Hölderian Version of Donsker-Prohorov's Invariance Principle

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Abstract—**The weak convergence of a sequence of stochastic processes is classically studied in the Skorohod space $D[0,1]$ or $C[0,1]$ but the weak Hölder convergence offers more continous functionals than $C[0,1]$ for statistical applications. We study the weak convergence of stochastic processes in Hölder spaces and using some results of tightness proved in these spaces, we obtain a Hölderian version of Donsker-Prohorov's invariance principle. First for the polygonal interpolation of the partial sums process, generalizing Lamperti's invariance principle to the non-stationary case and similar results are proved for the convolution smoothing of the partial sums process.**

Keywords: ***stochastic processes, tightness, Hölder space, invariance principle, Brownian motion.***

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1 Introduction

In parametric and non parametric statistics, many statistical applications (estimation, testing hypothesis,...) are based on continuous functionals of paths of processes and solved using the weak convergence of stochastic processes. The weak convergence of a sequence $(\xi_n, n \geq 1)$ of stochastic processes in some functional space provides results about the asymptotic distribution of continuous functionals of the paths. Since the Hölder spaces are topologically embedded in the spaces $C[0,1]$ of continuous functions and in the Skorokhod space $D[0,1]$, they support more continuous functionals. From this point of view, the alternative framework of Hölder spaces gives functional limit theorems of a wider scope. This choice may be relevant as soon as the paths of ξ_n and the limit process (like e.g. the Brownian motion and the Brownian bridge) share some Hölder regularity. The first result in this direction seems to be Lamperti's Hölderian invariance principle [6] for the (centered and normalized) polygonal partial sums processes. This result was completed in recent contributions by Račkauskas and Suquet [8, 10] who extended it to the case of adaptive self-normalized partial sums processes and proposed a necessary and sufficient condition for a generalized form of Lamperti's invariance principle. Some statistical applications of weak Hölder convergence are proposed by the same authors [9, 11].

We consider a sequence $(X_j)_{j\geq 1}$ of independent random variables not necessarily identically distributed with $EX_j = 0$ and $EX_j^2 = \sigma_j^2$. We denote ξ_n the random polygonal lines obtained by linear interpolation between the points $\left(\frac{j}{n}, \frac{S_j}{s_n}\right)$ where $S_j = \sum_{k=1}^{j} X_j$ and $s_n = \sqrt{\sigma_1^2 + \cdots + \sigma_n^2}$.

When the X_j are identically distributed with $EX_j^2 = \sigma^2$, the Donsker-Prohorov's invariance principle establishes then the $C[0,1]$ weak convergence of ξ_n to the Brownian motion W. The invariance principle in the Banach Hölder space $H_\alpha[0,1]$ has been established by Lamperti. Kerkyacharian and Roynette have derived it again using the Faber-Schauder basis of triangular functions.

Theorem 1 (Lamperti [6]) *Let $(X_j)_{j\geq 1}$ be a sequence of independent identically distributed random variables with $EX_j = 0$ and $E|X_j|^2 = \sigma^2$. Suppose that for some constant $\gamma > 2$, $E|X_j|^\gamma < \infty$.*

For all $n \in N^*$, $0 \leq j < n$, define

$$\xi_n(t,\omega) = \frac{1}{s_n}\sum_{k=1}^{k=j} X_k(\omega) + \frac{nt - j}{s_n} X_{j+1}(\omega), \quad \frac{j}{n} \leq t < \frac{j+1}{n}.$$

Then the sequence $(\xi_n)_{n\geq 1}$ converges weakly to the Brownian motion W in H_α^0 for all $\alpha < \frac{1}{2} - \frac{1}{\gamma}$.

Using some results of tightness proved in these spaces, Hamadouche [3] has extended this result to dependent random variables(α-mixing and association) and has proved the weak convergence in H_α^0 of the convolution smoothed process to the Brownian motion.

Our aim is to extend Lamperti's theorem to polygonal and convolution smoothed partial sums process with a sequence of independent random variables $(X_j)_{j\geq 1}$ not necessarily identically distributed. The polygonal smoothing

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is sometimes rough, like in the study of weak convergence of empirical and quantile processes in [2, 4]. Thus it is interesting to study the convolution smoothing which is useful in statistical applications, estimation of density, etc. Also we use the classical definition of the partial sums process, which is useful in the literature, for a non-stationary sequence of independent random variables instead of the adaptative construction used in [10] because for the general case of triangular array of random variables with uneven variances, the convergence of finite-dimensional distributions to Brownian motion does not immediately follow from the central limit theorem.

In Section 2, we recall the Banach Hölder space $H_\alpha[0,1]$ and its closed subspace $H_\alpha^0[0,1]$. We consider stochastic processes with paths in $H_\alpha^0[0,1]$ and treat them as random elements of $H_\alpha^0[0,1]$. We give some results of the weak convergence and tightness. Our invariance principles are presented in Section 3. We extend the Donsker-Prohorov's theorem for the independent random variables not necessarily identically distributed and prove the weak convergence in $H_\alpha^0[0,1]$ of the polygonal smoothed of the partial sums process to the Brownian motion. Similar result is proved for the convolution smoothed process.

