and 3.

Step 1 : $y''(x) = \sum_{i=1}^{m} a_i h_i(x)$

Step 2: $y(x) = \sum_{i=1}^{m} a_i P_{2,i}(x) + \sum_{\sigma=0}^{2-0-1} \frac{1}{\sigma!}(x-0)^{\sigma} y_0^{(\sigma)}$

$$= \sum_{i=1}^{m} a_i P_{2,i}(x) + 1 + xy_0'$$

where $y_0'$ is unknown. $y_0'$ can be found by consider $y(1) = 1.667433$.

$$y(1) = \sum_{i=1}^{m} a_i P_{2,i}(1) + 1 + (1)y_0' = 1.667433,$$

Hence, $y_0' = -\sum_{i=1}^{m} a_i P_{2,i}(1) + 0.667433$

Step 3 : From $y''(x) + y(x) = \sin x + x\cos x$, we obtain

$$\sum_{i=1}^{m} a_i h_i(x) + \sum_{i=1}^{m} a_i P_{2,i}(x) + 1 + x\left[-\sum_{i=1}^{m} a_i P_{2,i}(1) + 0.667433\right] = \sin x + x\cos x$$

Hence, $\sum_{i=1}^{m} a_i \left[h_i(x) + P_{2,i}(x) - xP_{2,i}(1)\right]$

$$= \sin x + x\cos x - 1 - 0.667433x$$

By carry out Step 4 to 5, we obtained the results as in Table 4.

Table 4 : Numerical solution for Example 4

| $x$ ( /32) | Solution | Exact solution | Absolute error |
|---|---|---|---|
| 1 | 1.036301 | 1.030767 | 0.005534 |
| 3 | 1.101311 | 1.089496 | 0.011815 |
| 5 | 1.159368 | 1.144700 | 0.014668 |
| 7 | 1.209068 | 1.196643 | 0.012425 |
| 9 | 1.257571 | 1.245594 | 0.011977 |
| 11 | 1.298038 | 1.291819 | 0.006219 |
| 13 | 1.336772 | 1.335577 | 0.001195 |
| 15 | 1.372508 | 1.377118 | 0.004610 |
| 17 | 1.415362 | 1.416676 | 0.001314 |
| 19 | 1.449621 | 1.454467 | 0.004846 |
| 21 | 1.482400 | 1.490681 | 0.008281 |
| 23 | 1.516283 | 1.525485 | 0.009202 |
| 25 | 1.549619 | 1.559012 | 0.009393 |
| 27 | 1.583481 | 1.591364 | 0.007883 |
| 29 | 1.616938 | 1.622605 | 0.005667 |
| 31 | 1.650668 | 1.652763 | 0.002095 |

## V. Conclusion

The designation of matrices $P$ is shown. The algorithm and procedure have been applied to use Haar wavelet method in solving ODEs. The result is comparable to the exact solution.

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