Mathematical Expressions to Describe the Relative Power Deviation and Capacity Factor of Wave Height Datasets

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Abstract—**Wave energy, like other sources of renewable energy, is receiving more interest than ever before. The challenge for an emerging wave energy sector is to address the variability in wave characteristics using careful site selection, and design conversion systems with high part load efficiency. This paper tackles the first of these objectives by developing simple mathematical expressions, based on a single parameter of the univariate gamma distribution, to describe the relative power deviation and 'capacity factor' of wave height datasets.**

Index Terms—**gamma distribution, ocean wave energy, capacity factor.**

I. Introduction

Ocean waves represent a decayed form of solar energy arising from the action of wind and characterized by a spectrum comprising wave height, period and direction. These characteristics are variable over time and depend on lunar and solar forces, temperature gradients, shadowing and depth to seabed [1]. Wave power levels differ from open sea to coastal zones and may be in the order of a few watts thru to a megawatt or more per meter crest length [2].

Due to the inherent variability of wave characteristics wave energy conversion has traditionally been greeted by considerable skepticism (see for example [2]). Increasing attention given to waves and swell as an alternate and renewable source of power coincides with global concerns over diminishing fossil fuel reserves. Since wave energy conversion systems cannot be continuously operated at their nominal rating—like conventional coal or gas plants can—the challenge for an emerging wave energy sector is to address irregularities in wave characteristics using careful site selection, and design energy conversion systems with high part load efficiency.

This paper tackles the first of these aforementioned objectives by providing a method of assessing the natural dependability of a wave height dataset. Here 'dependability' is quantified in terms of relative power deviation and capacity factor (defined as the predicted ratio of mean power over nominal power). The expressions developed in this work are an extension of the gamma (probability) distribution which is generally well suited to wave height data [3].

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II. Mathematical Development

A. Preliminary assumptions regarding wave power

The total kinetic and potential energy per unit surface area of a simplistic ocean wave is given by (1) [4]. Here ρ_w is the density of water (kg/m^3) and H is the wave height.

$$E = E_k + E_p = \frac{\rho_w\, g\, H^2}{8} \quad (J/m^2) \tag{1}$$

Waves with different wavelengths combine together and travel at specific group velocities [4]. This leads to expressions of wave power given by (2) and (3) where crest width, wave period and group velocity (incorporating wave period) are represented by L, T and c_g respectively. Expression (3) suggests that group velocity is half the wave propagation velocity and applies to deep waters only.

$$\frac{P}{L} = c_g\left(\frac{\rho_w\, g\, H^2}{8}\right) \quad (Watts\ /m) \tag{2}$$

$$\frac{P}{L} \approx \frac{\rho_w\, g^2\, H^2 T}{32\pi} \quad (Watts\ /m) \tag{3}$$

The variability of group velocity can be ignored since its effect on wave power is only minor. This supposition is also made by Denniss [3] and verified—to an extent—within the same paper. Re-arranging (2) and neglecting variations in c_g leads to (4) where by suitable choice of crest width (L) bracketed constants can be forced, for arguments sake, to equal unity.

$$P = \left(\frac{1}{8}\rho_w\, g\, c_g\, L\right) H^2 \quad (Watts) \tag{4}$$

$$\therefore \quad when \quad \frac{\rho_w\, g\, c_g}{8}\cdot L = \frac{\rho_w\, g\, c_g}{8}\cdot\frac{8}{\rho_w\, g\, c_g} = 1 \tag{5}$$

$$P = H^2 \tag{6}$$

According to (6) the mean, relative deviation, capacity factor, and so on, of extractable wave power can be evaluated by the square of wave height which is unaffected by constants (5). Assumption (6) is a critical one.

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B. Univariate gamma distribution

Gamma probability density of single and continuous variable *x*, such as wave height data, is given by (7). When expressed in terms of Euler's generalized factorial (8) the function has a frequency plot known as a gamma distribution. Parameter *n* is usually referred to as the distribution *shape* and parameter μ as the *rate* (or *scale*).

$$f_G(x;n,\mu)=\left(\frac{\mu^n}{\Gamma(n)}\right)x^{n-1}e^{-\mu x} \quad (0\le x\le\infty) \tag{7}$$

$$\Gamma(n)=\int_0^\infty v^{n-1}e^{-v}\,dv \tag{8}$$

The expectation of a continuous distribution is its 1st moment about the origin and can be found using (9). The m^{th} derivative (at $M = 0$) of this function has a corresponding expectation (10).