2 Random elements in Hölder space

We study stochastic processes with Hölderian paths as random elements of the functional space $H_\alpha[0,1]$. We observe directly the whole path, which corresponds to select at random a function ξ with distribution P_ξ.

2.1 Definitions

We define the Hölder space $H_\alpha[0,1]$ $(0<\alpha\leq 1)$ as the space of functions f vanishing at 0 such that

$$\|f\|_\alpha = \sup_{0<|t-s|\leq 1}\frac{|f(t)-f(s)|}{|t-s|^\alpha}<\infty. \tag{1}$$

Define the Hölderian modulus of continuity of f by

$$w_\alpha(f,\delta) = \sup_{0<|t-s|\leq \delta}\frac{|f(t)-f(s)|}{|t-s|^\alpha} \tag{2}$$

and the subspace $H_\alpha^0[0,1]$ of $H_\alpha[0,1]$ by

$$f\in H_\alpha^0 \Leftrightarrow f\in H_\alpha \text{ and } \lim_{\delta\to 0} w_\alpha(f,\delta)=0. \tag{3}$$

$(H_\alpha,\|.\|_\alpha)$ is a non-separable Banach space.

$(H_\alpha^0,\|.\|_\alpha)$ is a separable Banach space.

$\left(H_\alpha,\|.\|_\beta\right)$ is separable for $0<\beta<\alpha$ and is topologically embedded in H_β.

2.2 Weak convergence in Hölder space

The concept of weak convergence of probability measures can be formulated for the general metric space. We use this theory to obtain a whole class of limit theorems for functions of the partial sums $S_1,\dots,S_n,\dots$

2.3 Tightness

For the tightness, it is more convenient to work with $H_\alpha^0[0,1]$ which is separable instead of $H_\alpha[0,1]$. As the canonical injection of $H_\alpha^0[0,1]$ in $H_\alpha[0,1]$ is continuous, weak convergence in the former implies weak convergence in the later. A first sufficient condition for the tightness in $H_\alpha^0[0,1]$ is given by

Theorem 2 (Kerkyacharian-Roynette [5]) *Let $(\xi_n)_{n\geq1}$ be a sequence of processes vanishing at 0 and suppose there are $\gamma>0$, $\delta>0$ and $c>0$ such that*

$$\forall\lambda>0,\ P\left(|\xi_n(t)-\xi_n(s)|^\gamma\right)\leq\frac{c}{\lambda^\gamma}|t-s|^{1+\delta}. \tag{4}$$

Then the sequence $(\xi_n)_{n\geq1}$ is tight in $H_\alpha^0[0,1]$ for $0<\alpha<\frac{\delta}{\gamma}$.

The following corollary gives the moments version of the precedent theorem.

Corollary 3 (Lamperti [6]) *Let $(\xi_n)_{n\geq1}$ be a sequence of processes vanishing at 0. Suppose there are $\gamma>0$, $\delta>0$ and $c>0$ such that*

$$E|\xi_n(t)-\xi_n(s)|^\gamma\leq c|t-s|^{1+\delta}. \tag{5}$$

Then the sequence $(\xi_n)_{n\geq1}$ is tight in $H_\alpha^0[0,1]$ for $0<\alpha<\frac{\delta}{\gamma}$.

The sufficient and necessary condition which can be useful to test the optimally of certain results is given by the Hölder version of Ascoli's theorem.

Theorem 4 (Račkauskas, Suquet [7]) *Let $(\xi_n)_{n\geq1}$ be a sequence of random elements of $H_\alpha^0[0,1]$. $(\xi_n)_{n\geq1}$ is tight if and only if*

$$\forall\ \varepsilon>0,\ \lim_{\delta\to0}\sup_{n\geq1}P\left(w_\alpha(\xi_n,\delta)\geq\varepsilon\right)=0. \tag{6}$$

For more flexibility in the handling of moment inequalities, we use this following result.

Theorem 5 (Hamadouche [3]) *Let $(\xi_n)_{n\geq 1}$ be a sequence of random elements of $H_\alpha^0[0,1]$, satisfying the following conditions*

a) there exist constants $a>1$, $b>1$, $c>0$ and a sequence of positive numbers $(a_n)\searrow 0$ such that

$$E\left|\xi_n(t)-\xi_n(s)\right|^a \leq c\left|t-s\right|^b, \tag{7}$$

for all $|t-s|\geq a_n$, $0\leq s,t\leq 1$ and $n\geq 1$.

b) For any $\varepsilon>0$,

$$\lim_{n\to\infty} P\left(\omega_\alpha(\xi_n,a_n)>\varepsilon\right)=0. \tag{8}$$

Then for all $\alpha<a^{-1}(\min(a,b)-1)$, $(\xi_n)_{n\geq 1}$ is tight in $H_\alpha^0[0,1]$.