$$M(t)=\int_0^\infty e^{tx}f(x)\,dx \tag{9}$$

$$EX_m=\frac{d^m}{d^m x}M(0) \tag{10}$$

Hence, for a gamma distribution,

$$M(t)=\left(\frac{\mu^n}{\Gamma(n)}\right)\int_0^\infty x^{n-1}e^{(t-\mu)x}\,dx \tag{11}$$

$$=\left(\frac{\mu^n}{\Gamma(n)}\right)\left(\frac{\Gamma(n)}{(\mu-t)^n}\right) \tag{12}$$

$$=\left(\frac{\mu}{\mu-t}\right)^n \quad \{\text{m.g.f.}\}. \tag{13}$$

Therefore,

$$EX_1=\frac{d}{dx}M(0) \tag{14}$$

$$EX_1=\frac{n\,\mu^n}{(\mu-0)^{n+1}}=\frac{n}{\mu} \tag{15}$$

$$EX_2=\frac{(n+1)\,n\,\mu^n}{(\mu-0)^{n+2}}=\frac{(n+1)\,n}{\mu^2} \tag{16}$$

$$EX_m=\frac{d^m}{d^m x}M(0)=\quad\cdots\quad=\frac{\Gamma(m+n)}{\Gamma(n)\,\mu^m} \tag{17}$$

Following on, the variance or squared-spread of *x* is found using (18),

$$VX_m=EX_{2m}-(EX_m)^2. \tag{18}$$

Noting (15) and (16),

$$VX_1=EX_2-(EX_1)^2 \tag{19}$$

$$=\frac{(n+1)\,n}{\mu^2}-\left(\frac{n}{\mu}\right)^2 \tag{20}$$

$$=\frac{n}{\mu^2} \tag{21}$$

$$VX_2=EX_4-(EX_2)^2 \tag{22}$$

$$=\frac{(n+3)(n+2)(n+1)\,n\,\mu^n}{(\mu-0)^{n+4}}-\left(\frac{(n+1)\,n}{\mu^2}\right)^2 \tag{23}$$

$$=\frac{n(2n+2)(2n+3)}{\mu^4}. \tag{24}$$

Alternatively, gamma distribution variance for any order moment (m=1, 2, 3 …) is expressed as,

$$VX_m=\frac{\Gamma(2m+n)}{\Gamma(n)\,\mu^{2m}}-\left(\frac{\Gamma(m+n)}{\Gamma(n)\,\mu^m}\right)^2 \tag{25}$$

$$=\frac{\Gamma(n)\cdot\Gamma(2m+n)-\Gamma(m+n)^2}{\Gamma(n)^2\,\mu^{2m}}. \tag{26}$$

Standard deviation (σ) is defined as the square root of variance.

$$\sigma_{x_m}=\sqrt{VX_m}=\sqrt{EX_{2m}-(EX_m)^2} \tag{27}$$

Applying (27) to a gamma distribution,

$$\sigma_{x1}=\sqrt{VX_1} \tag{28}$$

$$=\frac{\sqrt{n}}{\mu} \tag{29}$$

$$\sigma_{x_2}=\sqrt{VX_2} \tag{30}$$

$$=\frac{\sqrt{n(2n+2)(2n+3)}}{\mu^2}. \tag{31}$$

Or in more general terms, and following from (26),

$$\sigma_{x_m}=\sqrt{\frac{\Gamma(n)\cdot\Gamma(2m+n)-\Gamma(m+n)^2}{\Gamma(n)^2\,\mu^{2m}}} \tag{32}$$

$$=\frac{\sqrt{\Gamma(n)\cdot\Gamma(2m+n)-\Gamma(m+n)^2}}{\Gamma(n)\,\mu^m}. \tag{33}$$

A distribution has relative deviation defined by the ratio between its standard deviation and associated moment. This property is important when comparing datasets.