3 Invariance principles in Hölder space

We consider the sequence $(X_j)_{j\geq 1}$ of independent random variables, not necessarily identically distributed with $EX_j=0$ and $\sigma_j^2=EX_j^2$. Denote $s_n=\sqrt{\sigma_1^2+\cdots+\sigma_n^2}$, $S_i=\sum_{k=1}^i X_k$ and $S_0=0$. We suppose that there exist $\gamma>2$, $m>0$ and $M>0$ such that $\forall\ j\geq 1$

$$E\left|X_j\right|^\gamma \leq M<\infty \quad \text{and} \quad m\leq \sigma_j^2=E\left|X_j\right|^2 \underset{j\to\infty}{\longrightarrow} \sigma^2. \tag{9}$$

3.1 Polygonal smoothing of partial sums process

We extend here the Donsker-Prohorov's theorem for independent random variables not necessarily identically distributed which is the first extension of Lamperti's invariance principle.

Theorem 6 *Let $(X_j)_{j\geq 1}$ be a sequence of independent random variables, not necessarily identically distributed with $EX_j=0$ and $EX_j^2=\sigma_j^2$ and satisfying (9).*

Define for all $n\in N^*$, $0\leq j<n$,

$$\xi_n(t,\omega)=\frac{1}{s_n}\sum_{k=1}^{k=j}X_k(\omega)+\frac{nt-j}{s_n}X_{j+1}(\omega), \tag{10}$$

for all $\frac{j}{n}\leq t<\frac{j+1}{n}$.

Then the sequence $(\xi_n)_{n\geq 1}$ converges weakly to the Brownian motion W in $H_\alpha^0[0,1]$ for all $\alpha<\frac{1}{2}-\frac{1}{\gamma}$.

Proof. We apply Theorem 5, with $a_n=\frac{1}{n}$.

Tightness of distributions $P_n=P\xi_n^{-1}$.

By Corollary 3, it is sufficient, to prove that under the assumptions of Theorem 6

$$E\left|\xi_n(t)-\xi_n(s)\right|^\gamma \leq K\left|t-s\right|^{1+\delta} \quad \text{with } 1+\delta=\frac{\gamma}{2}>1.$$

First case: $\frac{j}{n}\leq s\leq t\leq\frac{j+1}{n}$.

We have

$$|\xi_n(t)-\xi_n(s)|=\left|\frac{1}{s_n}n(t-s)X_{j+1}(\omega)\right|.$$

Under the assumption (9) it is easy to see that there exists a constant $M'=M^{\frac{2}{\gamma}}$ such that

$$m<E\left|X_j\right|^2\leq M', \quad \forall j\geq 1. \tag{11}$$

By the last inequalities, we deduce

$$|\xi_n(t)-\xi_n(s)|\leq\left|\frac{1}{\sqrt{nm}}n(t-s)X_{j+1}\right|=$$

$$\left(\frac{n}{m}\right)^{\frac{1}{2}}\left|(t-s)X_{j+1}\right|.$$

Then

$$E\left|\xi_n(t)-\xi_n(s)\right|^\gamma\leq E\left|\left(\frac{n}{m}\right)^{\frac{1}{2}}(t-s)X_{j+1}\right|^\gamma\leq$$

$$m^{-\frac{\gamma}{2}}M\left|t-s\right|^{\frac{\gamma}{2}},$$

since $n|t-s|\leq 1$ and $E\left|X_{j+1}\right|^\gamma<M$.

Finally

$$E\left|\xi_n(t)-\xi_n(s)\right|^\gamma\leq K\left|t-s\right|^{1+\delta}, \tag{12}$$

with $K=m^{-\frac{\gamma}{2}}M$ and $1+\delta=\frac{\gamma}{2}>1$.

Second case: $\frac{j-1}{n}\leq s\leq\frac{j}{n}\leq\frac{j+k}{n}\leq t\leq\frac{j+k+1}{n}, k=0,1,...,n-j-1$.

By triangular inequality,

$$|\xi_n(t)-\xi_n(s)|\leq\left|\xi_n(t)-\xi_n(\frac{j+k}{n})\right|+\left|\xi_n(\frac{j+k}{n})-\xi_n(\frac{j}{n})\right|$$

$$+\left|\xi_n(\frac{j}{n})-\xi_n(s)\right|.$$

By Jensen's inequality we have

$$\begin{aligned} E\left|\xi_n(t)-\xi_n(s)\right|^\gamma \quad &\leq \quad 3^{\gamma-1}\Bigg(E\left|\xi_n(t)-\xi_n(\frac{j+k}{n})\right|^\gamma \\ &+E\left|\xi_n(\frac{j+k}{n})-\xi_n(\frac{j}{n})\right|^\gamma \\ &+E\left|\xi_n(\frac{j}{n})-\xi_n(s)\right|^\gamma\Bigg). \end{aligned}$$

The first and the third terms can be treated as in the precedent case, thus there exist some constants K_1 and K_3 such that

$$E\left|\xi_n(t)-\xi_n(\frac{j+k}{n})\right|^\gamma \leq K_1\left|t-s\right|^{1+\delta} \qquad (13)$$

and

$$E\left|\xi_n(\frac{j}{n})-\xi_n(s)\right|^\gamma \leq K_3\left|t-s\right|^{1+\delta}. \qquad (14)$$

For the middle term, we have

$$E\left|\xi_n(\frac{j+k}{n})-\xi_n(\frac{j}{n})\right|^\gamma = E\left|\frac{1}{s_n}S_{j+k}-\frac{1}{s_n}S_j\right|^\gamma \leq \left(\frac{1}{nm}\right)^{\frac{\gamma}{2}} E\left|S_{j+k}-S_j\right|^\gamma.$$

By the Marcinkiewicz-Zygmund's inequality, it follows

$$\begin{aligned} E\left|\xi_n(\frac{j+k}{n})-\xi_n(\frac{j}{n})\right|^\gamma \quad &\leq \quad \left(\frac{1}{nm}\right)^{\frac{\gamma}{2}} C_\gamma . \\ &\left(\sum_{i=1}^k E\left|X_i\right|^2\right)^{\frac{\gamma}{2}} \\ &\leq \quad \left(\frac{1}{nm}\right)^{\frac{\gamma}{2}} C_\gamma\left(kM'\right)^{\frac{\gamma}{2}} = \\ &C_\gamma\left(\frac{M'}{m}\right)^{\frac{\gamma}{2}}\left(\frac{k}{n}\right)^{\frac{\gamma}{2}}. \end{aligned}$$

Since $|t-s| \geq \frac{j+k}{n}-\frac{j}{n}=\frac{k}{n}$,

$$E\left|\xi_n(\frac{j+k}{n})-\xi_n(\frac{j}{n})\right|^\gamma \leq C_\gamma\left(\frac{M'}{m}\right)^{\frac{\gamma}{2}}\left|t-s\right|^{\frac{\gamma}{2}}.$$

We deduce that there exist constants $K_2=C_\gamma\left(\frac{M'}{m}\right)^{\frac{\gamma}{2}}$ and $\delta>0$ such that

$$E\left|\xi_n(\frac{j+k}{n})-\xi_n(\frac{j}{n})\right|^\gamma \leq K_2\left|t-s\right|^{1+\delta}. \qquad (15)$$

With the three inequalities (13), (14) and (15) we obtain

$$E\left|\xi_n(t)-\xi_n(s)\right|^\gamma \leq 3^{\gamma-1}\left(K_1+K_2+K_3\right)\left|t-s\right|^{1+\delta}.$$

So, there exists a constant $K=3^{\gamma-1}\left(K_1+K_2+K_3\right)$ such that

$$E\left|\xi_n(t)-\xi_n(s)\right|^\gamma \leq K\left|t-s\right|^{1+\delta} \quad \text{with} \quad 1+\delta=\frac{\gamma}{2}>1.$$

Thus by Corollary 3, the sequence of distributions $(P_n)_{n\geq1}$ of processes ξ_n is tight in $H_\alpha^0[0,1]$ for any $0<\alpha<\frac{\delta}{\gamma}=\frac{1}{2}-\frac{1}{\gamma}$.

Convergence of the finite-dimensional distributions.

To show that the finite-dimensional distributions of the ξ_n converges to those of W, we consider first a point s and must prove that $\xi_n(s) \overset{D}{\longrightarrow} W_s$.

With the definition of ξ_n, we have

$$\left|\xi_n(s)-\frac{S_{[ns]}}{s_n}\right| = (ns-[ns])\frac{1}{s_n}X_{[ns]+1} \leq \frac{1}{s_n}X_{[ns]+1}.$$

It suffices to prove that $\frac{1}{s_n}X_{[ns]+1} \overset{P}{\longrightarrow} 0$ and $\frac{S_{[ns]}}{s_n} \overset{D}{\longrightarrow} W_s$.

For $\varepsilon>0$, the Bienaymé-Tchebychev's inequality implies that

$$P\left(\frac{1}{s_n}X_{[ns]+1}\geq\varepsilon\right) \leq \frac{1}{nm}\frac{M'}{\varepsilon^2} \underset{n\to\infty}{\to} 0$$

since $nm \leq s_n^2$ and $VAR(X_j) \leq M'$, $\forall\ j\geq1$. Then $\frac{1}{s_n}X_{[ns]+1} \overset{P}{\longrightarrow} 0$.

We know that by the Lindberg's theorem, $\frac{S_n}{s_n}$ converges in distribution to the normal law $\mathcal{N}$ if

$$\lim_{n\to\infty}\sum_{k=1}^n\frac{1}{s_n^2}\int_{|X_k|\geq\varepsilon s_n} X_k^2\,dP=0.$$

With the assumption $|X_k|\geq\varepsilon s_n$, we have $1\leq\frac{|X_k|^\delta}{|\varepsilon s_n|^\delta}$

$$\begin{aligned} \sum_{k=1}^n\frac{1}{s_n^2}\int_{|X_k|\geq\varepsilon s_n} X_k^2\,dP \quad &\leq \quad \sum_{k=1}^n\frac{1}{s_n^2}\int_{|X_k|\geq\varepsilon s_n}\frac{\left|X_k\right|^{2+\delta}}{\left|\varepsilon s_n\right|^\delta}\,dP \\ &\leq \quad \frac{M}{\varepsilon^\delta m^{\frac{2+\delta}{2}}}\frac{1}{n^{\frac{\delta}{2}}} \underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Then $\frac{S_{[ns]}}{s_n} \overset{D}{\longrightarrow} \mathcal{N}$. Using the independence of random variables and the assumption (9), it is easy to prove that

$$\lim_{n\longrightarrow\infty} VAR\left[\frac{S_{[ns]}}{s_n}\right]=s \quad \text{and} \quad E\left(\frac{S_{[ns]}}{s_n}\right)=0,$$

so $\frac{S_{[ns]}}{s_n} \xrightarrow{D} W_s$.