$$\psi_m = \frac{\sigma_{x_m}}{EX_m} \tag{34}$$

For a gamma distribution,

$$\psi_1 = \frac{\sigma_{x_1}}{EX_1} = \frac{\sqrt{n} \cdot \mu}{n \cdot \mu} \tag{35}$$

$$= \frac{\sqrt{n}}{n} \tag{36}$$

$$\psi_2 = \frac{\sigma_{x_2}}{EX_2} = \frac{\dfrac{\sqrt{n(2n+2)(2n+3)}}{\mu^2}}{\dfrac{(n+1)n}{\mu^2}} \tag{37}$$

$$= \frac{\sqrt{n(2n+2)(2n+3)}}{(n+1)n}. \tag{38}$$

In a generalized form,

$$\psi_m = \frac{\dfrac{\sqrt{\Gamma(n) \cdot \Gamma(2m+n) - \Gamma(m+n)^2}}{\Gamma(n)\mu^m}}{\dfrac{\Gamma(m+n)}{\Gamma(n)\mu^m}} \tag{39}$$

$$= \frac{\sqrt{\Gamma(n) \cdot \Gamma(2m+n) - \Gamma(m+n)^2}}{\Gamma(m+n)}. \tag{40}$$

Both (38) and (40, with m=2) describe the relative power deviation of a wave height dataset.

C. Capacity factor and the '10% rule'

Capacity factor is broadly defined by the ratio of an energy systems average power output over its maximum possible output in the same time period. This quantity is dimensionless and usually expressed in percentile fashion.

$$\text{CF} = \frac{P_{AVG}}{P_R} \tag{41}$$

Capacity factor of a wave height dataset can be deduced probabilistically using its relative power deviation and an upper cut-off limit assigned to its first order density integral. Limiting the cumulative integral—or sum probability—from 0 thru 0.9 is known as applying the '10% rule' [3]. This form of ideal or theoretical capacity attempts to balance the objectives of maximizing generator output and minimizing part load losses [3]. While useful when comparing datasets it should be emphasized that such an approach ignores the part-load problem by assuming all energy below the rated point can be converted into electricity at an equal level of efficiency.

For example, a gamma density function with n=4 and μ=3 has rated power corresponding to significant wave height at 2.227 metres. The distance (γ_{r1}) between rated wave height (H_R) and density function mean (EX_1) is expressed in units of standard deviation.

$$\gamma_{r1} = \frac{H_R - EX_1}{\sigma_{x1}} \tag{42}$$

In this example,

$$\gamma_{r1} = \frac{2.227 - 1.333}{(\sqrt{4}/3)} \approx 1.34. \tag{43}$$

Result (43) is a property of the gamma distribution. In other words, and irrespective of parameter values, distance between mean and rated height limited by the '10% rule' will always equal approximately 1.34 standard deviations. Hence it follows that,

$$H_R = \frac{1.34 \cdot \sqrt{n} + n}{\mu}. \tag{44}$$

Therefore,

$$CF_2 = \frac{P_{AVG}}{P_R} = \frac{EX_2}{H_R{}^2} \tag{45}$$

$$= \frac{\dfrac{(n+1)n}{\mu^2}}{\left(\dfrac{1.34\sqrt{n}+n}{\mu}\right)^2} \tag{46}$$

$$= \frac{(n+1)n}{(1.34\sqrt{n}+n)^2}. \tag{47}$$

Capacity factor of wave height data can also be resolved using distance γ_{r2} and relative deviation for second order moment as follows.

$$\gamma_{r2} = \frac{(H_R{}^2 - EX_2)}{\sigma_{x2}} \tag{48}$$

$$= \frac{(1.34\sqrt{n}+n)^2 - (n+1)n}{\sqrt{n(2n+2)(2n+3)}} \tag{49}$$

$$P_R = H_R{}^2 = P_{AVG}\ (1 + \gamma_{r2} \cdot \psi_2) \tag{50}$$

$$= P_{AVG}\left(1 + \frac{(1.34\sqrt{n}+n)^2 - (n+1)n}{(n+1)n}\right). \tag{51}$$

Substituting (51) into general equation (41),

$$CF_2 = \frac{P_{AVG}}{P_{AVG}\ (1+\gamma_{r2}\cdot\psi_2)} \tag{52}$$

$$= \frac{P_{AVG}}{P_{AVG}\ \left(1+\frac{(1.34\sqrt{n}+n)^2-(n+1)n}{(n+1)n}\right)} \tag{53}$$

$$= \frac{(n+1)n}{(1.34\sqrt{n}+n)^2}. \tag{54}$$

In a generalized form,

$$CF_m = \frac{\Gamma(m+n)}{\Gamma(n)(1.34\sqrt{n}+n)^m}. \tag{55}$$

III. Conclusion

It has been shown that gamma distributions have higher order moments which are easy to manipulate mathematically. When applied to a wave height dataset, the result is a very simple analytical expression for both relative power deviation and capacity factor—which are solely determined by the order of their moment and gamma shape parameter (*n*).

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