Consider now two points s and t with $s < t$. By the same arguments applied to the triangular array $(X_i, [ns] < i \leq [nt])$, we obtain

$$\frac{S_{[nt]} - S_{[ns]}}{s_n} \xrightarrow{D} W_t - W_s.$$

The two components of $\left(\frac{S_{[ns]}}{s_n}, \frac{S_{[nt]}}{s_n} - \frac{S_{[ns]}}{s_n}\right)$ are independent by the independence of (X_j). Since $\mathbb{R}$ is separable, it follows that

$$\left(\frac{S_{[ns]}}{s_n}, \frac{S_{[nt]}}{s_n} - \frac{S_{[ns]}}{s_n}\right) \xrightarrow{D} (W_s, W_t - W_s).$$

On the other hand

$$\left|(\xi_n(t) - \xi_n(s)) - \left(\frac{S_{[nt]}}{s_n} - \frac{S_{[ns]}}{s_n}\right)\right| \leq \left|\frac{1}{s_n} X_{[nt]+1}\right| + \left|\frac{1}{s_n} X_{[ns]+1}\right|.$$

We have shown that $\frac{1}{s_n} X_{[ns]+1} \overset{P}{\to} 0$ consequently $\frac{1}{s_n} X_{[nt]+1} \overset{P}{\to} 0$ so it follows that $\left|(\xi_n(t) - \xi_n(s)) - \left(\frac{S_{[nt]}}{s_n} - \frac{S_{[ns]}}{s_n}\right)\right| \xrightarrow{P} 0$ and hence $|\xi_n(t) - \xi_n(s)| \xrightarrow{D} (W_t - W_s)$. Since $\xi_n(s)$ and $\xi_n(t) - \xi_n(s)$ are independent by the independence of X_j, we deduce that

$$(\xi_n(s), \xi_n(t) - \xi_n(s)) \xrightarrow{D} (W_s, W_t - W_s).$$

We conclude that $(\xi_n(s), \xi_n(t)) \xrightarrow{D} (W_s, W_t)$ since the function h defined by $(x, y) \mapsto (x, x + y)$ is continuous.

We treat a set of three or more points in the same way, and hence the finite-dimensional distributions converges properly. This achieves the proof of Theorem 6.

3.2 Convolution smoothing of partial sums process

We recall here some results and some assumptions used by Hamadouche [3] for the convolution smoothed process in $H_\alpha[0, 1]$. We consider the Donsker-Prohorov's normalized partial sums process

$$\xi_n(t) = \frac{1}{s_n} S_{[nt]}(t), \quad t \in [0,1]. \tag{16}$$

For the sake of convenience, we shall use both following expressions of ξ_n

$$\xi_n(t) = \frac{1}{s_n} \sum_{i=1}^{n} S_i 1_{\left[\frac{i}{n}, \frac{i+1}{n}\right[}(t), \tag{17}$$

$$\xi_n(t) = \frac{1}{s_n} \sum_{k=1}^{n} X_k 1_{\left[\frac{k}{n}, 1\right]}(t). \tag{18}$$

Let K be a probability density on the real line $\mathbb{R}$ such that

$$\int_{\mathbb{R}} |u| K(u)\, du < \infty \tag{19}$$

and $(b_n)_{n\geq 1}$ a sequence of positive numbers such that $\lim_{n\to\infty} b_n = 0$ and

$$\frac{1}{b_n} = O\left(n^{\frac{\tau}{2}}\right), \quad 0 < \tau < \frac{1}{2}. \tag{20}$$

We define the sequence $(K_n)_{n\geq 1}$ of convolution kernels by

$$K_n(t) = \frac{1}{b_n} K\left(\frac{t}{b_n}\right), \qquad t \in \mathbb{R}. \tag{21}$$

Lemma 7 (Hamadouche [3]) *Let f be a bounded measurable function with support in $[0, 1]$ and K a convolution kernel satisfying*

$$K \in L^1([-1, 1]) \cap L^{\frac{1}{2}}([-1, 1]), \tag{22}$$

$$|K(x) - K(y)| \leq \alpha(K) |x - y|, \quad x, y \in [-1, 1], \tag{23}$$

for some constant $\alpha(K)$. Then the restriction to $[0\ ,1]$ of $(f * K) - (f * K)(0)$ is in $H_{\frac{1}{2}}[0, 1]$.

We consider the smoothed partial sums process defined by

$$\zeta_n(t) = (\xi_n * K_n)(t) - (\xi_n * K_n)(0) \qquad t \in [0, 1]. \tag{24}$$

The term $(\xi_n * K_n)(0)$ is subtracted in order to have a process with paths vanishing at zero.

Theorem 8 *Let $(X_j)_{j\geq 1}$ be a sequence of independent random variables, not necessarily identically distributed with $EX_j = 0$ and $EX_j^2 = \sigma_j^2$ and satisfying (9). Suppose that the convolution kernels K_n satisfy (19), (21), (22) and (23). Then the sequence of smoothed partial sums process ζ_n defined by (24) converges weakly to the Brownian motion W in $H_\alpha^0[0,1]$ for all $\alpha < \frac{1}{2} - \max\left(\tau, \frac{1}{\gamma}\right)$.*

Proof. By lemma 7, ζ_n is in $H_\alpha^0[0, 1]$ for all $\alpha < \frac{1}{2}$. We apply Theorem 5 with $a_n = \frac{1}{n}$. We recall that

$$(\xi_n * K_n)(t) = \int_{\mathbb{R}} \xi_n(t - u) K_n(u)\, du =$$

$$\int_{\mathbb{R}} \xi_n(u) K_n(t-u)\, du.$$

Tightness. Using Theorem 5 with $a_n = \frac{1}{n}$, we study separately the cases $t - s \geq \frac{1}{n}$ and $t - s < \frac{1}{n}$. Without loss of generality we can assume that $t > s$.

First case: $t - s \geq \frac{1}{n}$.

$$E|\zeta_n(t) - \zeta_n(s)|^\gamma = E|\xi_n * K_n(t) - \xi_n * K_n(s)|^\gamma =$$
$$E\left|\frac{1}{s_n}\int_{\mathbb{R}}\left(\sum_{k=[n(s-u)]+1}^{[n(t-u)]} X_k\right) K_n(u)\, du\right|^\gamma =$$
$$E\left|E_{K_n}\left(\frac{1}{s_n}\sum_{k=[n(s-u)]+1}^{[n(t-u)]} X_k\right)\right|^\gamma.$$

Applying the Jensen's inequality whith respect to $K_n(u)\, du$ we obtain

$$E|\zeta_n(t) - \zeta_n(s)|^\gamma \leq E\left(E_{K_n}\left|\frac{1}{s_n}\sum_{k=[n(s-u)]+1}^{[n(t-u)]} X_k\right|^\gamma\right).$$

By the Fubini's theorem we have

$$E|\zeta_n(t) - \zeta_n(s)|^\gamma \leq E_{K_n}\left(E\left|\frac{1}{s_n}\sum_{k=[n(s-u)]+1}^{[n(t-u)]} X_k\right|^\gamma\right)$$
$$\leq \int_{\mathbb{R}} E\left|\frac{1}{s_n}\sum_{k=[n(s-u)]+1}^{[n(t-u)]} X_k\right|^\gamma K_n(u)\, du.$$

Using the Marcinkiewicz-Zygmund's inequality for the moments of sums of the independent random variables we obtain

$$E|\zeta_n(t) - \zeta_n(s)|^\gamma \leq$$

$$\int_{\mathbb{R}}\left(\frac{1}{s_n}\right)^\gamma C_\gamma\left(\sum_{k=[n(s-u)]+1}^{[n(t-u)]} E|X_k|^2\right)^{\frac{\gamma}{2}} K_n(u)\, du.$$

Using the assumptions (9) and (11), we deduce that

$$E|\zeta_n(t) - \zeta_n(s)|^\gamma \leq$$
$$\int_{\mathbb{R}}\left(\frac{1}{\sqrt{nm}}\right)^\gamma C_\gamma\left(\sum_{k=[n(s-u)]+1}^{[n(t-u)]} M'\right)^{\frac{\gamma}{2}} K_n(u)\, du \leq$$
$$\int_{\mathbb{R}}\left(\frac{1}{\sqrt{nm}}\right)^\gamma C_\gamma M_\gamma\left([n(t-u)] - [n(s-u)]\right)^{\frac{\gamma}{2}} K_n(u)\, du.$$

Since $[n(t-u)] - [n(s-u)] \leq n(t-s) + 2$ and $|t-s| \geq \frac{1}{n}$,

$$E|\zeta_n(t) - \zeta_n(s)|^\gamma \leq$$
$$\int_{\mathbb{R}} (mn)^{-\frac{\gamma}{2}} C_\gamma M_\gamma (n|t-s| + 2)^{\frac{\gamma}{2}} K_n(u)\, du \leq$$
$$\int_{\mathbb{R}} m^{-\frac{\gamma}{2}} C_\gamma M_\gamma \left(|t-s| + \frac{2}{n}\right)^{\frac{\gamma}{2}} K_n(u)\, du \leq$$
$$\int_{\mathbb{R}} m^{-\frac{\gamma}{2}} C_\gamma M_\gamma (|t-s| + 2|t-s|)^{\frac{\gamma}{2}} K_n(u)\, du \leq$$
$$\int_{\mathbb{R}} m^{-\frac{\gamma}{2}} C_\gamma M_\gamma (3|t-s|)^{\frac{\gamma}{2}} K_n(u)\, du \leq$$
$$m^{-\frac{\gamma}{2}} C_\gamma M_\gamma (3|t-s|)^{\frac{\gamma}{2}} \int_{\mathbb{R}} K_n(u)\, du \leq$$
$$m^{-\frac{\gamma}{2}} C_\gamma M_\gamma 3^{\frac{\gamma}{2}} |t-s|^{\frac{\gamma}{2}}.$$

Hence there exists a constant $C'_\gamma = m^{-\frac{\gamma}{2}} C_\gamma M_\gamma 3^{\frac{\gamma}{2}}$ such that

$$E|\zeta_n(t) - \zeta_n(s)|^\gamma \leq C'_\gamma |t-s|^{\frac{\gamma}{2}}.$$

Second case: $0 \leq t - s < \frac{1}{n}$.

$$|\zeta_n(t) - \zeta_n(s)| = |\xi_n * K_n(t) - \xi_n * K_n(s)| =$$
$$\left|\int_{\mathbb{R}} \frac{1}{s_n}\sum_{k=1}^{n} X_k \left(K_n(t-u) - K_n(t-s)\right) 1_{\left[\frac{k}{n},1\right]}(u)\, du\right| \leq$$
$$\int_{\mathbb{R}} \frac{1}{s_n}\sum_{k=1}^{n} |X_k|\, |K_n(t-u) - K_n(t-s)|\, 1_{\left[\frac{k}{n},1\right]}(u) du \leq$$
$$\int_{\mathbb{R}} \frac{1}{s_n}\sum_{k=1}^{n} |X_k| \frac{1}{b_n}\left|K\left(\frac{t-u}{b_n}\right) - K\left(\frac{s-u}{b_n}\right)\right| 1_{\left[\frac{k}{n},1\right]}(u)\, du.$$

Using Assumption (23) of Lemma 7, it follows

$$|\zeta_n(t) - \zeta_n(s)| \leq$$
$$\int_{\mathbb{R}} \frac{1}{s_n}\sum_{k=1}^{n} |X_k| \frac{1}{b_n}\alpha(K)\left|\frac{t-s}{b_n}\right| 1_{\left[\frac{k}{n},1\right]}(u)\, du \leq$$
$$\frac{1}{s_n}\sum_{k=1}^{n} |X_k| \frac{1}{b_n}\alpha(K)\frac{|t-s|}{b_n}\int_{\mathbb{R}} 1_{\left[\frac{k}{n},1\right]}(u)\, du \leq$$
$$\frac{1}{\sqrt{nm}}\sum_{k=1}^{n} |X_k| \frac{1}{b_n^2}\alpha(K)|t-s|\left(1 - \frac{k}{n}\right) \leq$$
$$\frac{1}{\sqrt{nm}}\frac{1}{b_n^2}\alpha(K)\sum_{k=1}^{n} |X_k|\, |t-s|,$$

since $\left(1 - \frac{k}{n}\right) < 1$ and hence

$$\frac{|\zeta_n(t) - \zeta_n(s)|}{|t-s|^\alpha} \leq \frac{1}{\sqrt{nm}}\frac{1}{b_n^2}\alpha(K)\sum_{k=1}^{n} |X_k|\, |t-s|^{1-\alpha}.$$

As a result,

$$w_\alpha\left(\zeta_n,\frac{1}{n}\right)=\sup_{|t-s|\le\frac{1}{n}}\frac{|\zeta_n(t)-\zeta_n(s)|}{|t-s|^\alpha}\le$$
$$\frac{1}{\sqrt{m}}\frac{1}{b_n^2}\alpha(K)\,n^{-\frac{1}{2}}\sum_{k=1}^{n}|X_k|\left|\frac{1}{n}\right|^{1-\alpha}\le$$
$$\frac{1}{\sqrt{m}}\frac{1}{b_n^2 n^{\frac{1}{2}-\alpha}}\alpha(K)\frac{1}{n}\sum\nolimits_{k=1}^{n}|X_k|\,.$$

Now to prove that $w_\alpha\left(\zeta_n,\frac{1}{n}\right)\overset{P}{\to}0$, it suffices to show that

$$\frac{1}{\sqrt{m}}\frac{1}{b_n^2 n^{\frac{1}{2}-\alpha}}\alpha(K)\frac{1}{n}\sum_{k=1}^{n}|X_k|\overset{P}{\to}0.$$

By the Markov's inequality we have

$$P\left[\frac{\alpha(K)}{\sqrt{m}}\frac{1}{b_n^2 n^{\frac{1}{2}-\alpha}}\frac{1}{n}\sum_{k=1}^{n}|X_k|>\delta\right]\le$$

$$\frac{1}{\delta}\frac{\alpha(K)}{\sqrt{m}}\frac{1}{b_n^2 n^{\frac{1}{2}-\alpha}}\frac{1}{n}\sum_{k=1}^{n}E|X_k|\,.$$

The assumption (9) and the Schwartz's inequality give $E|X_k|\le M'^{\frac{1}{2}}$ so

$$P\left[\frac{\alpha(K)}{\sqrt{m}}\frac{1}{b_n^2 n^{\frac{1}{2}-\alpha}}\frac{1}{n}\sum\nolimits_{k=1}^{n}|X_k|>\delta\right]\le$$

$$\frac{1}{\delta}\frac{\alpha(K)}{\sqrt{m}}\frac{1}{b_n^2 n^{\frac{1}{2}-\alpha}}\frac{n\sqrt{M'}}{n}\underset{n\to\infty}{\to}0\,.$$

Using the assumption (20) we deduce that

$$\frac{1}{b_n^2}n^{a-\frac{1}{2}}=\left(\frac{1}{b_n}n^{\frac{1}{2}\left(\alpha-\frac{1}{2}\right)}\right)^2\underset{n\to\infty}{\longrightarrow}0\quad\text{if}\quad\alpha<\frac{1}{2}-\tau.$$

We conclude about tightness by Theorem 5, noting that its hypothesis are satisfied for $a=\gamma$, $b=\frac{\gamma}{2}$, $c=C'_\gamma$ and $a_n=\frac{1}{n}$. We obtain then the tightness of (ζ_n) in $H_\alpha^0[0,1]$ for all $\alpha<\frac{1}{2}-\tau$ and $\alpha<\frac{1}{2}-\frac{1}{\gamma}$ so for all $\alpha<\frac{1}{2}-\max\left(\tau,\frac{1}{\gamma}\right)$.

Convergence of the finite-dimensional distributions of $\{\zeta_n,\ n\ge1\}$. By Theorem 6, the finite-dimensional distributions of ξ_n converge to those of the Brownian motion, it will be the same for those of ζ_n if we prove for instance the convergence to zero of $E|\zeta_n(t)-\xi_n(t)|^2$ for all $t\in[0,1]$.

$$E|\zeta_n-\xi_n(t)|^2=E\left|\int_R\xi_n(t-u)K_n(u)\,du-\xi_n(t)\right|^2$$
$$=E\left|\int_R\frac{1}{s_n}\left(S_{[n(t-u)]}-S_{[nt]}\right)K_n(u)\,du\right|^2$$
$$=E\left|\int_R\frac{1}{s_n}\sum_{i=[nt]+1}^{[n(t-u)]}X_iK_n(u)\,du\right|^2$$
$$=E\left|E_k\left[\frac{1}{s_n}\sum_{i=[nt]+1}^{[n(t-u)]}X_i\right]\right|^2.$$

Applying the Jensen's inequality with respect to K_n, we obtain

$$E|\xi_n*K_n(t)-\xi_n(t)|^2\le E\left[E_k\left|\frac{1}{s_n}\sum_{i=[nt]+1}^{[n(t-u)]}X_i\right|^2\right].$$

By the Fubini's theorem we have

$$E|\xi_n*K_n(t)-\xi_n(t)|^2\le E_{K_n}\left[E\left|\frac{1}{s_n}\sum_{i=[nt]+1}^{[n(t-u)]}X_i\right|^2\right]$$
$$\le\int_{\mathbb{R}}E\left|\frac{1}{s_n}\sum_{i=[nt]+1}^{[n(t-u)]}X_i\right|^2K_n(u)\,du.$$

Applying again the Marcinkiewiez-Zygmund's inequality, it follows

$$E|\xi_n*K_n(t)-\xi_n(t)|^2\le$$
$$c\int_{\mathbb{R}}\left[\left(\frac{1}{s_n}\right)^2\sum_{i=[nt]+1}^{[n(t-u)]}E\left|X_i^2\right|\right]K_n(u)du\le$$
$$c\int_{\mathbb{R}}\frac{M'}{nm}(n|u|+2)K_n(u)du$$
$$(\text{since }([n(t-u)]-[nt])\le(n|u|+2))$$
$$\le c\frac{M'}{m}\int_{\mathbb{R}}\left(|u|+\frac{2}{n}\right)K_n(u)du\le$$
$$c'\int_{\mathbb{R}}\left(|u|+\frac{2}{n}\right)K_n(u)du\quad\text{with }c'=c\frac{M'}{m}.$$

By (22) and noting $v=\frac{u}{b_n}$, it follows that

$$E|\xi_n*K_n(t)-\xi_n(t)|^2\le$$
$$c'\int_{\mathbb{R}}\left(b_n|v|+\frac{2}{n}\right)\frac{1}{b_n}K(v)b_ndv\le$$
$$c'\left(b_n\int_{\mathbb{R}}|v|K(v)dv+\frac{2}{n}\right).$$

Since $\int_{\mathbb{R}} |v|\, K(v)dv < \infty$ and b_n goes to zero as n goes to infinity, we deduce that $\xi_n(t) * K_n(t) - \xi_n(t)$ goes to 0 in $L^2(\Omega)$ for all $t \in [0,1]$. In particular for $t = 0$, $E\left|\xi_n * K_n(0)\right|^2$ goes to 0 as n goes to infinity. We have finally, for all $t \in [0,1]$

$$E\left|\zeta_n(t) - \xi_n(t)\right|^2 =$$
$$E\left|(\xi_n * K_n)(t) - (\xi_n * K_n)(0) - \xi_n(t)\right|^2 \leq$$
$$2\left(E\left|(\xi_n * K_n)(t) - \xi_n(t)\right|^2 + E\left|(\xi_n * K_n)(0)\right|^2\right) \underset{n\to\infty}{\to} 0.$$

Hence

$$\zeta_n(t) - \xi_n(t) \overset{L^2}{\to} 0 \text{ and implies } \zeta_n(t) - \xi_n(t) \overset{P}{\to} 0$$

so

$$\sum_{i=1}^{k} \left|\zeta_n(t_i) - \xi_n(t_i)\right|^2 = \|\zeta_n - \xi_n\|^2_{\mathbb{R}^k} \overset{P}{\to} 0.$$

This achieves the proof of the convergence of the finite-dimensional distribution of ζ_n. The Theorem 8 is then proved.

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